Wake Forest University May 22, 2019

Noise-Induced Tipping in a Periodically Forced System: The Noise Drift Balanced Regime

John A. Gemmer
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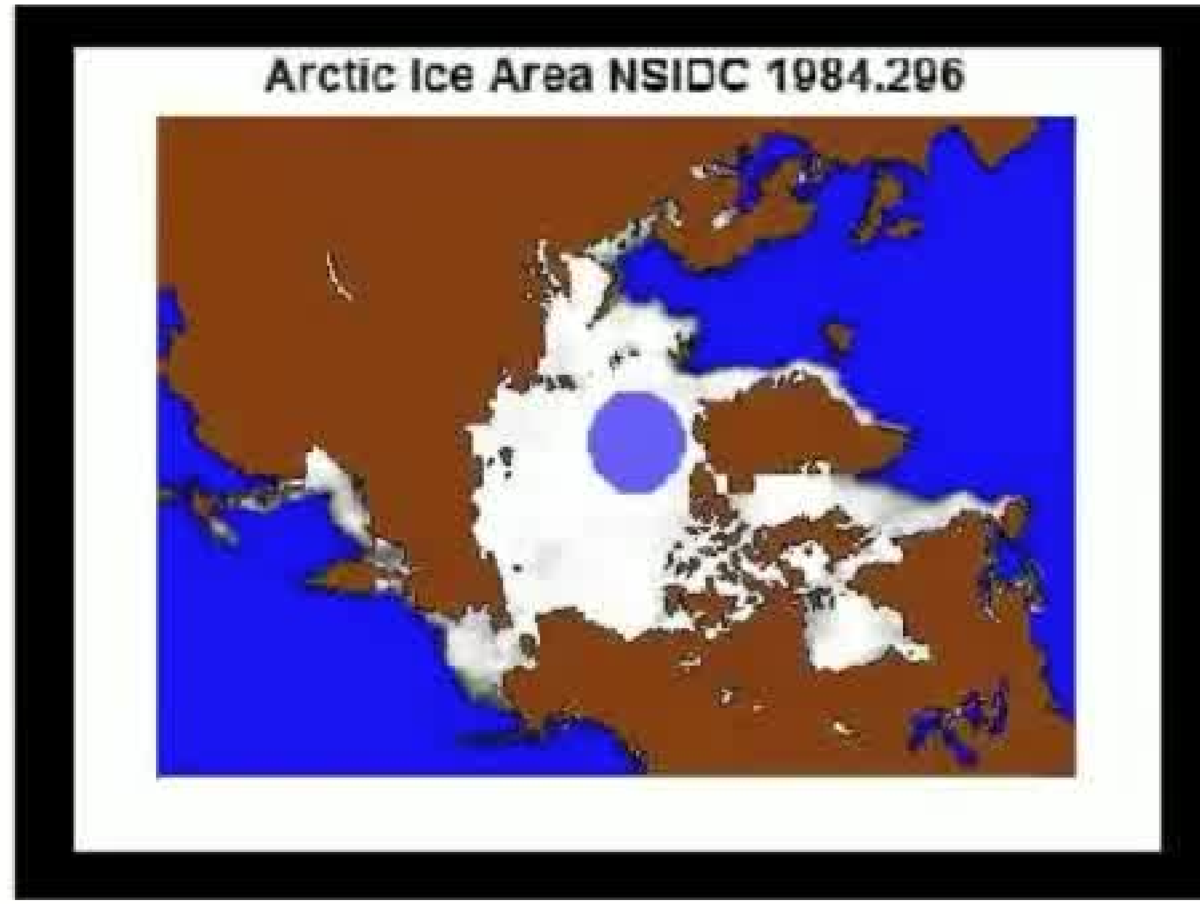

Collaborators:

1. Yuxin Chen (Northwestern University)
2. Alexandria Volkening (MBI)
3. Mary Silber (University of Chicago)
4. Jessica Zanetell (University of Arizona)

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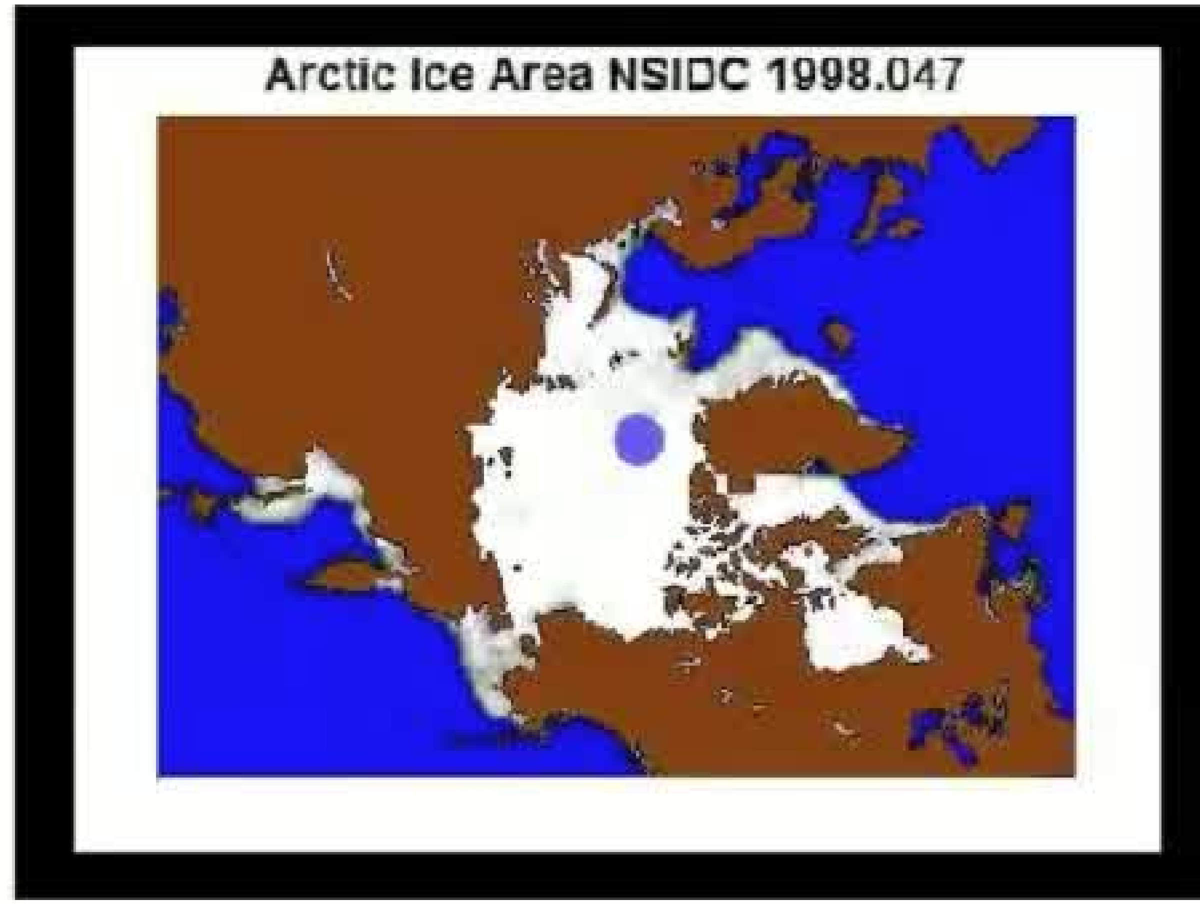

