Determining the Critical Spectrum Using Lin's Method

MS167: Recent Advances in the Stability of Travelling Waves

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Set-up

Goal: employ Lin's method to determine spectral (and non-linear) stability of traveling waves in singularly perturbed reaction-diffusion systems.

Outline:

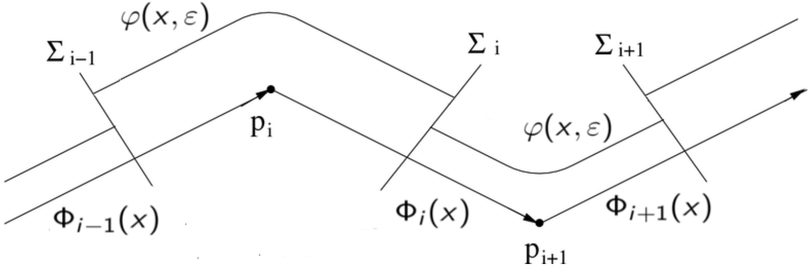
- Discussion of Lin's method
- Application to travelling waves in **regularly** perturbed RD-systems
  - Existence
  - Stability
- Application to travelling waves in **singularly** perturbed RD-systems
  - Existence
  - Stability
    - (Oscillatory) pulses in FHN equations
    - Periodic pulse solutions in slowly nonlinear RD-systems

Lin's method

Classical setting [Lin '90]

- **ODE** $\partial_x \varphi = f(\varphi, \varepsilon)$, ε small parameter;
- **Heteroclinics** $\Phi_i(x)$ connecting fixed points p_i and p_{i+1} for $\varepsilon = 0$;
- **Codim-1 planes** Σ_i through $\Phi_i(0)$ orthogonal to $\dot{\Phi}_i(0)$.

$\Rightarrow \exists$ **piecewise continuous solution** $\varphi(x, \varepsilon)$ close to $\Phi_* := \bigcup_i \Phi_i[\mathbb{R}]$:



Lyapunov-Schmidt reduction: $\varphi(x, \varepsilon)$ is smooth $\Leftrightarrow$ jumps $J_i(\varepsilon)$ vanish.

Lin's method

Technical core:

- Rewrite as inhomogeneous problem:

$$\begin{aligned}\partial_x\varphi &= f(\varphi,\varepsilon)\\ &= Df(\Phi_i(x),0)\varphi + f(\varphi,\varepsilon) - Df(\Phi_i(x),0)\varphi \\ &= A_i(x)\varphi + \underbrace{g_i(x,\varepsilon)}_{\text{'small'}};\end{aligned} \qquad (*)$$

- Establish **exponential dichotomies** for $\partial_x\varphi = A_i(x)\varphi$ on $\mathbb{R}_\pm$ (fixed points p_i hyperbolic with consistent splitting!);

- Yields **variation of constant formulas** on $\mathbb{R}_\pm$:
  - Solution $\varphi_i(x,\varepsilon)$ to $(*)$ on $[-a_i, a_i]$ with **boundary values**

    $$P_{i,\pm}[\varphi_i(\pm a_i,\varepsilon)] = \ldots;$$

  - **Explicit** expression for jump $J_i(\varepsilon) := \varphi_i(0^+,\varepsilon) - \varphi_i(0^-,\varepsilon)$.

- Solution $\varphi(x,\varepsilon)$: **concatenate** φ_i's.

Lin's method

Extensions beyond ODEs to:

- Parabolic PDEs *[Sandstede,...]*
- Elliptic PDEs *[Peterhof, Sandstede, Scheel,...]*
- (Spatially) discrete systems/LDE *[Knobloch, Mallet-Paret, Georgi,...]*
- Mixed type FDE *[Harterich, Hupkes, Mallet-Paret, Verduyn-Lunel,...]*
- ...

