Use of the Bayesian Approximation Error Approach to Account for Model Discrepancy: The Robin Problem Revisited

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SIAM CSE19

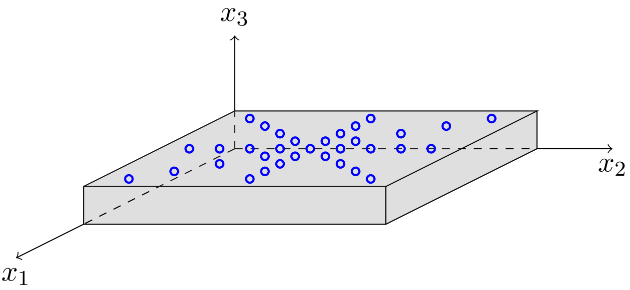
Overview

1. Motivation
2. Inverse Problems in the Bayesian Framework
3. Model discrepancy and the Bayesian approximation error (BAE) approach
4. Recovery of the Robin coefficient
5. Numerical Examples
6. Current & Future Work
7. Conclusion and Discussion

The Robin problem

Find $\beta(x)$ on inaccessible part of domain from noisy measurements of u on accessible part of domain

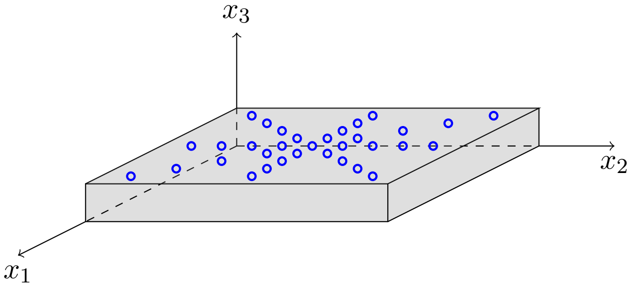
$$\begin{aligned} -\Delta u(\boldsymbol{x}) &= 0 && \text{in } \Omega, \\ \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{t}} &= g(\boldsymbol{x}) && \text{on } \Gamma_{\mathrm{t}} \\ \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{b}} + e^{\beta(\boldsymbol{x})} u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{b}} \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{s}}, \end{aligned} \tag{1}$$



The Robin problem revised

Find $\beta(x)$ on inaccessible part of domain from noisy measurements of u on accessible part of domain without knowing $a(x)$

$$
\begin{aligned}
-\nabla \cdot (e^{a(\boldsymbol{x})} \nabla u(\boldsymbol{x})) &= 0 && \text{in } \Omega, \\
e^{a(\boldsymbol{x})} \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{t}} &= g(\boldsymbol{x}) && \text{on } \Gamma_{\mathrm{t}} \\
e^{a(\boldsymbol{x})} \nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{b}} + e^{\beta(\boldsymbol{x})} u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{b}} \\
u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{s}}.
\end{aligned}
\tag{2}
$$



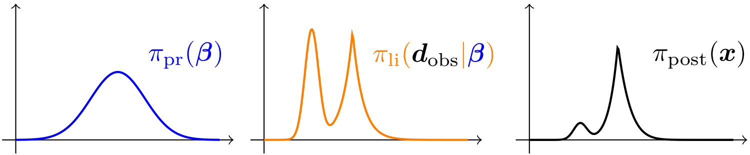
The Bayesian View of Inverse Problems

Inverse problem[1]**:** With observed data, $\boldsymbol{d}$, and parameter to observable map $\boldsymbol{f}$, find $\boldsymbol{\beta}$ given

$$\boldsymbol{d} = \boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a}) + \boldsymbol{e}$$

1. All unknowns are taken to be random variables.
2. Solution to the inverse problem is posterior probability density.

$$\pi_{\text{post}}(\boldsymbol{\beta}) = \pi(\boldsymbol{\beta}|\boldsymbol{d}) = \frac{\pi_{\text{pr}}(\boldsymbol{\beta})\pi_{\text{li}}(\boldsymbol{d}|\boldsymbol{\beta})}{\pi(\boldsymbol{d})} \propto \pi_{\text{pr}}(\boldsymbol{\beta})\pi_{\text{li}}(\boldsymbol{d}|\boldsymbol{\beta})$$



[1]Details in for example: J. Kaipio, E. Somersalo, *Statistical and Computational Inverse Problems*, Springer, 2005