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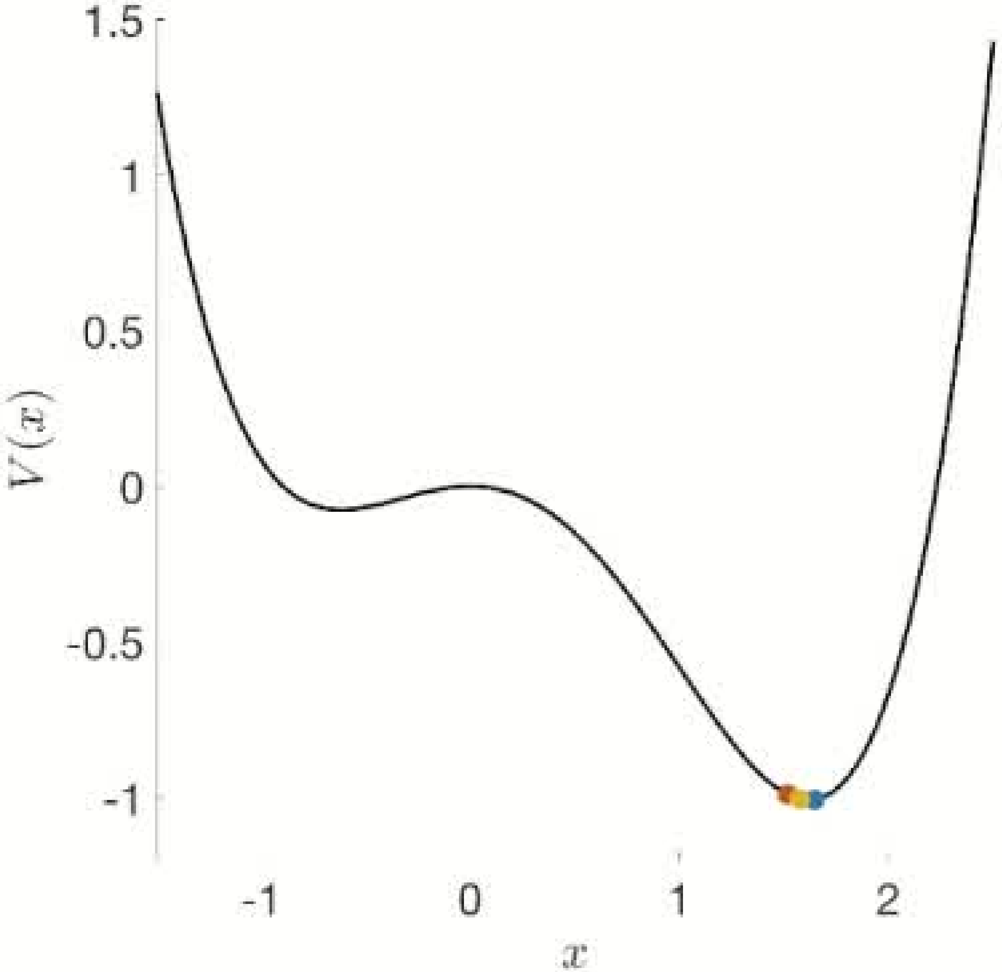
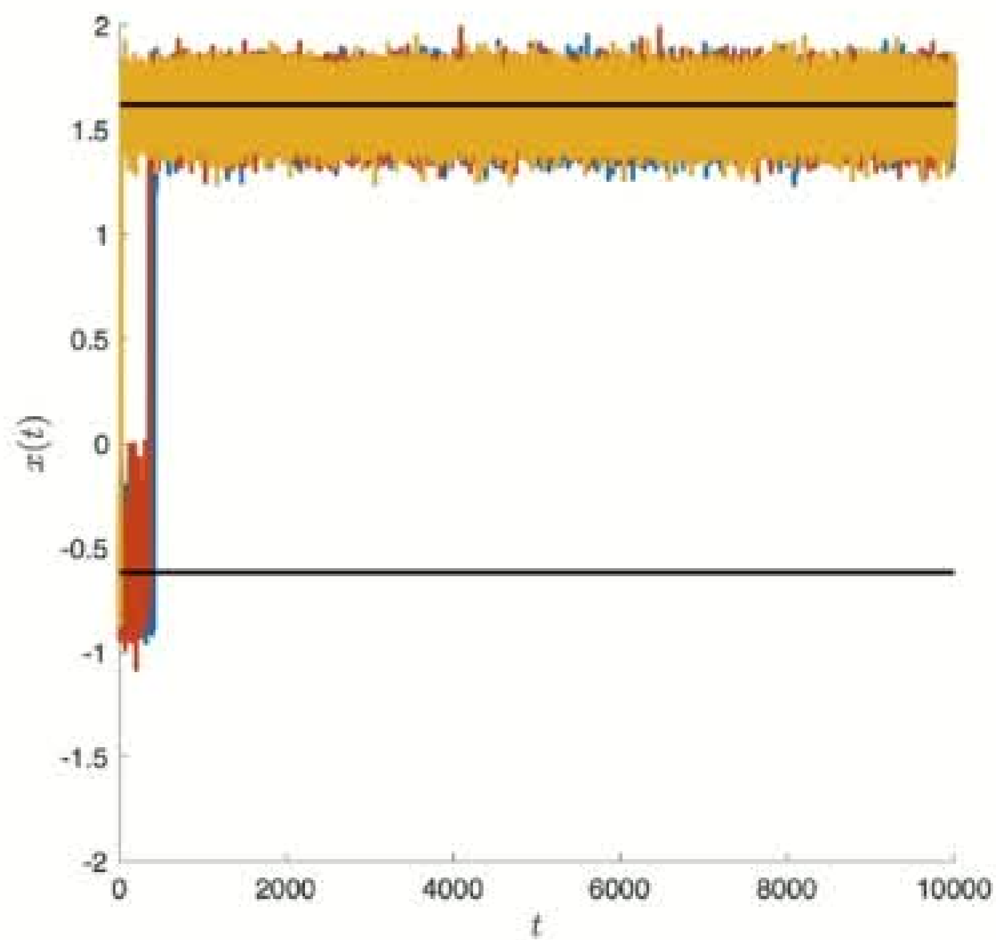
Numerical Experiment

$$dx = -\nabla V(x(t)) + \sigma dW = (x - x^3 + x^2)dt + \sigma dW$$





σ	$\mathbb{E}[\tau_{lu}]$	$\mathbb{E}[\tau_{ul}]$
0.2	237	2.5×10^{22}
0.15	4500	2.6×10^{39}
0.1	2×10^7	1.1×10^{88}

$$\mathbb{E}[\tau_{lu}] \sim C_{lu} \exp(C_{lu}/\sigma^2)$$

$$\mathbb{E}[\tau_{ul}] \sim C_{ul} \exp(C_{ul}/\sigma^2)$$

Toy Model

$$dx_t = \underbrace{\varepsilon^{-1}\left(x - x^3 + \alpha\right)dt}_{\text{Double Well Potential}} + \underbrace{A\varepsilon^{-1}\cos(2\pi t)dt}_{\text{Periodic Forcing}} + \underbrace{\sigma dW_t}_{\text{Noise}}$$

Rescaled Version:

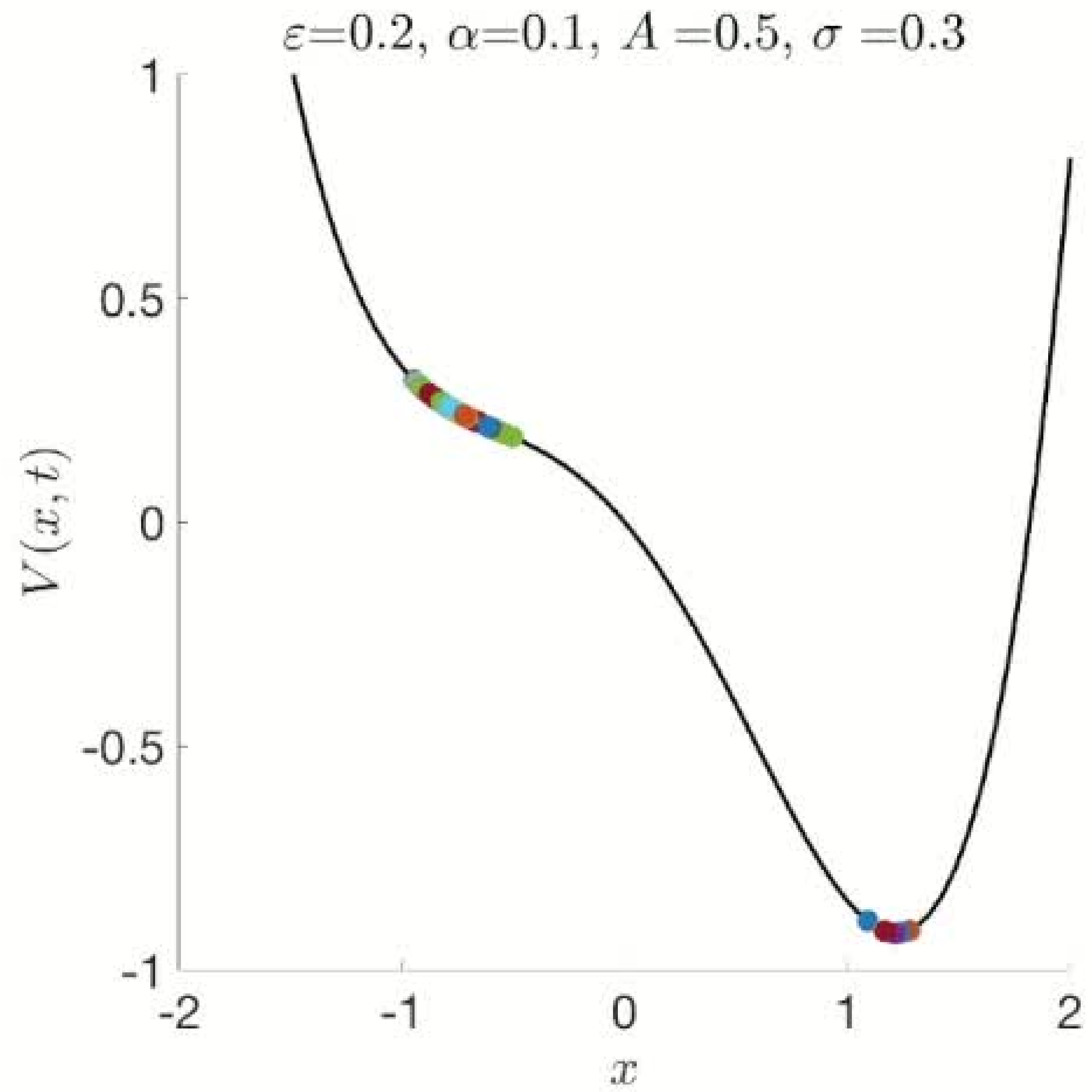
$$dx_\tau = (x - x^3 + \alpha)d\tau + A\cos(2\pi\varepsilon\tau)d\tau + \varepsilon^{\frac{1}{2}}\sigma dW_\tau$$

