The dynamics of unorganized segregation

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[1] Joint work with David Haw (Imperial College)

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MS137 Modelling female and minority representation in society

University of BRISTOL

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Schelling, T.C. *Models of segregation.*
The American Economic Review **59**:488–493 (1969)
Schelling, T.C. *Dynamic model of segregation.*
Journal of Mathematical Sociology **1**:143–186 (1971)

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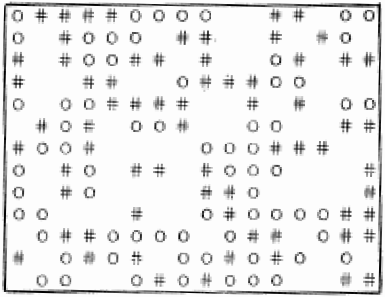
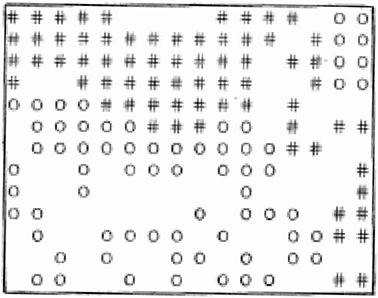
- Conversations with film director Stanley Kubrick led to movie "Dr Strangelove" (*Schelling's dilemma*).

Schelling's Spatial Proximity Model (SPM)

- His Spatial Proximity Model (SPM) is an early agent-based model.
- Two groups distributed in random order on chessboard.
- Jump to empty square if fewer than half your neighbours are same as you (notion of *tolerance*).
- Leads to (self-organized) segregation in almost all cases.





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Haw, D.J. and Hogan, S.J.
A dynamical systems model of unorganized segregation.
Journal of Mathematical Sociology **42**:113–127 (2018)

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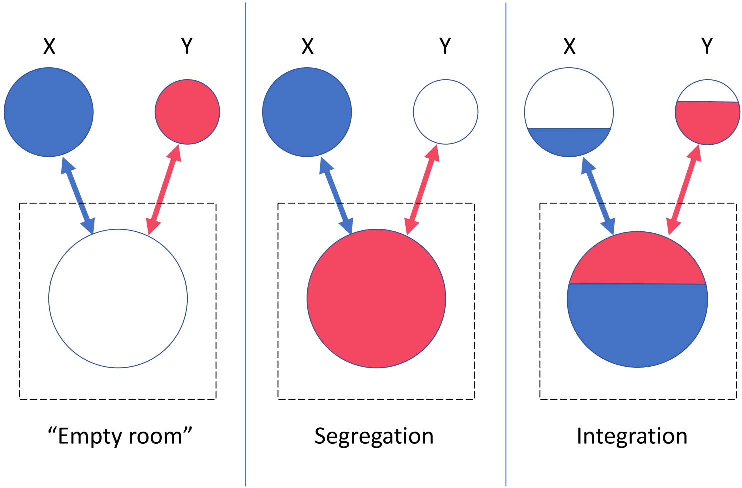
BNM: Within a group and between groups, different people can have different tolerances, as follows:

- Two groups X, Y of different sizes in one neighbourhood (Y is the minority).
- Everyone is concerned about the ethnic composition of the neighbourhood.
- People will stay in the neighbourhood until their own *limiting tolerance ratio* is reached.
- Limiting tolerance ratio is monotone decreasing (the *most* tolerant are the first to enter and last to leave; the *least* tolerant are the last to enter and the first to leave).

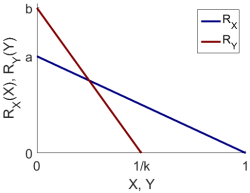
BNM - one neighbourhood, inc. reservoirs*



* "Places where colour does not matter" (Schelling)

BNM - tolerance

- Limiting tolerance ratios given by $Y/X = R_X(X)$ & $X/Y = R_Y(Y)$



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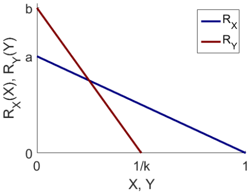


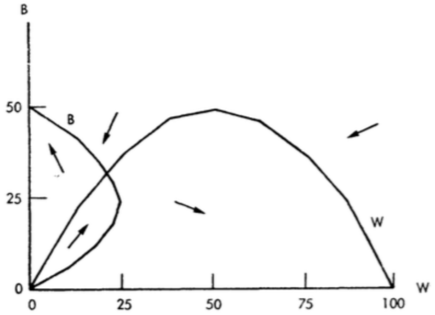
X-population scaled to 1 and $k > 1$, as Y is the minority.