The MAP Estimate

- **The maximum a posteriori estimate:** The point in parameter space that maximises the posterior probability density function
- Gaussian prior, mean $\boldsymbol{\beta}_*$ and covariance $\boldsymbol{\Gamma}_\beta = (\boldsymbol{A}^T\boldsymbol{A})^{-1}$
- Additive Gaussian noise in the measurements, $\boldsymbol{e} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Gamma}_e)$, then the posterior density is[2]

$$\pi_{\text{post}}(\boldsymbol{\beta}) \propto \exp\left\{-\frac{1}{2}\left(\|\boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a}) - \boldsymbol{d}\|^2_{\boldsymbol{\Gamma}_{\mathrm{e}}^{-1}} + \|\mathbf{A}(\boldsymbol{\beta} - \boldsymbol{\beta}_*)\|^2\right)\right\}$$

and

$$\boldsymbol{\beta}_{\text{MAP}} := \arg\min_{\boldsymbol{\beta}\in\mathbb{R}^n}\left\{\frac{1}{2}\left(\|\boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a}) - \boldsymbol{d}\|^2_{\boldsymbol{\Gamma}_{\mathrm{e}}^{-1}} + \|\mathbf{A}(\boldsymbol{\beta} - \boldsymbol{\beta}_*)\|^2\right)\right\}$$

[2] Details in for example: J. Kaipio, E. Somersalo, *Statistical and Computational Inverse Problems*, Springer, 2005

Prior Densities

We employ a *weighted* squared inverse elliptic operator as our prior covariance operator[3]:
$\boldsymbol{A} = \boldsymbol{K}\boldsymbol{G}^{-1}$ where

$$K_{ij} = \alpha \int_{\Gamma_\mathrm{b}} \left(\gamma \nabla\phi_i \cdot \nabla\phi_j + \phi_i\phi_j\right)\, d\mathbf{x} + \int_{\partial\Gamma_\mathrm{b}} \kappa\phi_i\phi_j\, d\mathbf{s},$$
$$G_{ij} = 4\pi\gamma\alpha^2\sqrt{K_{ij}^{-1}}\delta_{ij}, \quad i, j \in \{1, 2, \ldots, n\},$$

with $\alpha > 0$, $\gamma > 0$ and $\kappa \geq 0$ controlling variance and correlation structure, and δ_{ij} is the Kronecker delta.

- Other approaches include use of *Aristotelian boundary conditions*[4]

[3] Details in: Y. Daon, G. Stadler, *Mitigating the influence of the boundary on PDE-based covariance operators*, Inverse Problems and Imaging, 2017

[4] Details in: D. Calvetti, J. Kaipio, E. Someralo, *Aristotelian prior boundary conditions*, International Journal of Mathematics and Computer Science, 2006

Model Discrepancy and the BAE Approach

The Bayesian approximation error[5] (BAE) approach has been used to account for uncertainties and discrepancies in many models.

Ingredients:

1. Let $\boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a})$ an *accurate* forward problem
2. Let $\boldsymbol{g}_{a_*}(\boldsymbol{\beta})$ a *coarse/approximative* forward problem with auxiliary/nuisance parameter(s) $\boldsymbol{a}$ set to $\boldsymbol{a}_*$.

Notice,

$$
\begin{aligned}
\boldsymbol{d} &= \boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a}) + \boldsymbol{e} = \boldsymbol{g}_{a_*}(\boldsymbol{\beta}) + \boldsymbol{e} + \left(\boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{a}) - \boldsymbol{g}_{a_*}(\boldsymbol{\beta})\right) \\
&= \boldsymbol{g}_{a_*}(\boldsymbol{\beta}) + \boldsymbol{e} + \boldsymbol{\varepsilon}(\boldsymbol{\beta}) = \boldsymbol{g}_{a_*}(\boldsymbol{\beta}) + \boldsymbol{\nu}(\boldsymbol{\beta}),
\end{aligned}
$$

- $\boldsymbol{\varepsilon}(\boldsymbol{\beta})$: *Approximation errors* accounts for model discrepancies
- $\boldsymbol{\nu}(\boldsymbol{\beta})$: *Total errors* accounts for all errors

[5] Introduced in: J. Kaipio, E. Somersalo, *Statistical and Computational Inverse Problems*, Springer, 2005

Calculating the statistics of $\varepsilon(\beta)$

$\varepsilon(\beta) = f(\beta, a) - g_{a_*}(\beta)$

- In general cannot be computed analytically
- Gaussian approximation of $\pi(\varepsilon, \beta)$
- Approximate ε and β as uncorrelated $\rightarrow$ *enhanced error model*[6]
- Total error is then Gaussian with $\nu \sim \mathcal{N}(\varepsilon_*, \Gamma_e + \Gamma_\varepsilon)$