$\varepsilon \sim$ frequency

$\alpha \sim$ biasing term

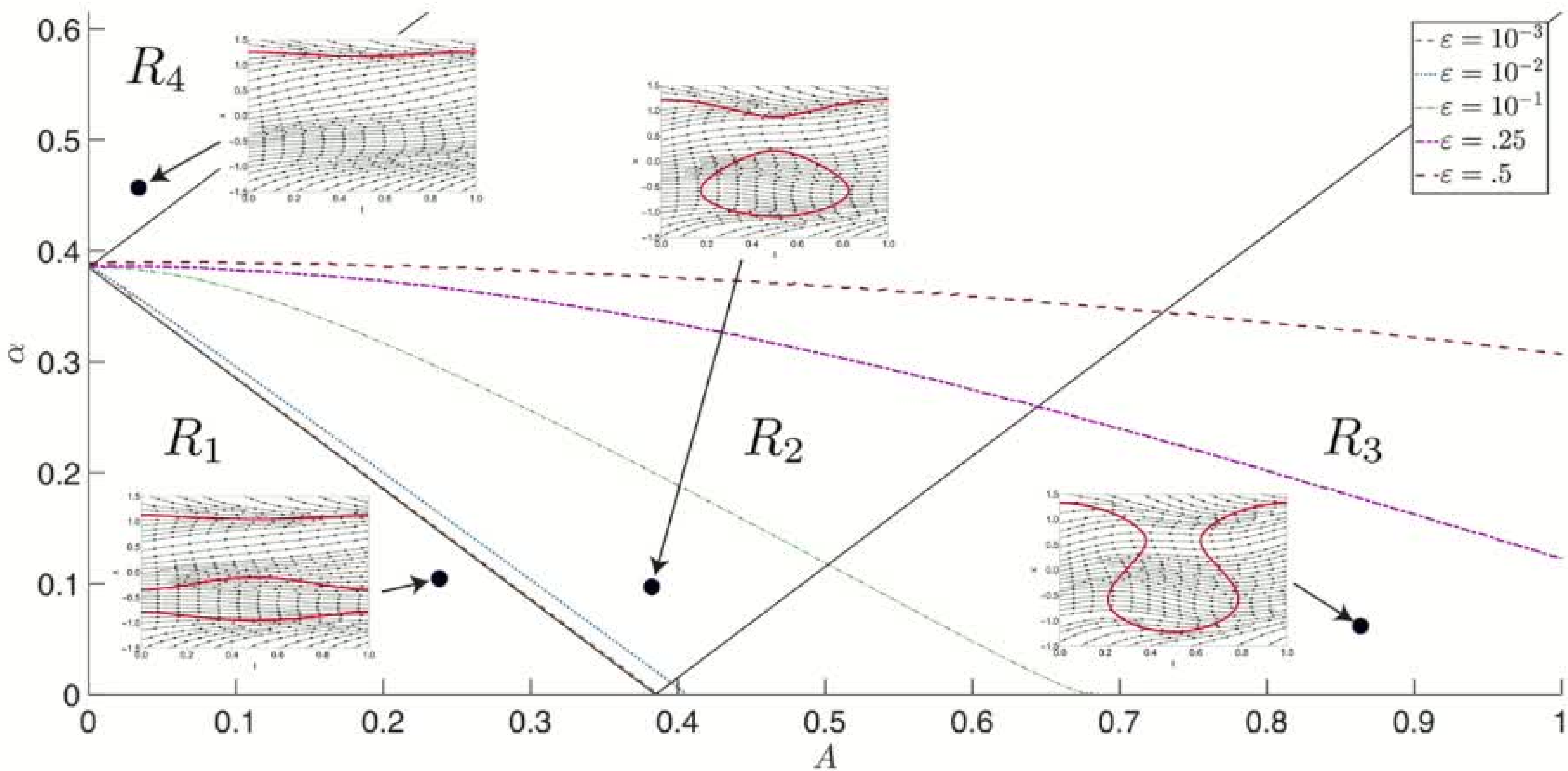
$A \sim$ forcing strength

$\alpha \sim$ noise strength



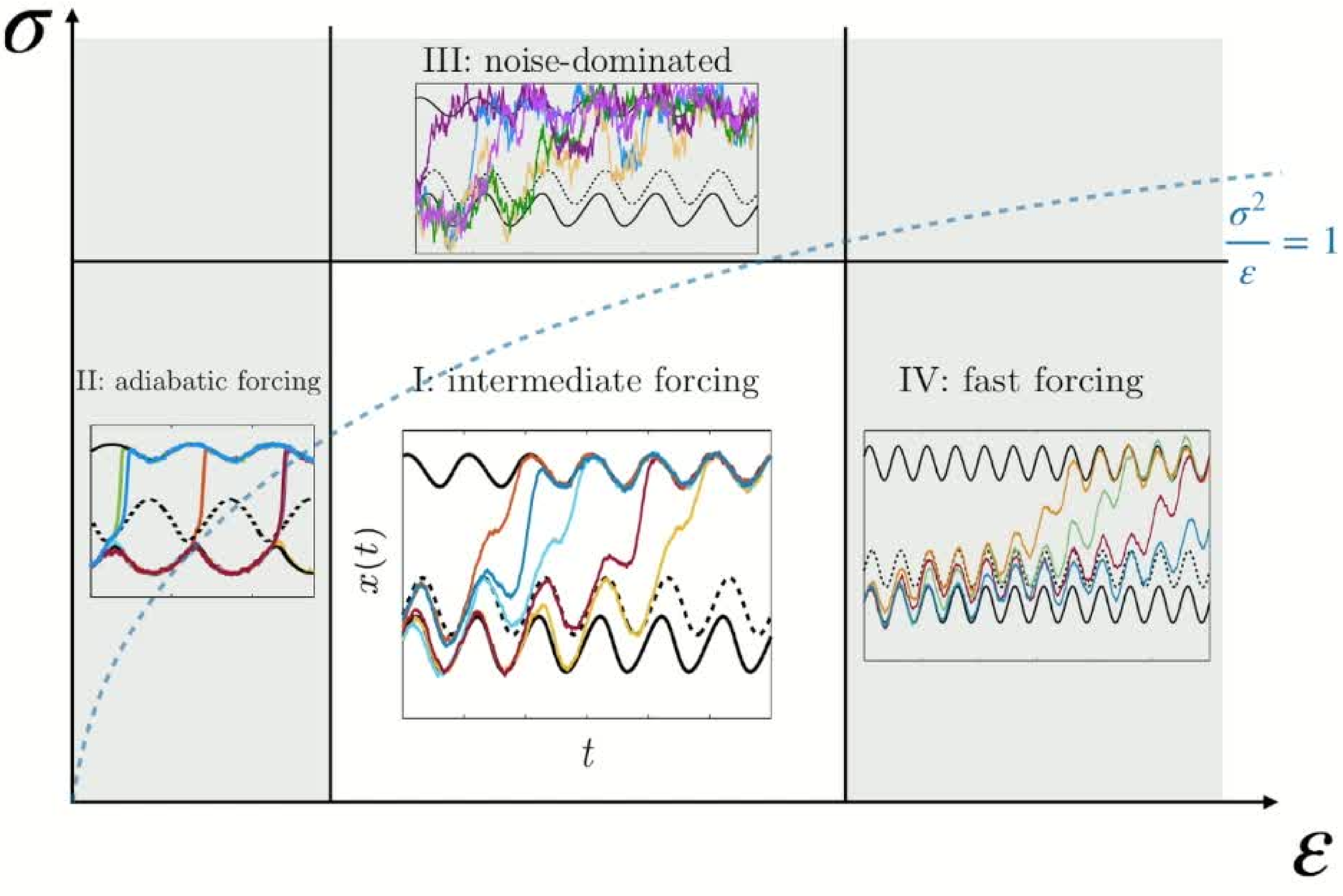
Parameter Regions

$$dx_t = \underbrace{\varepsilon^{-1}\left(x - x^3 + \alpha\right)dt}_{\text{Double Well Potential}} + \underbrace{A\varepsilon^{-1}\cos(2\pi t)dt}_{\text{Periodic Forcing}} + \underbrace{\sigma dW_t}_{\text{Noise}}$$