Application to travelling waves

Travelling-wave solution $w_{\mathrm{tw}}(x,t) = W_{\mathrm{tw}}(x-ct)$ to **regularly perturbed** RD-system

$$w_t = \mathcal{D}w_{xx} + N(w,\varepsilon), \qquad w(x,t) \in \mathbb{R}^n, \quad \varepsilon \text{ small parameter.}$$

Co-moving frame $\zeta = x - ct$: $W_{\mathrm{tw}}(\zeta)$ solves

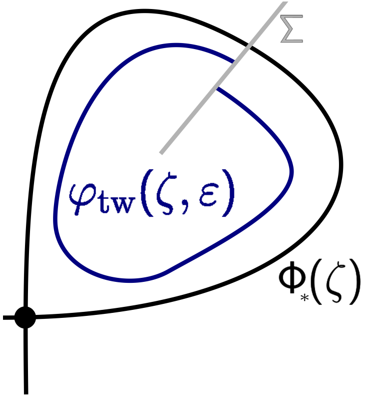
$$0 = \mathcal{D}w_{\zeta\zeta} + cw_\zeta + N(w,\varepsilon) \quad \Leftrightarrow \quad \varphi_\zeta = f(\varphi,\varepsilon). \qquad (*)$$

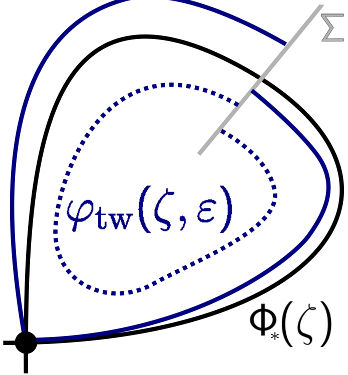
Assume $(*)$ at $\varepsilon = 0$ admits **(chain of) homo-/heteroclinics** $\Phi_*(\zeta)$.

Constructions using Lin's method [Sandstede et al.]

- Periodic wave trains close to homoclinic pulse solution Φ_*;
- N-pulses bifurcating from primary pulse Φ_*;
- N-fronts close to heteroclinic loop Φ_*;
- etc.

Application to travelling waves





Periodic wave train $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$ accompanying homoclinic pulse Φ_*.

Travelling 2-pulse $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$ bifurcating from primary pulse Φ_*.

Application to travelling waves - Existence

Lin's method

Yields **piecewise continuous solution** $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$ for $|\varepsilon| \ll 1$ to

$$\begin{aligned}\varphi_\zeta &= f(\varphi,\varepsilon)\\ &= Df(\Phi_i(\zeta),0)\varphi + f(\varphi,\varepsilon) - Df(\Phi_i(\zeta),0)\varphi\\ &= A(\zeta)\varphi + g(\zeta,\varepsilon),\end{aligned}$$

Lyapunov-Schmidt: $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$ smooth $\Leftrightarrow$ jumps $J_i(\varepsilon)$ vanish.

Idea: apply Lin's method also to **eigenvalue problem** about $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$!

Application to travelling waves - Stability

Eigenvalue problem about $\Phi_i(\zeta)$

$$\psi_\zeta = \underbrace{(Df(\Phi_i(\zeta),0) + B(\lambda_*))}_{A(\zeta,\lambda_*)}\psi, \qquad B(\lambda) := \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Lin's method

Yields **piecewise continuous eigenfunction** $\psi(\zeta,\varepsilon,\lambda)$, $|\varepsilon|, |\lambda - \lambda_*| \ll 1$ to **eigenvalue problem** about $\varphi_{\mathrm{tw}}(\zeta,\varepsilon)$:

$$\begin{aligned} \psi_\zeta &= (Df(\varphi_{\mathrm{tw}}(\zeta,\varepsilon),\varepsilon) + B(\lambda))\,\psi \\ &= A(\zeta,\lambda_*)\psi + [Df(\varphi_{\mathrm{tw}}(\zeta,\varepsilon),\varepsilon) - Df(\Phi_i(\zeta),0) + B(\lambda-\lambda_*)]\,\psi \\ &= A(\zeta,\lambda_*)\psi + \underbrace{g(\zeta,\varepsilon,\lambda-\lambda_*)}_{\text{'small'}}, \end{aligned}$$