To calculate ε_* and Γ_ε (done *offline*):

- *Generate* r *samples from* $\pi(\beta, a)$
- *Compute* $\varepsilon^{(\ell)} = f(\beta^{(\ell)}, a^{(\ell)}) - g_{a_*}(\beta^{(\ell)})$, $\ell = 1, 2, \ldots, r$
- *Calculate*

$$\varepsilon_* = \frac{1}{r}\sum_{\ell=1}^{r} \varepsilon^{(\ell)} \quad \textit{and} \quad \Gamma_\varepsilon = \frac{1}{r-1}\sum_{\ell=1}^{r}(\varepsilon^{(\ell)} - \varepsilon_*)(\varepsilon^{(\ell)} - \varepsilon_*)^T$$

[6] Details in for example: J. Kaipio, V. Kolehmainen, *Approximate marginalization over modelling errors and uncertainties in inverse problems*, Bayesian Theory and Applications, 2013

What Have we Accomplished?

An updated likelihood
⇒An updated MAP estimate:

$$\boldsymbol{\beta}_{\text{MAP}}^{\text{BAE}} := \arg\min_{\boldsymbol{\beta} \in \mathbb{R}^n} \{\mathcal{J}(\boldsymbol{\beta})\}$$

with

$$\begin{aligned}\mathcal{J}(\boldsymbol{\beta}) &= \frac{1}{2}\left(\|\boldsymbol{g}_{a_*}(\boldsymbol{\beta}) - \boldsymbol{d} + \boldsymbol{\nu}_*\|^2_{\boldsymbol{\Gamma}_\nu^{-1}} + \|\mathbf{A}\left(\boldsymbol{\beta} - \boldsymbol{\beta}_*\right)\|^2\right) \\ &= \frac{1}{2}\left(\|\mathcal{B}\boldsymbol{u} - \boldsymbol{d} + \boldsymbol{\nu}_*\|^2_{\boldsymbol{\Gamma}_\nu^{-1}} + \|\mathbf{A}\left(\boldsymbol{\beta} - \boldsymbol{\beta}_*\right)\|^2\right)\end{aligned}$$

$\mathcal{B}$ is the observation operator, and $\boldsymbol{u}$ is the FEM solution to the forward problem

$$\begin{aligned} -\nabla \cdot (e^{a_*}\nabla u(\boldsymbol{x})) &= 0 && \text{in } \Omega, \\ e^{a_*}\nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_\text{t} &= g(\boldsymbol{x}) && \text{on } \Gamma_\text{t} \\ e^{a_*}\nabla u(\boldsymbol{x}) \cdot \boldsymbol{n}_\text{b} + e^{\beta(\boldsymbol{x})} u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_\text{b} \\ u(\boldsymbol{x}) &= 0 && \text{on } \Gamma_\text{s}. \end{aligned}$$

Methods for Inversion

Use an inexact CG adjoint-based Gauss-Newton method

- Set up Lagrangian functional $\mathcal{L} : \mathcal{V} \times \mathcal{V} \times \mathcal{E} \to \mathbb{R}$ is

$$\mathcal{L}(u, p, \beta) := \mathcal{J} + \int_{\Omega} e^{a_*} \nabla u \cdot \nabla p \, d\boldsymbol{x} - \int_{\Gamma_{\mathrm{t}}} g p \, d\boldsymbol{s}_{\mathrm{t}} + \int_{\Gamma_{\mathrm{b}}} e^{\beta} u p \, d\boldsymbol{s}_{\mathrm{b}},$$

- Gradient of $\mathcal{J}$ found by requiring variations of $\mathcal{L}$ with respect to the forward potential u and the *adjoint potential* p vanish
- Results in following strong form of gradient $\mathcal{G}$,

$$\mathcal{G}(\beta) := \mathcal{A}^2 (\beta - \beta_*) + e^{\beta} u p$$

where u satisfies the forward problem, and p satisfies the adjoint Poisson problem for given u and β:

Methods for Inversion

The adjoint Poisson problem:

$$\begin{aligned} -\nabla \cdot (e^{a_*} \nabla p(\boldsymbol{x})) &= -\mathcal{B}^* \boldsymbol{\Gamma}_{\boldsymbol{\nu}}^{-1} (\mathcal{B} \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{d} + \boldsymbol{\nu}_*) && \text{in } \Omega, \\ e^{a_*} \nabla p(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{t}} &= 0 && \text{on } \Gamma_{\mathrm{t}}, \\ e^{a_*} \nabla p(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{b}} + e^{\beta(\boldsymbol{x})} p(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{b}}, \\ p(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{s}}, \end{aligned}$$