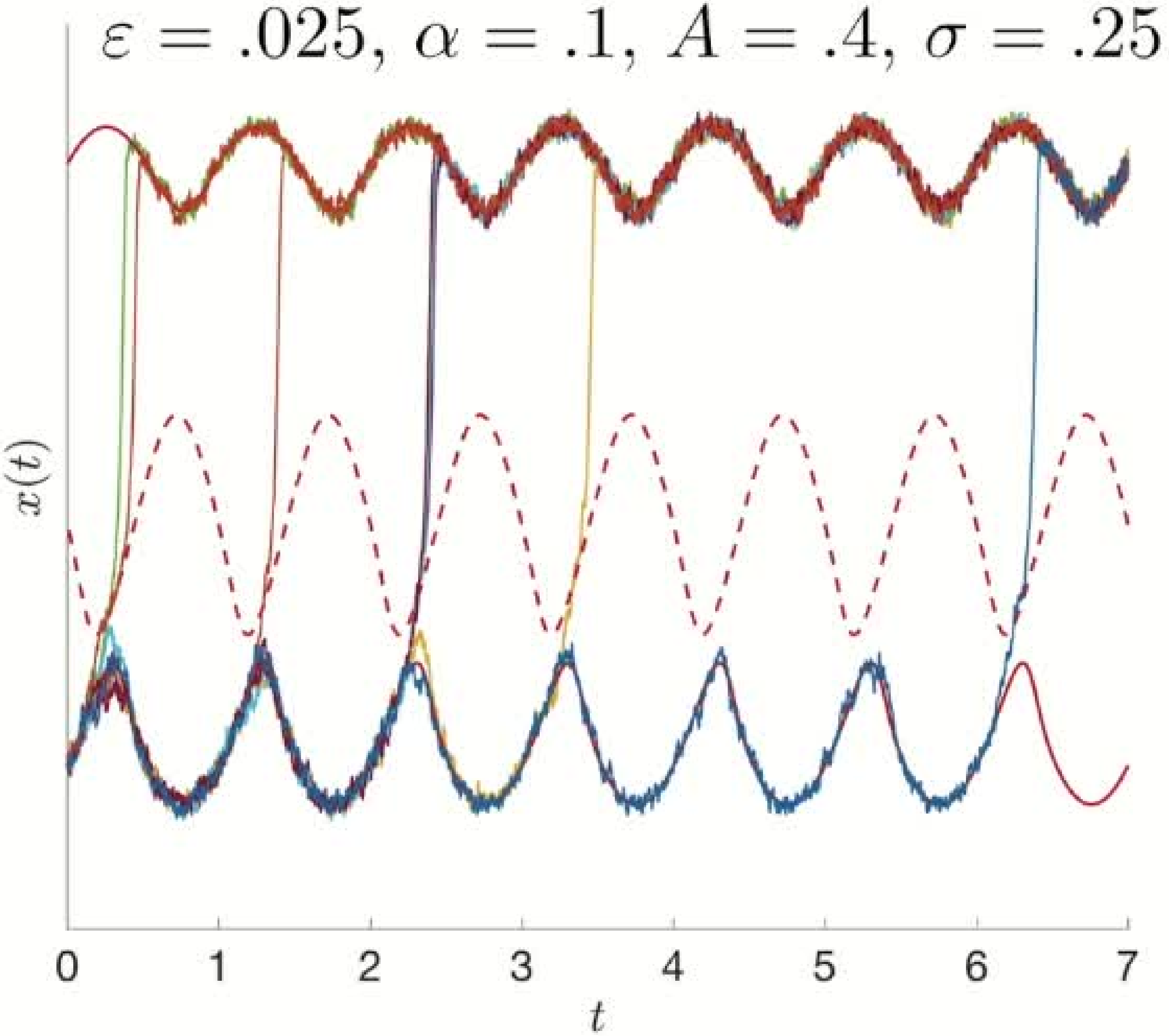
What governs tipping times?