- *Linear* $R_X(X), R_Y(Y)$ are parabolae in the (X, Y) plane:

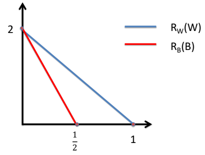
$$\begin{aligned} Y &= XR_X(X) = aX(1-X) \\ X &= YR_Y(Y) = bY(1-kY). \end{aligned}$$

Parameters $\alpha \equiv ak$, $\beta \equiv ab$ important in sequel.

BNM - Schelling example, with $(X, Y) = (W, B)$.



- Example from Schelling: $(a, b, k) = (2, 2, 2)$



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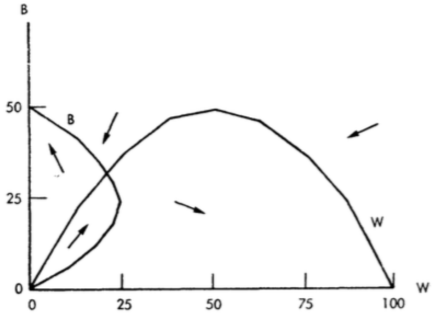

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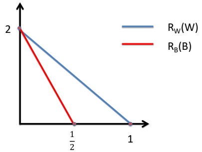
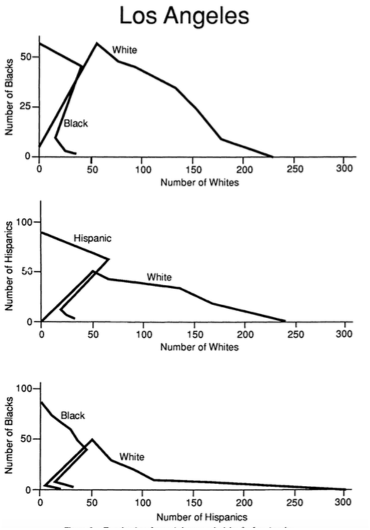


- 90 W tolerate 18 B,
  75 W tolerate 37.5 B,
  50 W tolerate 50 B,
  25 W tolerate 37.5 B.

BNM - Clark (1991) data.



- Clark (1991) collected data from telephone surveys.
- All respondents asked identical question: "Suppose you . . . have found a nice place. What mixture of neighbours would you prefer?"
- Results similar to Schelling assumptions, but with smaller overlap.
- See also Michelle Feng MS112 . . . yesterday.

BNM - DJH & SJH: key idea

- The tolerance parabolae are *nullclines*, corresponding to zero growth of the respective population, of a *Schelling dynamical system*.

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- The tolerance parabolae are *nullclines*, corresponding to zero growth of the respective population, of a *Schelling dynamical system*.
- In addition, the lines $X = 0$ and $Y = 0$ are nullclines.
- The *intersection* of nullclines are *equilibria* of the Schelling dynamical system, whose stability can be examined by standard methods.

Schelling dynamical system

- For linear tolerance schedules

$$\begin{aligned}\dot{X} &= [aX(1-X) - Y]\,X\\ \dot{Y} &= [bY(1-kY) - X]\,Y.\end{aligned}$$

Schelling dynamical system

- For linear tolerance schedules

$$\begin{aligned}\dot{X} &= [aX(1-X)-Y]\,X\\ \dot{Y} &= [bY(1-kY)-X]\,Y.\end{aligned}$$

- Rescale time $\hat{t} = at$, set $\hat{Y} = aY$ and drop hats. Then

$$\begin{aligned}\dot{X} &= [X(1-X)-Y]\,X\\ a\dot{Y} &= [\beta Y(1-\alpha Y)-X]\,Y\end{aligned}$$

where $\alpha \equiv ak > 0, \quad \beta \equiv ab > 0.$

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- Equilibria $(X, Y) = (X_e, Y_e)$ are (real, positive) solutions of $Y = X(1 - X)$ and $X = \beta Y(1 - \alpha Y)$.
- Clearly $(X_e, Y_e) = (0, 0), (1, 0), (0, \frac{1}{\alpha})$. These correspond to:
    i) the "empty room",
    ii) X-population only in the neighbourhood.
    iii) Y-population only in the neighbourhood .

Unlimited numbers - integrated equilibria

Integrated equilibria satisfy $X_e^3 + a_2 X_e^2 + a_1 X_e + a_0 = 0$, $Y_e = X_e(1 - X_e)$ where $a_2 \equiv -2, \quad a_1 \equiv \frac{1+\alpha}{\alpha}, \quad a_0 \equiv \frac{1-\beta}{\alpha\beta}$ and both $X_e, Y_e \neq 0$.