- Action of the Gauss-Newton approximation of the Hessian operator evaluated at β in the direction $\hat{\beta}$ is given by

$$\mathcal{H}(\beta)(\hat{\beta}) := \mathcal{A}^2 \hat{\beta} + e^{\beta} \hat{\beta} u \hat{p}$$

where $\hat{p}$ satisfies the *incremental adjoint Poisson problem* and $\hat{u}$ satisfies the *incremental forward Poisson problem*

The incremental adjoint Poisson problem

$$\begin{aligned}
-\nabla \cdot (e^{a_*} \nabla \hat{p}(\boldsymbol{x})) &= -\mathcal{B}^* \boldsymbol{\Gamma}_{\boldsymbol{\nu}}^{-1} \mathcal{B} \hat{u}(\boldsymbol{x}) && \text{in } \Omega \\
e^{a_*} \nabla \hat{p}(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{t}} &= 0 && \text{on } \Gamma_{\mathrm{t}} \\
e^{a_*} \nabla \hat{p}(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{b}} + e^{\beta(\boldsymbol{x})} \hat{p}(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{b}}, \\
\hat{p}(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{s}},
\end{aligned}$$

The incremental forward Poisson problem

$$\begin{aligned}
-\nabla \cdot (e^{a_*} \nabla \hat{u}(\boldsymbol{x})) &= 0 && \text{in } \Omega \\
e^{a_*} \nabla \hat{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{t}} &= 0 && \text{on } \Gamma_{\mathrm{t}} \\
e^{a_*} \nabla \hat{u}(\boldsymbol{x}) \cdot \boldsymbol{n}_{\mathrm{b}} + e^{\beta(\boldsymbol{x})} \hat{u}(\boldsymbol{x}) &= -\hat{\beta} e^{\beta(\boldsymbol{x})} u(\boldsymbol{x}) && \text{on } \Gamma_{\mathrm{b}}, \\
\hat{u}(\boldsymbol{x}) &= 0 && \text{on } \Gamma_{\mathrm{s}}.
\end{aligned}$$

Methods for Inversion

The resulting system to be solved (inexactly using CG) for the Gauss-Newton search direction, $\hat{\beta}$, is

$$\mathcal{H}(\beta)(\hat{\beta}) = -\mathcal{G}(\beta).$$

For joint inversion[7] we would also need to solve

$$\mathcal{H}_a(a)(\hat{a}) = -\mathcal{G}_a(a)$$

for $\hat{a}$, with

$$\begin{aligned} \mathcal{G}_a(a) &:= \mathcal{A}_a^2\,(a - a_*) + e^a \nabla u \cdot \nabla p \\ \mathcal{H}_a(a)(\hat{a}) &:= \mathcal{A}_a^2 \hat{a} + e^a \hat{a} \nabla u \cdot \nabla \hat{p} \end{aligned}$$

[7] Such an approach was used in an ice sheet problem, details in N. Petra et al., *An inexact Gauss-Newton method for inversion of basal sliding and rheology parameters in a nonlinear Stokes ice sheet model*, Journal of Glaciology, 2012

Computational Examples Set Up

- Domain $\Omega = [0, 1] \times [0, 1] \times [0, 0.01]$
- 33 point measurements on the top of the domain
- Avoid *inverse crimes* by using finer FEM discretisation to generate data than for inversions

	Mesh use	#Nodes	#Els	#Param
Example 1				
	Data synthesis	28,611	150,000	2,601
	Inversion	6,727	32,400	961
Example 2				
	Data synthesis	132,651	750,000	2,601
	Inversion	29,791	162,000	961

- 1% noise added to measurements: $\boldsymbol{\Gamma}_e = \delta_e^2 \boldsymbol{I}$.

Prior for β

- Covariance set by using $\alpha = 7$, $\gamma = 0.01$ and $\kappa = 0$
- Recall, $\boldsymbol{\Gamma}_\beta = (\boldsymbol{A}^T\boldsymbol{A})^{-1}$, with $\boldsymbol{A} = \boldsymbol{K}\boldsymbol{G}^{-1}$ where

$$K_{ij} = \alpha \int_{\Gamma_\mathrm{b}} \left(\gamma \nabla\phi_i \cdot \nabla\phi_j + \phi_i\phi_j\right) \, d\mathbf{x} + \int_{\partial\Gamma_\mathrm{b}} \kappa\phi_i\phi_j \, d\mathbf{s},$$

and $\boldsymbol{G}^{-1}$ effectively homogenises the variance of the prior.