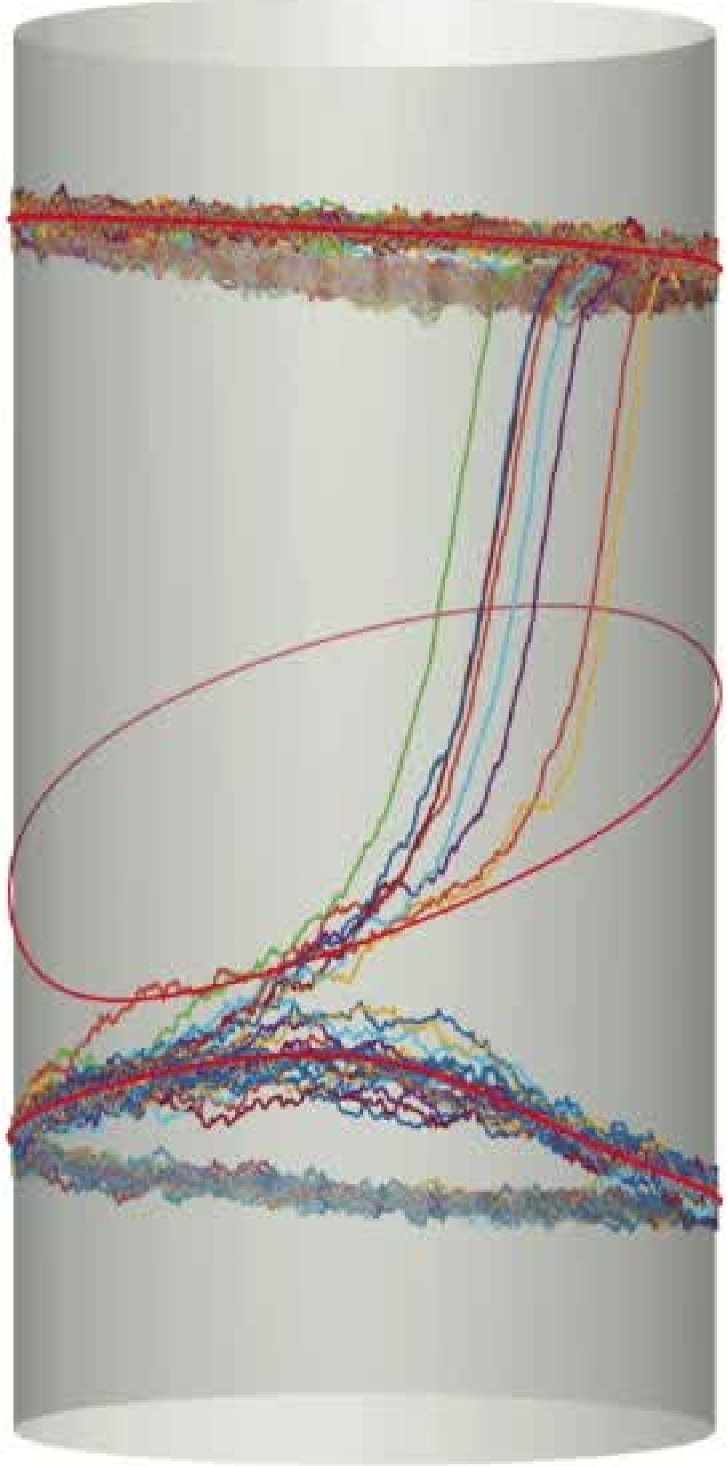
Parameter Regimes



Realizations

$$dx_t = \varepsilon^{-1}(x - x^3 + \alpha + A\cos(2\pi t))dt + \sigma dW$$



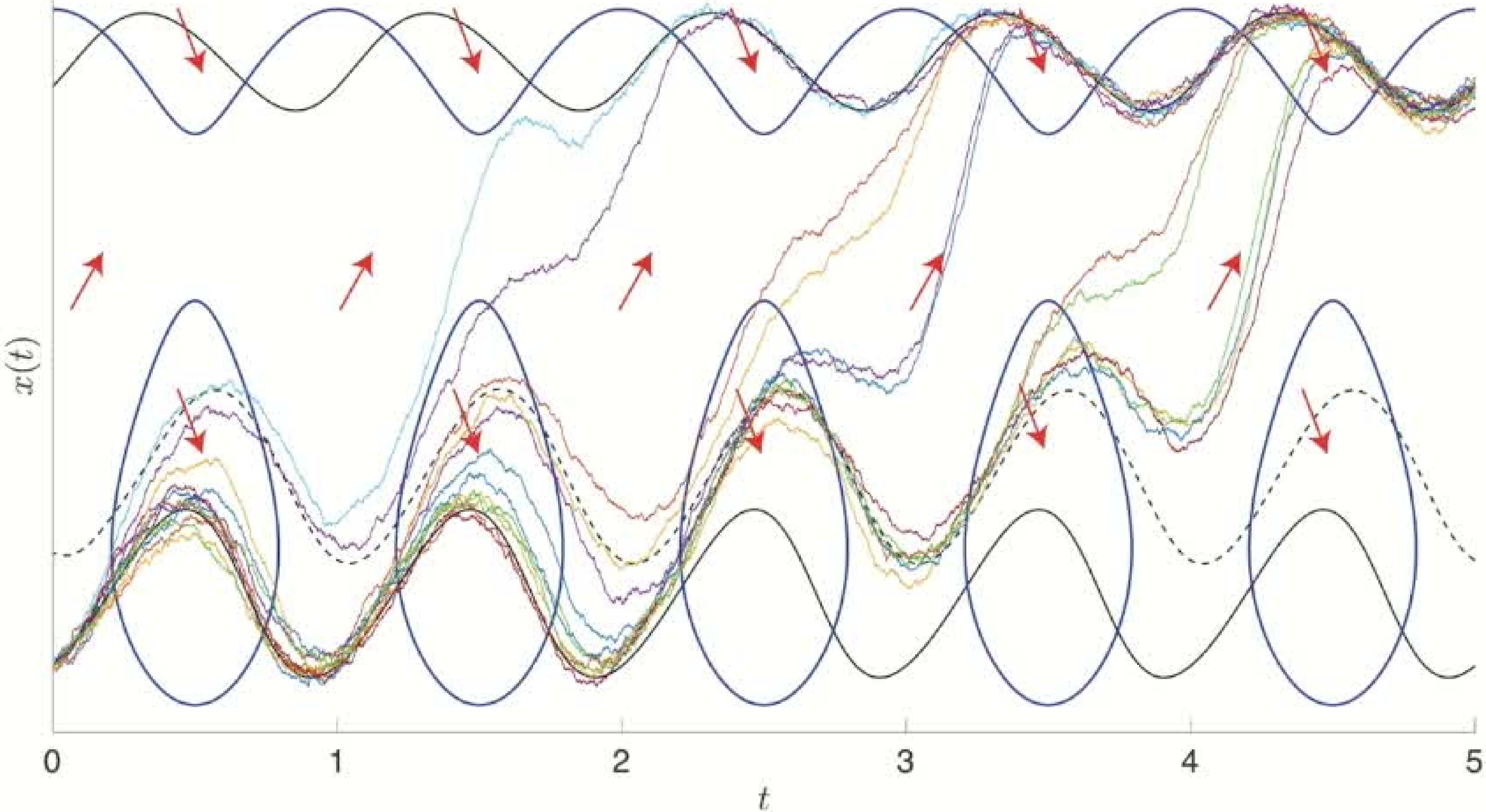


Quasistatic forcing

Realizations

$$dx_t = \varepsilon^{-1}(x - x^3 + \alpha + A\cos(2\pi t))dt + \sigma dW$$

$$\varepsilon = .25,\ \alpha = .25,\ A = .5,\ \sigma = .1.$$



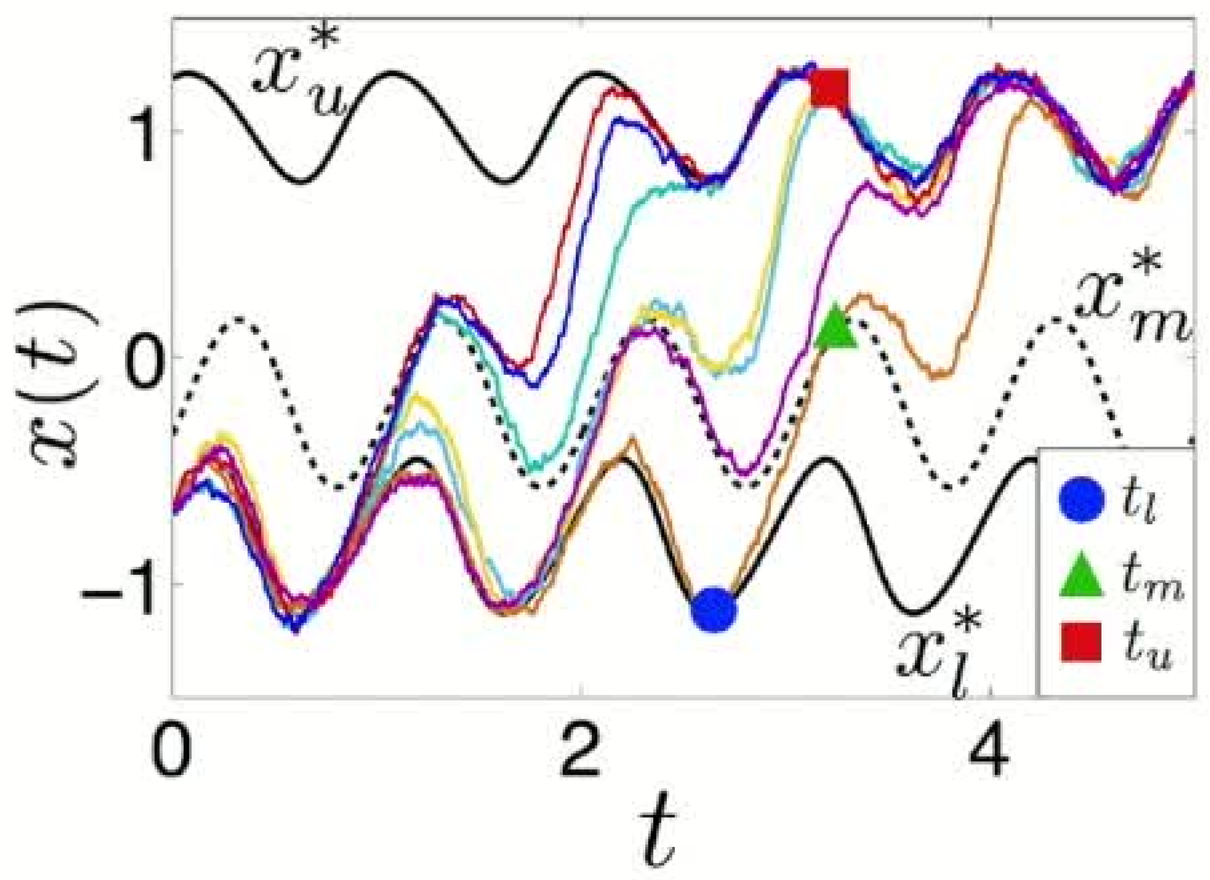
What role does the geometry of the flow play?