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- Cubic has *three* real roots when $\beta_-(\alpha) < \beta < \beta_+(\alpha)$ where

$$\beta_\pm(\alpha) = \frac{9\alpha - 2\alpha^2 \pm 2\sqrt{\alpha(\alpha-3)^3}}{4-\alpha}$$

provided $\alpha > 3$; *one* real root otherwise.

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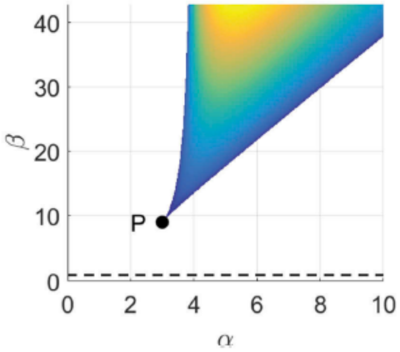
  provided $\alpha > 3$; *one* real root otherwise.
- $\beta = \beta_\pm(\alpha)$ is a *supercritical pitchfork*.

Unlimited numbers - integrated equilibria

- We have three real roots (inc. one stable integrated equilibrium) when (α, β) lies in the shaded region, where $P : (\alpha, \beta) = (3, 9)$:



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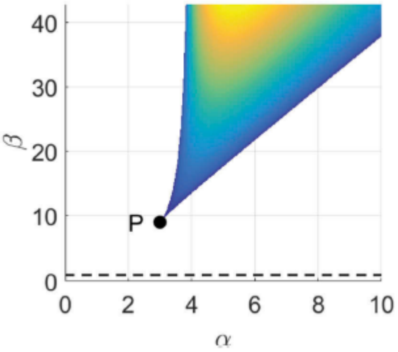

- Stable integration needs small minority, with high combined tolerance.

Neighbourhood tipping = basins of attraction

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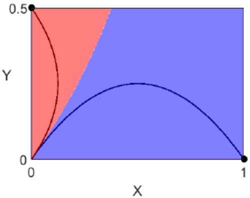

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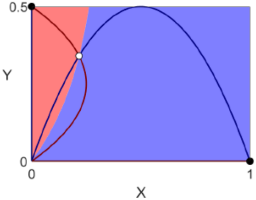
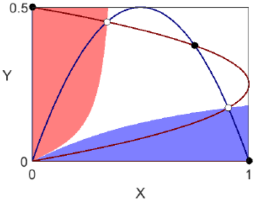


$(\alpha, \beta) = (4, 4)$



$(\alpha, \beta) = (4, 16)$

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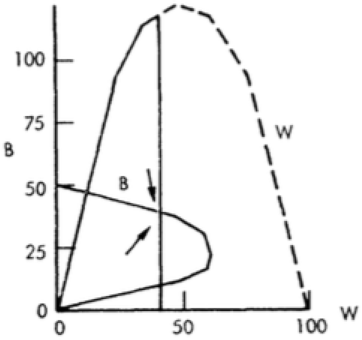
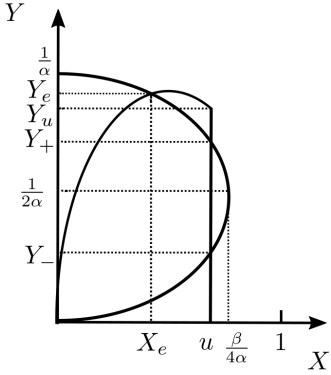
- This result formalizes and generalizes Schelling's results.
- Neighbourhood tipping is due to basins of attraction.

Limited numbers

- Schelling: "*limiting the numbers allowed to be present in the [neighbourhood] can sometimes produce [an integrated equilibrium].*" Figures are for limiting the X-population (keep out most intolerant).





Limited numbers - criteria

- Limited X-population: If we set a limit $X = u$, then we must have $u < \frac{\beta}{4\alpha}$ and we get new stable integrated equilibria for

$$\beta \in [\beta_-^u, \beta_+^u], \quad \beta_\pm^u = 2(\alpha \pm \sqrt{\alpha^2 - 2\alpha}).$$

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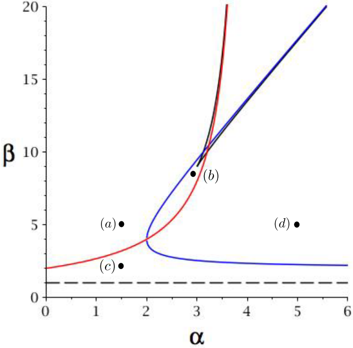
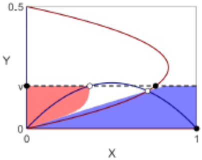
- Limited Y-population: If we set a limit $Y = v$, then we must have $v < \frac{1}{4}$ and we get new stable integrated equilibria for