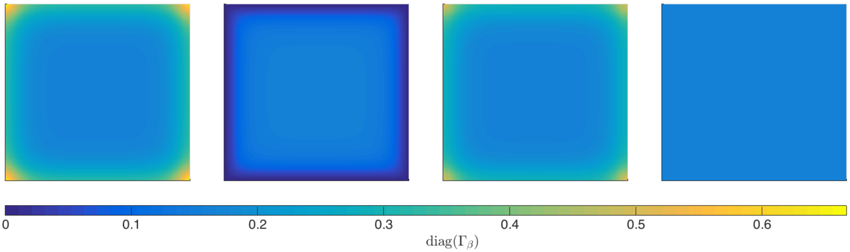


Figure: spatial variance of β for different boundary conditions

Prior for β

- Prior mean set as $\beta_* = 1$
- Same prior for $\boldsymbol{\beta}$ and true value, $\boldsymbol{\beta}_{\text{true}}$ used for both numerical examples

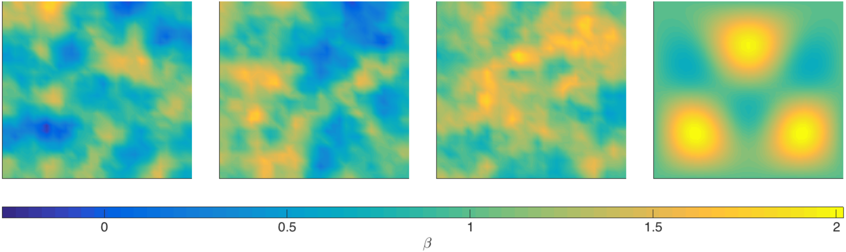


Figure: Three draws from $\pi_{\text{pr}}(\boldsymbol{\beta})$ and $\boldsymbol{\beta}_{\text{true}}$

- Mean and Covariance set using $a_* = 0$, $\alpha_a = 100$, $\gamma_a = 0.001$
- With $\boldsymbol{\Gamma}_a = (\boldsymbol{A}_a^T \boldsymbol{A}_a)^{-1}$, with $\boldsymbol{A}_a = \boldsymbol{K}$ where

$$K_{ij} = \alpha_a \int_\Omega \left(\gamma_a \nabla\phi_i \cdot \nabla\phi_j + \phi_i\phi_j\right) \, d\mathbf{x}$$

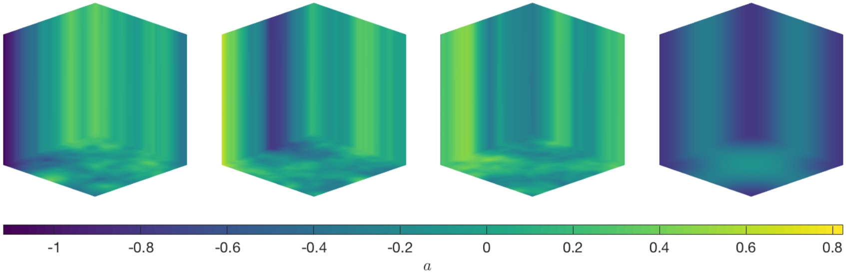


Figure: Three draws from $\pi_{\text{pr}}(a)$ and a_{true}

- $r = 1000$ samples drawn to calculate $\boldsymbol{\varepsilon}_*$ and $\mathbf{\Gamma}_\varepsilon$.
- Some components of $\mathbf{\Gamma}_\varepsilon$ are $\approx 100\times$ larger than those in $\mathbf{\Gamma}_e$
- Perhaps more important: Structured noise

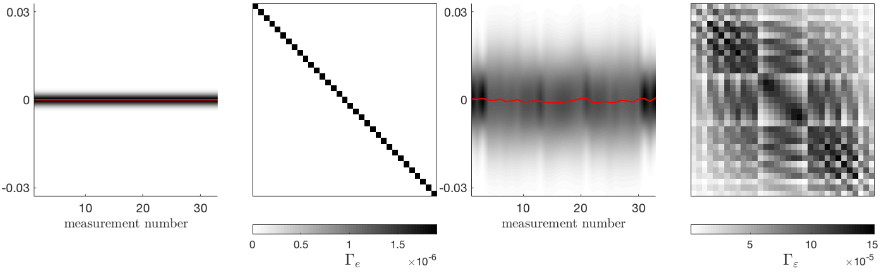


Figure: Statistics of the measurement errors and the approximation errors

Results for Example One

Results are compared for for the following

- **Reference case:** use the correct value of a in the model
- **Conventional error case:** neglect approximation errors
- **BAE case:** take into account approximation errors