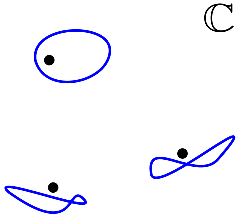
Lyapunov-Schmidt: $\lambda \in \sigma(\mathcal{L}_{\varphi_{\mathrm{tw}}(\cdot,\varepsilon)}) \Leftrightarrow \psi(\zeta,\varepsilon,\lambda)$ smooth

$\Leftrightarrow$ jumps $J_i(\varepsilon,\lambda-\lambda_*)$ vanish.

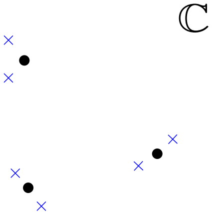
Application to travelling waves - Stability

Implicit function theorem:

- $\lambda_* \in \sigma(\mathcal{L}_{\Phi_i}) \implies \exists\ \lambda(\varepsilon) \in \sigma(\mathcal{L}_{\varphi_{\rm tw}(\cdot,\varepsilon)})$ with $\lim\limits_{\varepsilon\to 0} \lambda(\varepsilon) = \lambda_*$.
- λ_* away from $\bigcup_i \sigma(\mathcal{L}_{\Phi_i}) \implies \lambda_* \notin \sigma(\mathcal{L}_{\varphi_{\rm tw}(\cdot,\varepsilon)})$.



Spectrum of **periodic wave train** $\varphi_{\rm tw}(\zeta,\varepsilon)$ and limiting homoclinic pulse Φ_*.



Spectrum of **travelling 2-pulse** $\varphi_{\rm tw}(\zeta,\varepsilon)$ and limiting primary pulse Φ_*.

Leading order of $\lambda(\varepsilon)$ important if $\lambda_* \in i\mathbb{R}$ (translational invariance!).

Singularly perturbed problems

What happens if ε induces a **scale separation**?

I.e. what if we replace

$$w_t = \mathcal{D} w_{xx} + N(w, \varepsilon), \qquad \text{by} \qquad \begin{cases} u_t &= \mathcal{D}_1 u_{xx} + N_1(u, v, \varepsilon) \\ v_t &= \varepsilon^2 \mathcal{D}_2 v_{xx} + N_2(u, v, \varepsilon) \end{cases},$$

'regularly perturbed' by 'singularly perturbed'

with $0 < \varepsilon \ll 1$?

Is Lin's method still applicable (in stability analysis)?

Singularly perturbed problems - Existence

$$\begin{cases} u_t &= \mathcal{D}_1 u_{xx} + N_1(u,v,\varepsilon) \\ v_t &= \varepsilon^2 \mathcal{D}_2 v_{xx} + N_2(u,v,\varepsilon) \end{cases}, \qquad u(x,t) \in \mathbb{R}^m, v(x,t) \in \mathbb{R}^n.$$

Travelling waves in PDE $\Leftrightarrow$ solutions to **slow-fast system:**

$$\omega_\xi = \varepsilon f(\omega,\chi,\varepsilon),$$
$$\chi_\xi = g(\omega,\chi,\varepsilon),$$

Co-moving frame: $\xi = \varepsilon^{-1}x + ct$.

Singularly perturbed problems - Existence

$$\begin{cases} u_t &= \mathcal{D}_1 u_{xx} + N_1(u,v,\varepsilon) \\ v_t &= \varepsilon^2 \mathcal{D}_2 v_{xx} + N_2(u,v,\varepsilon) \end{cases}, \qquad u(x,t) \in \mathbb{R}^m, v(x,t) \in \mathbb{R}^n.$$

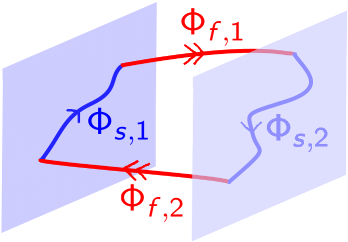
Geometric singular perturbation theory [Fenichel '79]

- Fast and slow reduced systems

$$\omega_\xi = 0, \qquad \chi_\xi = g(\omega,\chi,0). \qquad\qquad \omega_{\hat\xi} = f(\omega,\chi,0), \qquad 0 = g(\omega,\chi,0).$$

- Construct **singular orbit** $\Phi_* = \Phi_f \cup \Phi_s$ (is no solution to PDE for $\varepsilon = 0$!)