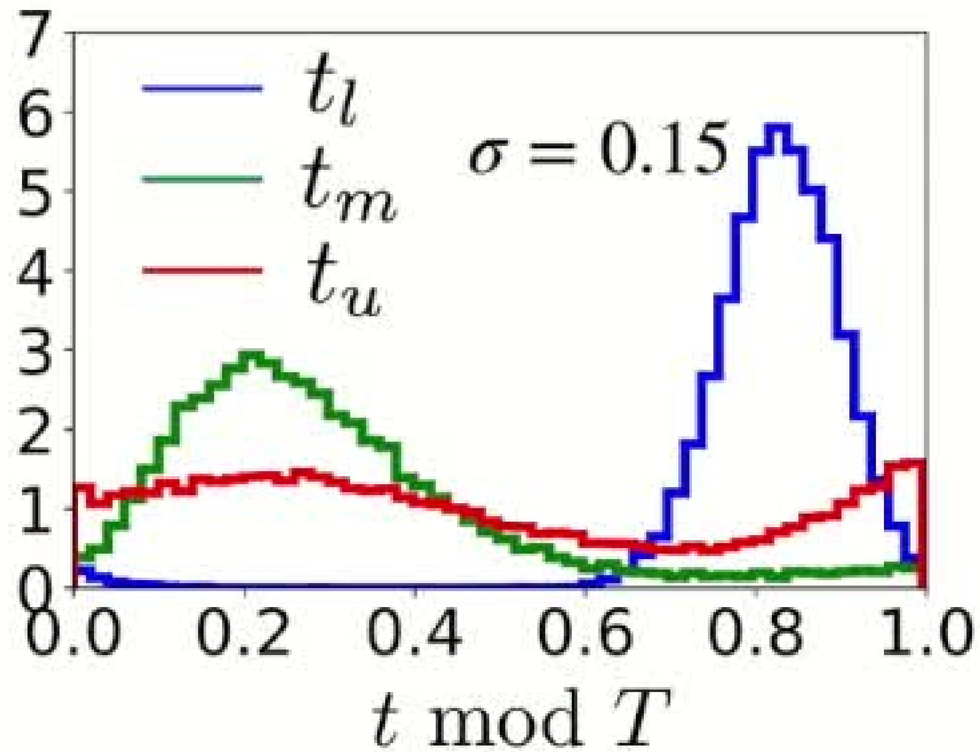
Tipping Times

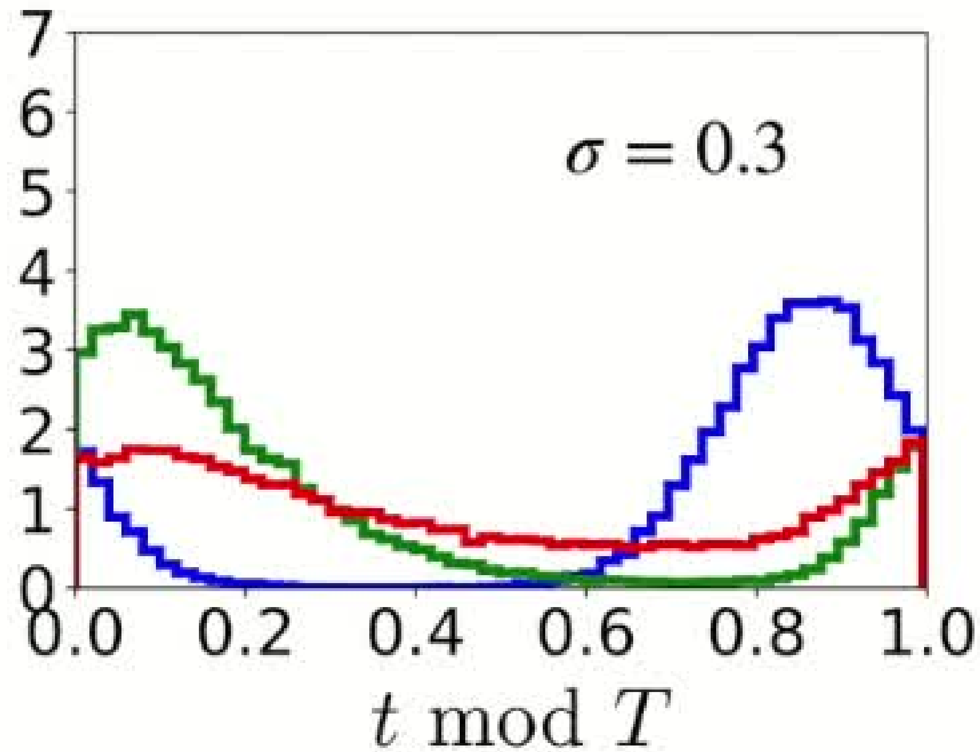
$$t_l = \max\{t \geq 0 : X_t \geq x_l^*(t)\}$$

$$t_m = \min\{t \geq 0 : X_t \geq x_m^*(t)\}$$

$$t_u = \min\{t \geq 0 : X_t \geq x_u^*(t)\}$$







Problem

Is there a preferred phase for transitioning between metastable states?

(Incomplete) Literature on noise-induced transition

Time independent system

1. Freidlin, Mark I., and Alexander D. Wentzell. *Random Perturbations of Dynamical Systems*. Vol. 260. Springer Science & Business Media, 2012.
2. Ren, Weiqing, and Eric Vanden-Eijnden. "Minimum action method for the study of rare events." *Communications on pure and applied mathematics* 57.5 (2004): 637-656.

Slowly time dependent system

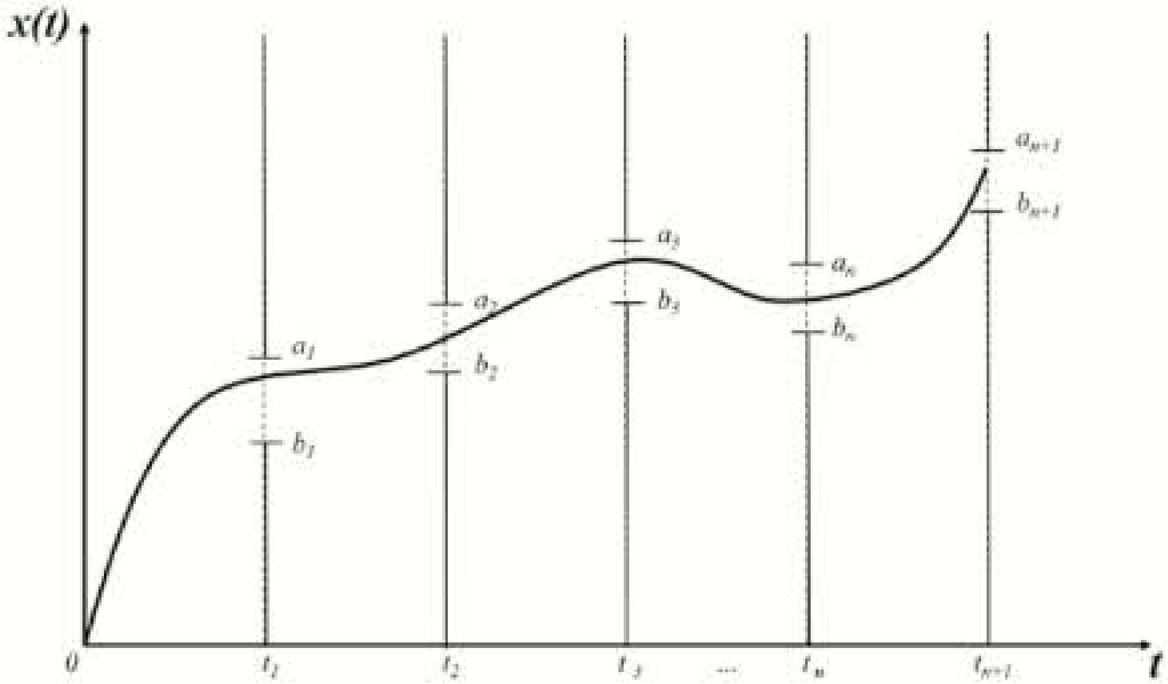
1. Berglund, Nils, and Barbara Gentz. *Noise-induced phenomena in slow-fast dynamical systems: a sample-paths approach*. Springer Science & Business Media, 2006.
2. Berglund, Nils, and Barbara Gentz. "Metastability in simple climate models: pathwise analysis of slowly driven Langevin equations." *Stochastics and Dynamics* 2.03 (2002): 327-356.