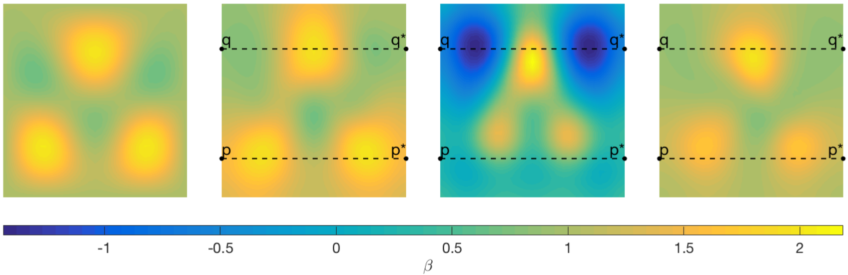


Figure: β_{true}, reference MAP, conventional error MAP, BAE MAP

Results for Example One

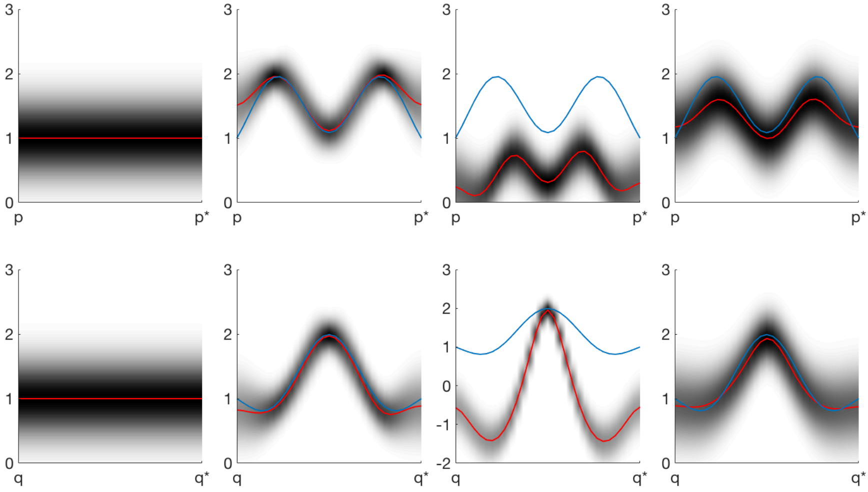


Figure: Prior and posterior variance: reference, conventional error, BAE

Example Two: The Anisotropic Case

- Mean and Covariance set using $a_* = 0$, $\alpha_a = 100$, $\boldsymbol{\gamma}_a = \text{diag}(10^{-2}, 10^{-2}, 10^{-8})$
- With $\boldsymbol{\Gamma}_a = (\boldsymbol{A}_a^T \boldsymbol{A}_a)^{-1}$, with $\boldsymbol{A}_a = \boldsymbol{K}$ where

$$K_{ij} = \alpha_a \int_\Omega (\boldsymbol{\gamma}_a \nabla \phi_i \cdot \nabla \phi_j + \phi_i \phi_j) \; d\mathbf{x}$$

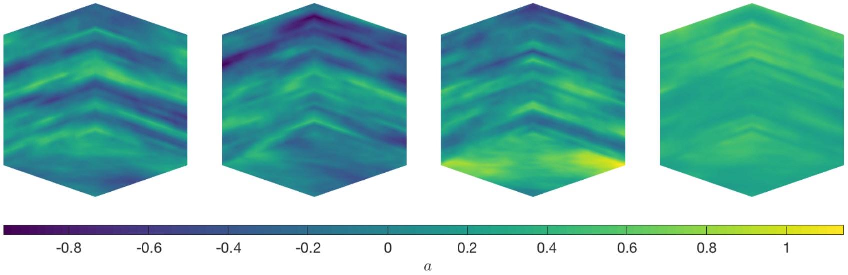
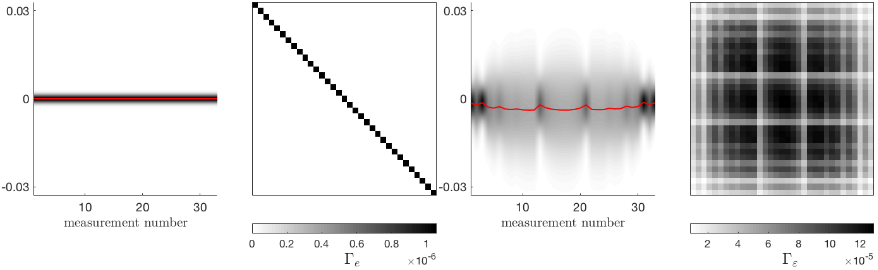