- Establish **actual solution** $\varphi_{\mathrm{tw}}(\xi,\varepsilon)$ in ε-vicinity.

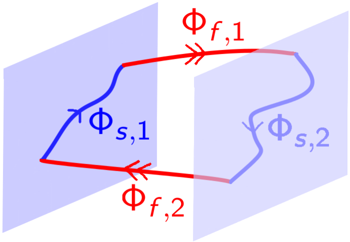


Singularly perturbed problems - Stability

Eigenvalue problems about $\Phi_* = \bigcup_i [\Phi_{f,i} \cup \Phi_{s,i}]$

$$\psi_\xi = A_f(\Phi_{f,i}(\xi), \lambda)\psi, \quad (\mathrm{F_i}), \qquad \psi_\xi = A_s(\Phi_{s,i}(\varepsilon\xi), \lambda)\psi, \quad (\mathrm{S_i}).$$

- Homoclinic/heteroclinic connection $\Phi_{f,i}$ $\Rightarrow$ **Exponential trichotomies** for $(\mathrm{F_i})$ on $\mathbb{R}_\pm$.
- Normal hyperbolicity with consistent splitting along **slowly varying** $\Phi_{s,i}$ $\Rightarrow$ **Exponential trichotomies** for $(\mathrm{S_i})$ on some interval.



Fixed points replaced by slow transitions!

Lin's method

Yields **piecewise continuous eigenfunction** $\psi(\xi, \varepsilon, \lambda)$ to

$$\psi_\xi = A(\varphi_{\mathrm{tw}}(\xi, \varepsilon), \varepsilon, \lambda)\psi.$$

Singularly perturbed problems - Stability

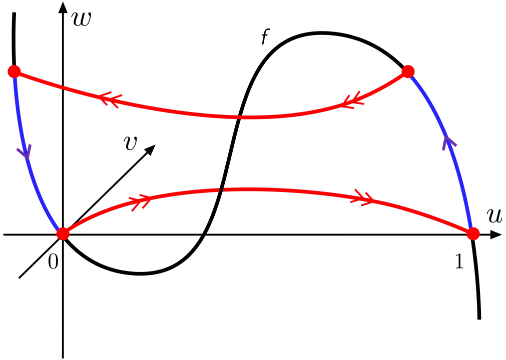
Two applications:

1. (Oscillatory) travelling pulses in FHN equations
2. Periodic pulse solutions in slowly nonlinear RD-systems

Singularly perturbed problems - Stability

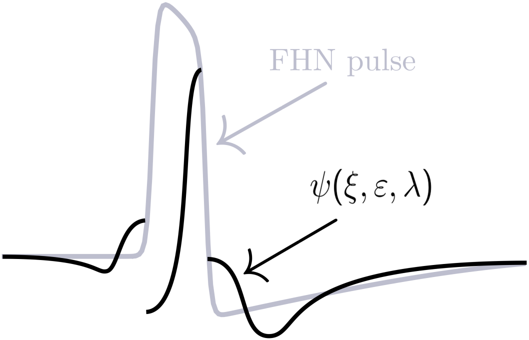
Application 1: (Oscillatory) traveling pulses in **FitzHugh-Nagumo equation** (nerve propagation) [Carter,Sandstede,BdR]

$$\begin{aligned} u_t &= u_{xx} + f(u) - w, \\ w_t &= \varepsilon(u - \gamma w). \end{aligned} \tag{FHN}$$



Singularly perturbed problems - Stability

Lin's method: exponentially localized, piecewise continuous eigenfunction $\psi(\xi, \varepsilon, \lambda)$ (in exponentially weighted space!)