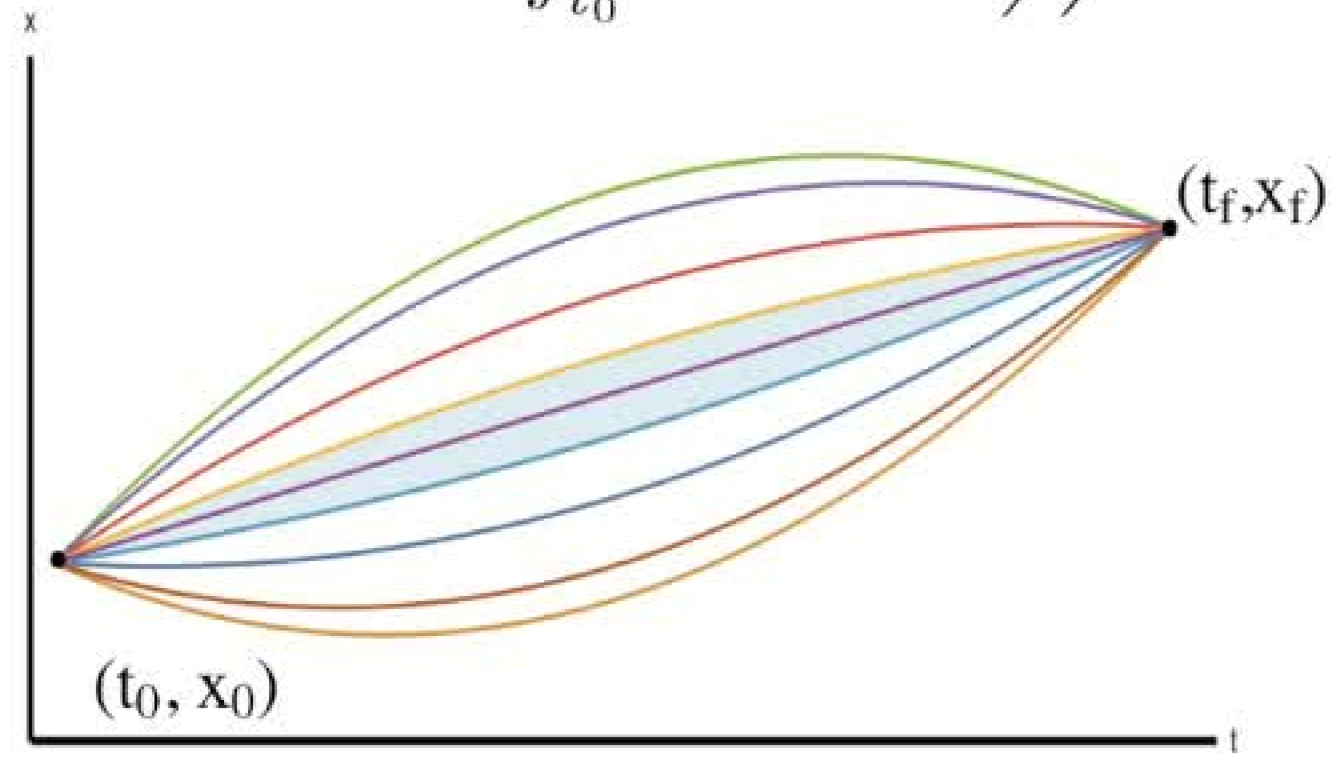
Least action method

1. Smelyanskiy, V. N., et al. "Fluctuations, escape, and nucleation in driven systems: logarithmic susceptibility." *Physical review letters* 79.17 (1997): 3113.
2. Dykman, M. I., et al. "Activated escape of periodically driven systems." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 11.3 (2001): 587-594.
3. Ritchie, Paul, and Jan Sieber. "Early-warning indicators for rate-induced tipping." *Chaos: An Interdisciplinary Journal of Nonlinear Science* 26.9 (2016): 093116.

Path Integral Formulation

$$P_\sigma = C \exp\left(-\frac{1}{2\sigma^2}\left(\int_{t_0}^{t_f} (\dot{x} - f(x,t))^2\, dt + \sigma^2 \int_{t_0}^{t_f} f_x(x,t)dt\right)\right) d[x]$$





Most probable path is the minimizer of the **Onsager-Machlup** functional $I_\sigma : \mathcal{A} \to \mathbb{R}$:

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} \underbrace{(\dot{x} - f(x(t),t))^2}_{\text{Flow deviation}} dt + \sigma^2 \int_{t_0}^{t_f} \underbrace{f_x(x(t),t)}_{\text{Floquet exp.}} dt.$$

$$\mathcal{A} = \{x \in W^{1,2}([t_0,t_f]) : x(t_0) = x_-^*(t_0) \text{ and } x(t_f) = x_+^*(t_f)\}$$

Miller, P. D. (2006). *Applied asymptotic analysis* (Vol. 75). American Mathematical Soc.

Ritchie, P., & Sieber, J. (2016). Early-warning indicators for rate-induced tipping. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, *26*(9), 093116.

Onsager-Machlup Functional

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} \underbrace{(\dot{x} - f(x(t),t))^2}_{\text{Flow deviation}} dt + \sigma^2 \int_{t_0}^{t_f} \underbrace{f_x(x(t),t)}_{\text{Floquet exp.}} dt.$$

Freidlin–Wentzell rate functional:

$$I_0[x(t)] = \int_{t_0}^{t_f} \underbrace{(\dot{x} - f(x(t),t))^2}_{\text{Flow deviation}} dt$$

Time varying potential:

$$V(x,t) = \varepsilon^{-1}\bar{V}(x,t) = \varepsilon^{-1}\left(\frac{x^4}{4} - \frac{x^2}{2} - \alpha x - A\cos(2\pi t)x\right)$$

Variational problem with multiple scales:

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} \left(\dot{x} + \varepsilon^{-1}\bar{V}_x(x(t),t)\right)^2 dt - \frac{\sigma^2}{\varepsilon}\int_{t_0}^{t_f} \bar{V}_{xx}\left(x(t),t\right) dt.$$

Interesting parameter regimes:

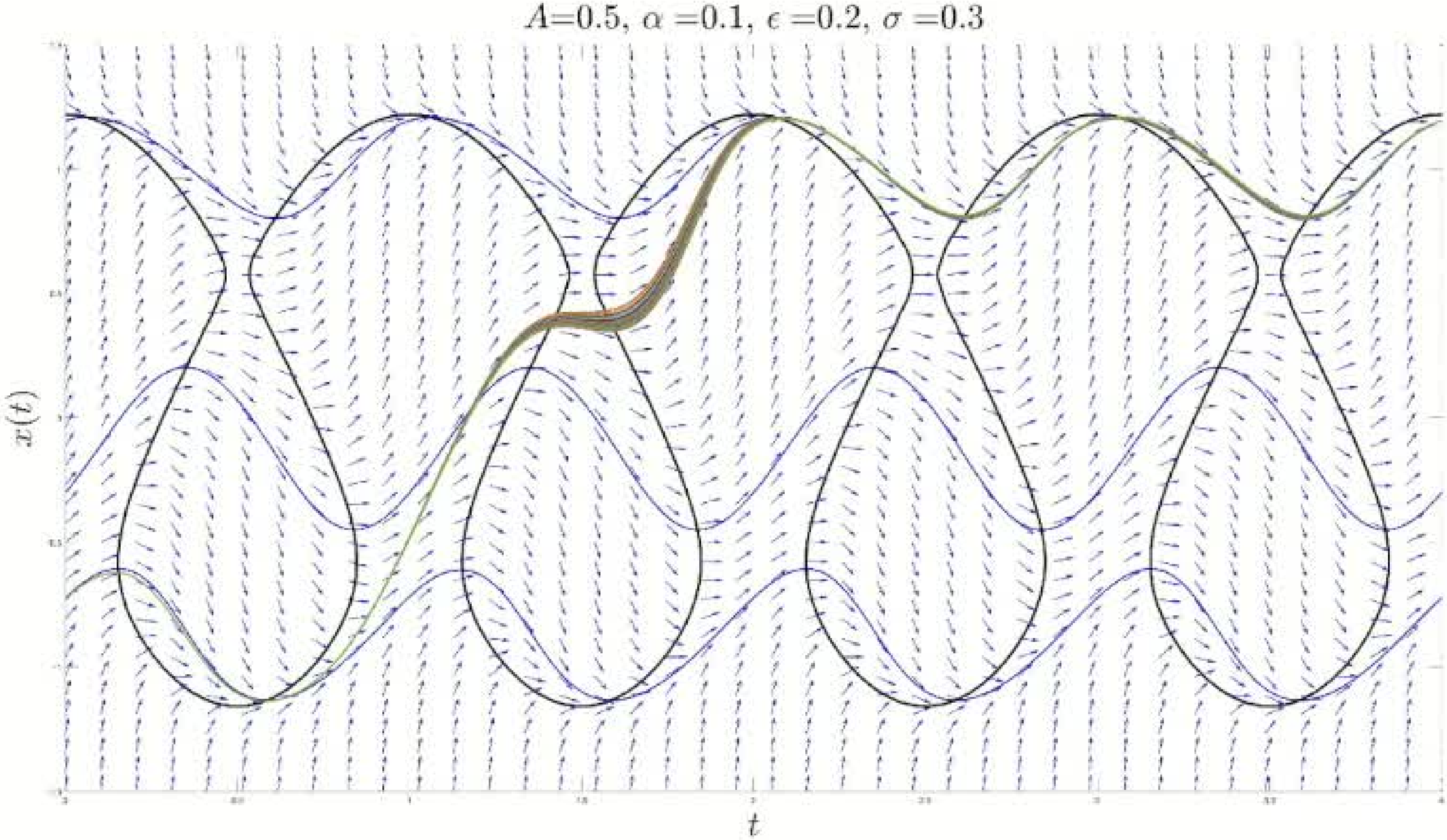
$$\varepsilon \ll \sigma^2 \ll 1 \qquad \varepsilon \sim \sigma^2 \ll 1$$