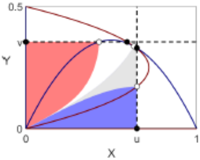
$$\beta > \beta^v, \quad \beta^v = \frac{8}{4 - \alpha}.$$

Limited numbers - basins of attraction

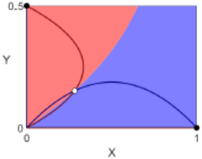




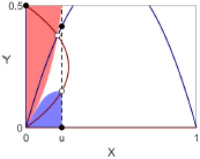
(a) $(\alpha, \beta) = (1.5, 5)$; $(a, b, k) = (\frac{3}{4}, \frac{20}{3}, 2)$.



(b) $(\alpha, \beta) = (2.9, 8.35)$; $(a, b, k) = (1.45, 5.76, 2)$.



(c) $(\alpha, \beta) = (1.5, 2)$; $(a, b, k) = (\frac{3}{4}, \frac{8}{3}, 2)$.



(d) $(\alpha, \beta) = (5, 5)$; $(a, b, k) = (\frac{5}{2}, 2, 2)$.

- Points $(a) - (d)$ have no stable integrated equilibria in the absence of population limitation.
- Limitation can not produce stable integrated population at (c).

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- For certain (α, β), can get many stable integrated equilibria.
- In other cases, limitation can not produce integration.

Two neighbourhoods ("two rooms") model

- Consider the situation in which X and Y populations are wholly contained within 2 neighbourhoods: $(X_i, Y_i), \;\; i = 1, 2$ denotes the (X, Y)-populations in neighbourhood i.

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- Assume people only care about the population mix of their own neighbourhood.

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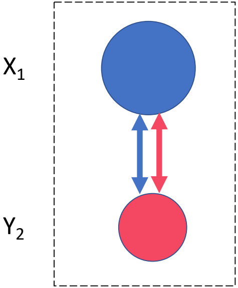
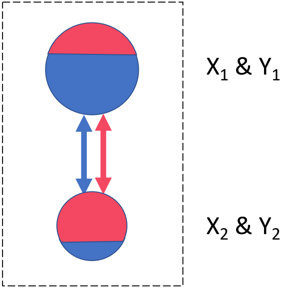


Two neighbourhoods - segregation



Two neighbourhoods - integration

Schelling dynamical system: two neighbourhoods

$$
\begin{aligned}
\frac{dX_1}{dt} &= a_1 X_1^2(1 - X_1) - X_1 Y_1 \\
&\quad - a_2 X_1(1 - X_1)^2 + (1 - X_1)(\frac{1}{k} - Y_1), \\
\frac{dY_1}{dt} &= b_1 Y_1^2(1 - kY_1) - X_1 Y_1 \\
&\quad - k b_2 Y_1(\frac{1}{k} - Y_1)^2 + (1 - X_1)(\frac{1}{k} - Y_1).
\end{aligned}
$$

Schelling dynamical system: two neighbourhoods

- Simplest case: linear tolerance schedules of X_1, X_2 and of Y_1, Y_2 identical.
- We find steady states (X_1^e, Y_1^e) by considering solutions of

$$\begin{aligned} Y_1 &= (1 - X_1)[\frac{1}{\alpha} - X_1 + 2X_1^2], \\ X_1 &= (1 - \alpha Y_1)[1 - \beta Y_1 + 2\alpha\beta Y_1^2]. \end{aligned}$$

- Since $a_1 = a_2 = a, b_1 = b_2 = b$, we have $\alpha = ka$, $\beta = ab$.

Schelling dynamical system: two neighbourhoods

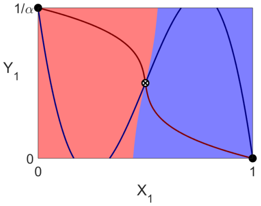
- By symmetry, $(X_1^e, Y_1^e) = (1,0), (0, \frac{1}{\alpha}),\ (\frac{1}{2}, \frac{1}{2\alpha})$ corresponding to

  i) all the X-population in neighbourhood 1 and all the Y-population in neighbourhood 2,
  ii) all the X-population in neighbourhood 2 and all the Y-population in neighbourhood 1,
  iii) both X, Y-populations *evenly* split between neighbourhoods 1 and 2.