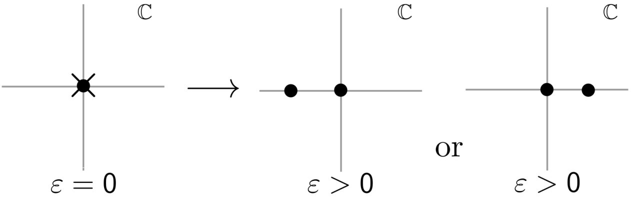


Lyapunov-Schmidt: λ is an eigenvalue $\Leftrightarrow$ jumps $J(\varepsilon, \lambda)$ vanish.

Singularly perturbed problems - Stability

Outcome Lin's method

- **Critical spectrum:** two eigenvalues converging to 0 as $\varepsilon \to 0$;
- Translational invariance $\Rightarrow$ one eigenvalue must be 0;
- **Leading order expression** critical eigenvalue $\lambda_c(\varepsilon) = \varepsilon\mathcal{M}_1 + \text{h.o.t.};$
- In **oscillatory regime** $\lambda_c(\varepsilon) = \varepsilon^{2/3}\mathcal{M}_2 + \text{h.o.t.}.$



In Evans-function analysis [Jones '84, Yanagida '85]

Sign critical eigenvalue via **parity argument**.

Singularly perturbed problems - Stability

Application 2: Periodic pulse solutions to general class of RD-systems [Doelman,Rademacher,BdR]

Model

$$\begin{cases} u_t &= \mathcal{D}_1 u_{yy} - H(u,v,\varepsilon) \\ v_t &= \varepsilon^2 \mathcal{D}_2 v_{yy} - G(u,v,\varepsilon)v \end{cases}, \quad u(y,t) \in \mathbb{R}^m, v(y,t) \in \mathbb{R}^n.$$

Allow for semi-strong interaction:

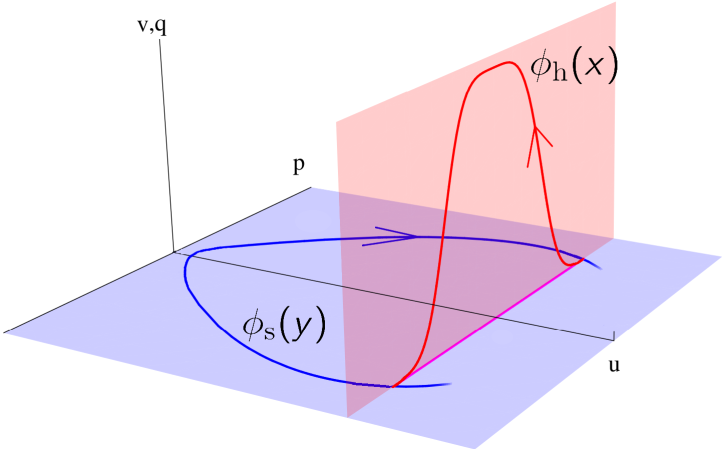
$$H(u,v,\varepsilon) = H_1(u,v,\varepsilon) + \varepsilon^{-1} H_2(u,v)v.$$

Motivation

- **Multi-dimensional, slowly-nonlinear** class;
- Includes **Gierer-Meinhardt** system (morphogenesis);
- If $n = 1$ having $H_2 \not\equiv 0$ can prevent solutions from being **unstable**.