Figure: Three draws from $\pi_{\text{pr}}(a)$ and a_{true}

Approximation Errors for Example Two

- Again, $r = 1000$ samples drawn to calculate $\boldsymbol{\varepsilon}_*$ and $\boldsymbol{\Gamma}_\varepsilon$.
- Some components of $\boldsymbol{\Gamma}_\varepsilon$ are $> 100\times$ larger than those in $\boldsymbol{\Gamma}_e$
- We have: Structured noise



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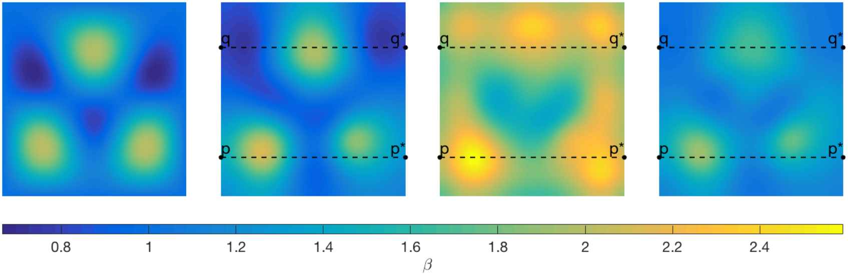


Figure: β_{true}, reference MAP, conventional error MAP, BAE MAP

Results for Example Two

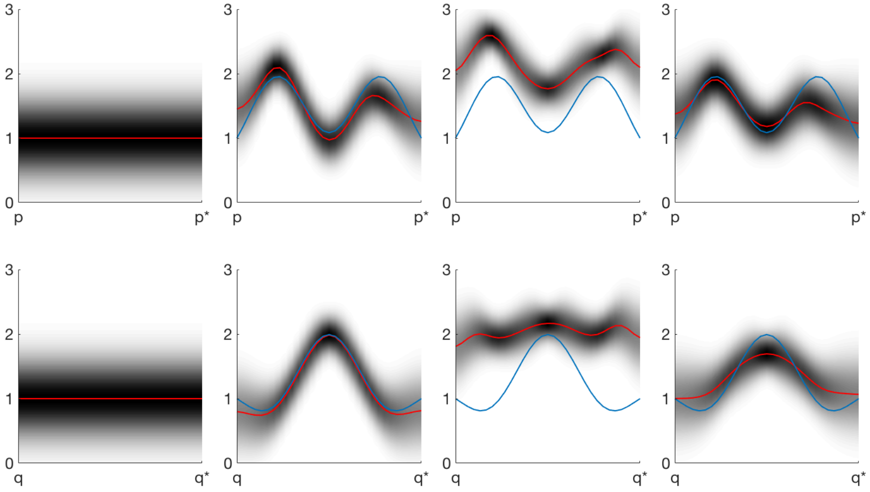


Figure: Prior and posterior variance: reference, conventional error, BAE

Computational Costs

- Gauss-Newton terminated when norm of gradient decreased by factor of 10^7
- CG iterations are terminated using Eisenstat-Walker condition

	MAP	#GN	#CG	avg.CG	#back	#Poisson
Example 1						
	REF	8	117	15	0	250
	CEM	11	101	10	4	228
	BAE	5	57	12	0	124
Example 2						
	REF	6	54	9	0	120
	CEM	8	95	12	0	206
	BAE	5	97	20	0	204

Current & Future Work: The Ice Sheet Problem

Consider the incompressible Stokes equations[8]

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}_{\boldsymbol{u}} &= \rho \boldsymbol{g} \qquad \boldsymbol{x} \in \Omega, \\ \nabla \cdot \boldsymbol{u} &= 0 \qquad \boldsymbol{x} \in \Omega, \end{aligned} \tag{3}$$

- Basal sliding coefficient β
- Glen's flow-law exponent parameter $\boldsymbol{n}$: Determines (non-)linearity

$$\boldsymbol{\sigma}_{\boldsymbol{u}} = -p\mathbf{I} + 2\eta(\boldsymbol{u}, \boldsymbol{n})\dot{\boldsymbol{\varepsilon}} \qquad \text{Cauchy stress tensor}$$

$$\eta(\boldsymbol{u}, \boldsymbol{n}) = \frac{1}{2} A^{-\frac{1}{\boldsymbol{n}}} \left(\dot{\boldsymbol{\varepsilon}}_{\parallel}(\boldsymbol{u}) + \epsilon \right)^{\frac{1-\boldsymbol{n}}{2\boldsymbol{n}}} \qquad \text{Effective viscosity}$$

$$\dot{\boldsymbol{\varepsilon}}(u) = \frac{1}{2} \left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T \right) \qquad \text{Strain rate tensor}$$