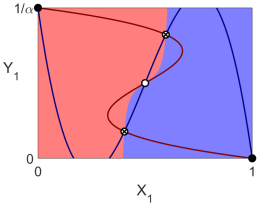
- Find stable integrated solutions when $\beta \in [\beta_-, \beta_+]$ where

$$\beta_\pm = \frac{4}{(\alpha - 8)} \left[\alpha^2 - 9\alpha \pm \sqrt{\alpha(\alpha - 6)^3} \right].$$

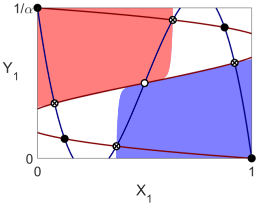
Schelling dynamical system: two neighbourhoods



$(\alpha, \beta) = (9, 16)$

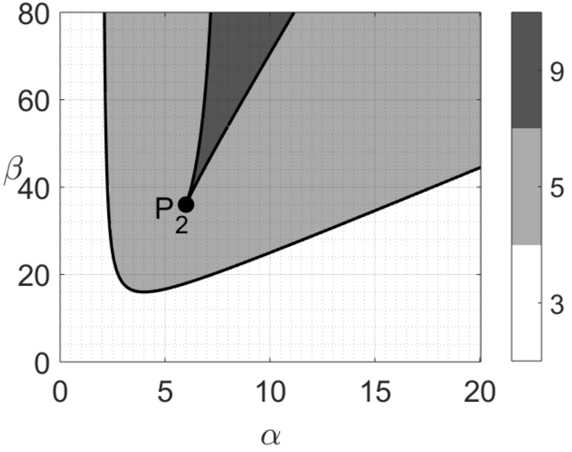


$(\alpha, \beta) = (9, 40)$

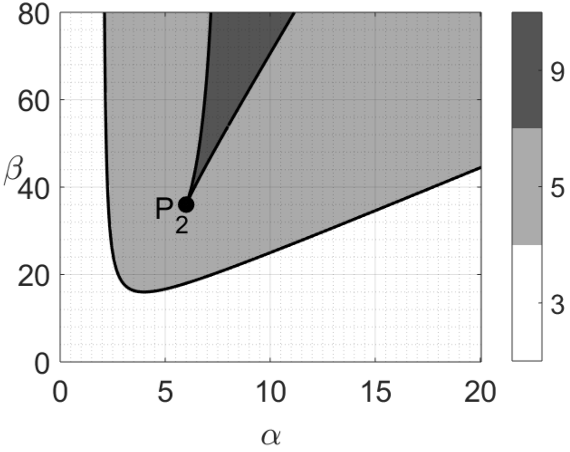


$(\alpha, \beta) = (9, 80)$

Two neighbourhoods - stable integrated equilibria

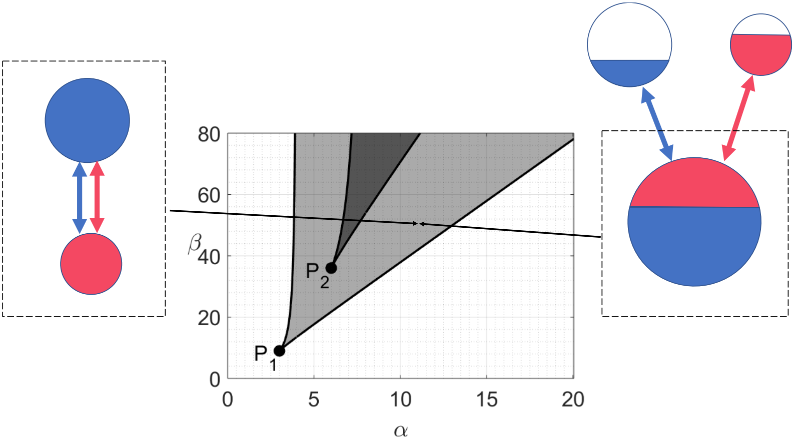


Two neighbourhoods - stable integrated equilibria



- Integration in two neighbourhoods needs tiny minority with very high combined tolerance: $P_2 = (6, 36)$.

1-to-2 neighbourhood



- Integration can be lost by changing number of neighbourhoods, despite no change in either population.

Conclusions

- Have turned Schelling's BNM into a dynamical system. Reproduced and generalised his results.
- For *unlimited* numbers in one neighbourhood, derived explicit criteria for stable integration.
- For *limited* numbers in one neighbourhood, shown exactly how to turn a segregated population into an integrated one.
- For two neighbourhoods model, derived explicit criteria for stable integration.
- Integration can be lost by changing number of neighbourhoods, despite no change in either population.