Singularly perturbed problems - Stability

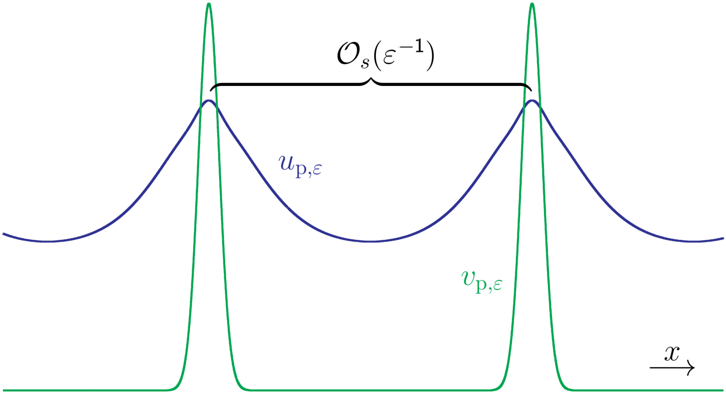
Singular periodic orbit:



Actual solution lies in ε-vicinity.

Singularly perturbed problems - Stability

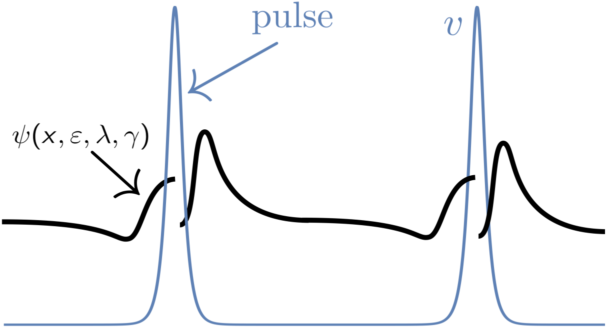
Far-from-equilibrium periodic with **localized** v-components and **non-localized** u-components.



Period is $2L_\varepsilon = \mathcal{O}_s(\varepsilon^{-1})$.

Singularly perturbed problems - Stability

Lin's method: γ-twisted, piecewise continuous eigenfunction $\psi(x, \varepsilon, \lambda, \gamma)$

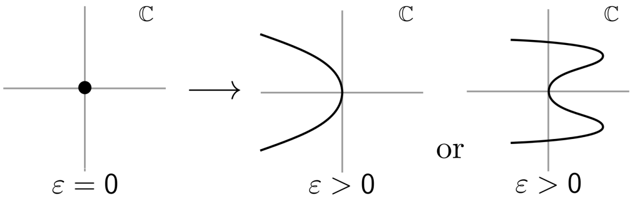


Jumps in terms of λ, ε and Floquet multiplier γ.

Singularly perturbed problems - Stability

Evans-function factorization via Ricatti transform

- **Explicit control** over spectrum in limit $\varepsilon \to 0$;
- **Curve of essential spectrum** that shrinks to 0 as $\varepsilon \to 0$;
- Control in limit $\varepsilon \to 0$ is **insufficient** to decide upon stability.



Complement with Lin's method

- **Expansion** critical curve $\lambda_c(\gamma, \varepsilon) = \varepsilon^2 \lambda_0(\gamma) + h.o.t.$;
- Yields **explicit** conditions for **nonlinear diffusive stability**.

Conclusion

Lin's method in stability analyses of travelling waves

- Yields **piecewise continuous eigenfunction** for each λ;
- **Lyapunov-Schmidt:** jump $J(\lambda, \varepsilon)$ vanishes $\Leftrightarrow$ λ in spectrum.

Extension from regular to singularly perturbed RD-systems:

- Fixed points replaced by **slow transitions**;
- Induces additional **slow eigenvalue problems**;
- Exponential dichotomies replaced by **trichotomies**;
- **Jumps** occur during fast transitions.

Questions

Thank you for the attention!

- B. de Rijk. Spectra and stability of spatially periodic pulse patterns II: the critical spectral curve, *submitted*
- P. Carter, B. de Rijk, B. Sandstede. Stability of travelling pulses with oscillatory tails in the FitzHugh-Nagumo system, *J. Nonlinear Sci.* 26-5 (2016), pp. 1369-1444
- B. de Rijk, A. Doelman, J.D.M. Rademacher. Spectra and stability of spatially periodic pulse patterns: Evans function factorization via Riccati transformation, *SIAM J. Math. Anal.* 48-1 (2016), pp. 61-121