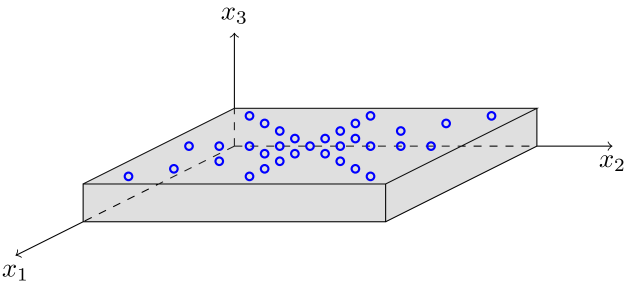
$$\dot{\boldsymbol{\varepsilon}}_{\parallel}(\boldsymbol{u}) = \frac{1}{2} \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}) : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}) \qquad \text{Second invariant}$$

[8] See for example N. Petra et al., *An inexact Gauss-Newton method for inversion of basal sliding and rheology parameters in a nonlinear Stokes ice sheet model*, Journal of Glaciology, 2012

The Ice Sheet Problem

$$
\begin{aligned}
\boldsymbol{u}|_{\Gamma_l} = \boldsymbol{u}|_{\Gamma_r} \quad \text{and} \quad \boldsymbol{\sigma_u n}|_{\Gamma_l} &= \boldsymbol{\sigma_u n}|_{\Gamma_r} && \boldsymbol{x} \in \Gamma_p \\
\boldsymbol{\sigma_u n} &= \boldsymbol{0} && \boldsymbol{x} \in \Gamma_t \\
\boldsymbol{u} \cdot \boldsymbol{n} &= 0 && \boldsymbol{x} \in \Gamma_b \\
\boldsymbol{T\sigma_u n} + e^{\beta}\boldsymbol{Tu} &= \boldsymbol{0} && \boldsymbol{x} \in \Gamma_b,
\end{aligned} \tag{4}
$$

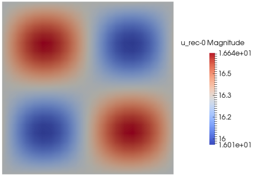
$$
\int_\Omega 2\eta(\boldsymbol{u}, \boldsymbol{n})\dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}) : \dot{\boldsymbol{\varepsilon}}(\boldsymbol{v}) - p\nabla \cdot \boldsymbol{v} - q\nabla \cdot \boldsymbol{u}\; d\boldsymbol{x} + \int_{\Gamma_b} e^{\beta}\boldsymbol{Tu} \cdot \boldsymbol{Tv}\; ds = \int_\Omega \rho \boldsymbol{g} \cdot \boldsymbol{v}\; d\boldsymbol{x}
$$



The Ice Sheet Problem

- Replace nonlinear Stokes model with linear counterpart [9]

$$\boldsymbol{f}(\boldsymbol{\beta}, \boldsymbol{n}) = \boldsymbol{g}_{\boldsymbol{n}_*}(\boldsymbol{\beta}) + \boldsymbol{\varepsilon}(\boldsymbol{\beta}), \quad \boldsymbol{n}_* = 1$$



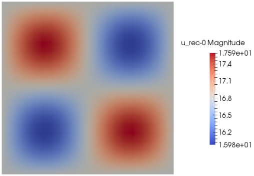


Figure: Nonlinear (left) and Linear (right) velocities for same β_{true},

[9] See R. Nicholson, O. Babaniyi, N. Petra, *Incorporating model discrepancy stemming from uncertain rheology in an inverse ice sheet flow problem*, in preperation

Summary

Conclusions:

- The Bayesian approximation error (BAE) approach allows use of simpler forward model while *premarginalising* over auxilliary parameters
- Consequential model discrepancy dealt with systematically.
- BAE approach results in an updated likelihood which fits in naturally to the existing framework.
- Neglecting model discrepancy can lead to infeasible results.
- Robin problem details in R. Nicholson, N. Petra, J. Kaipio, *Estimation of the Robin coefficient field in a Poisson problem with uncertain conductivity field*, Inverse Problems 34 (11) 2018
- Ice sheet problem details in R. Nicholson, O. Babaniyi, N. Petra, *Incorporating model discrepancy stemming from uncertain rheology in an inverse ice sheet flow problem*, in preperation
- Thank you