Solutions to Boundary Value Problem

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} (\dot{x} - f(x(t),t))^2\, dt + \sigma^2 \int_{t_0}^{t_f} f_x(x(t),t)dt.$$

Gradient Flow

$$\frac{\partial x}{\partial s} = -\frac{\delta I}{\delta x}$$



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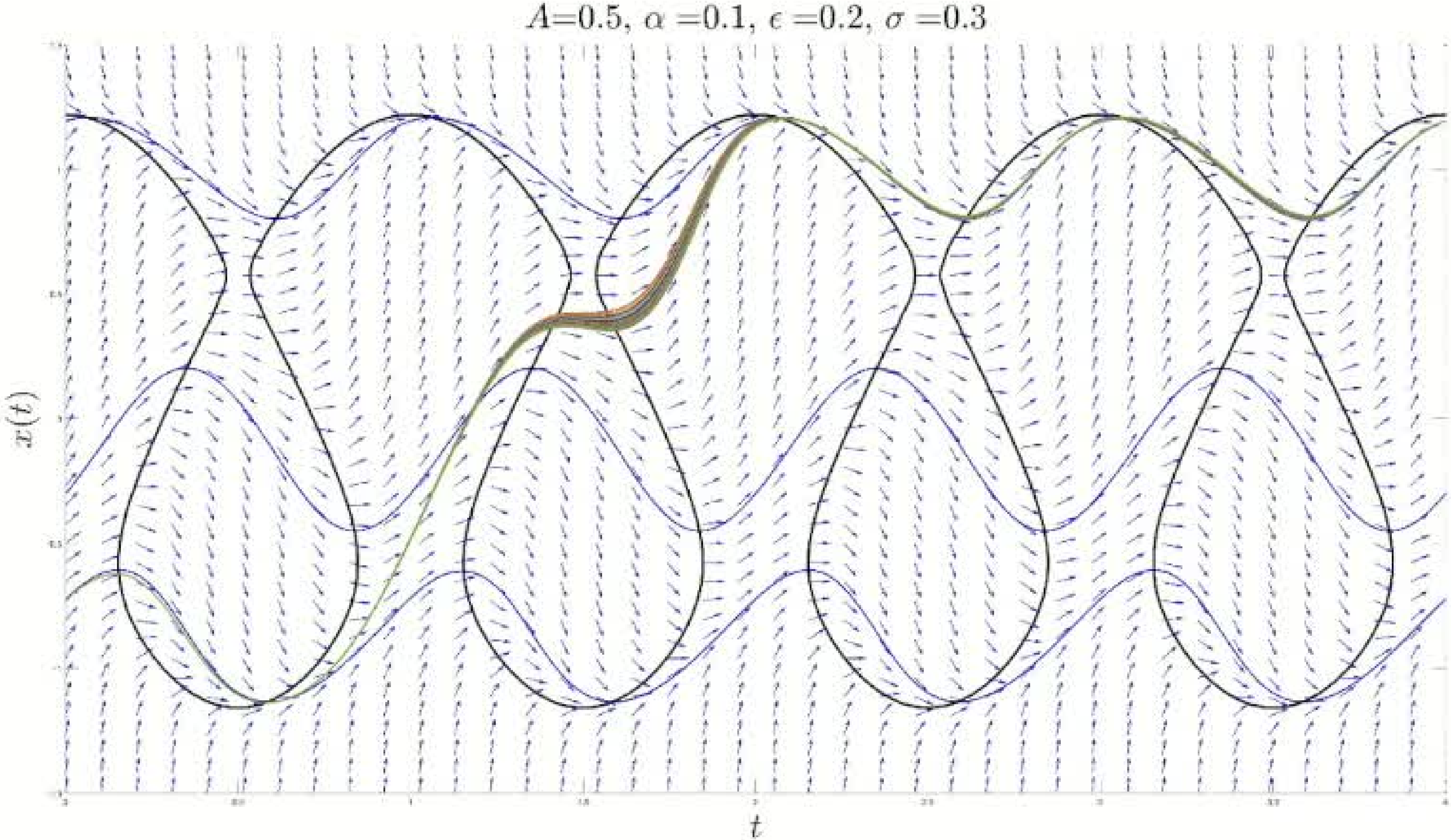

Solutions to Boundary Value Problem

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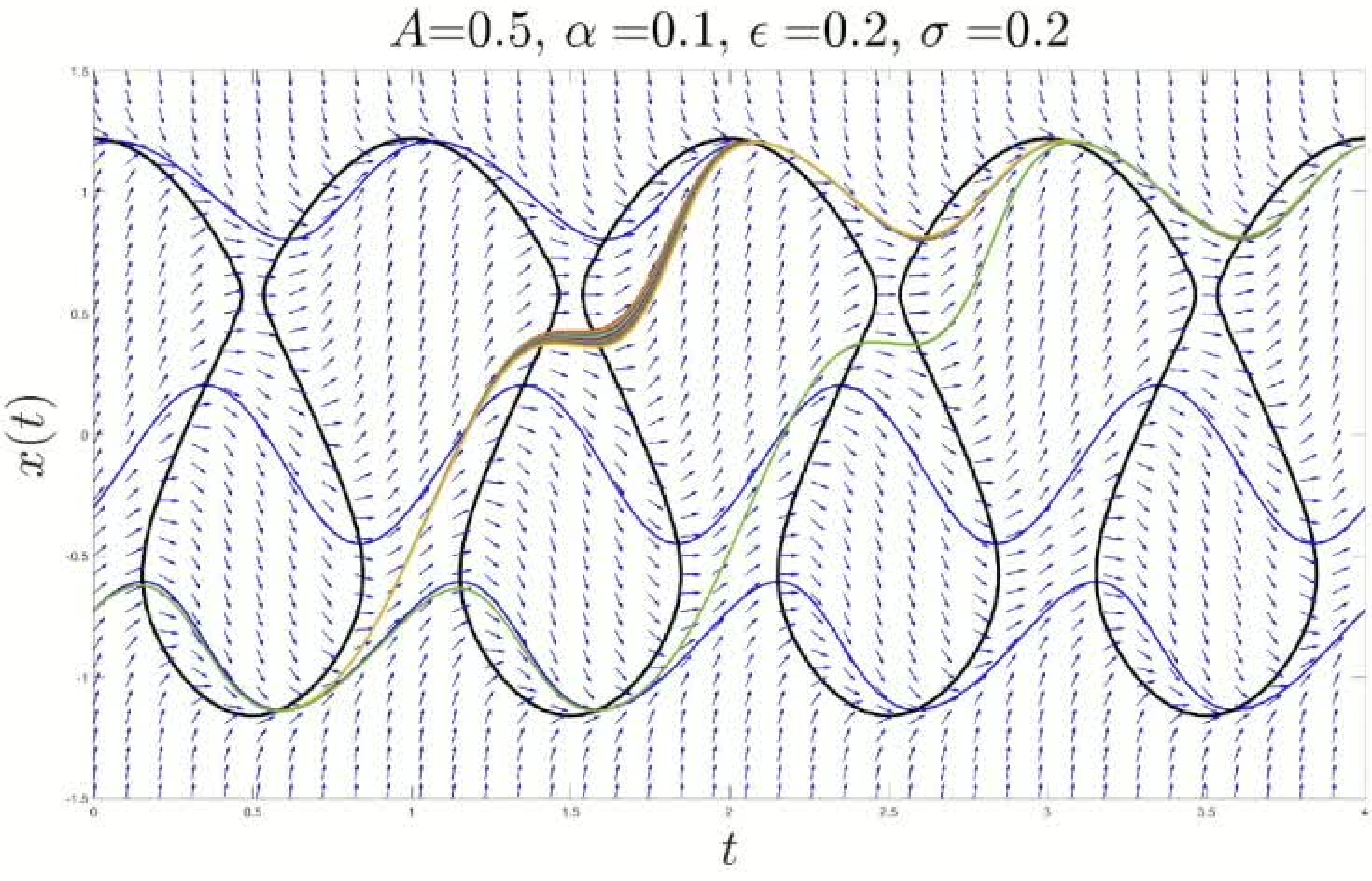
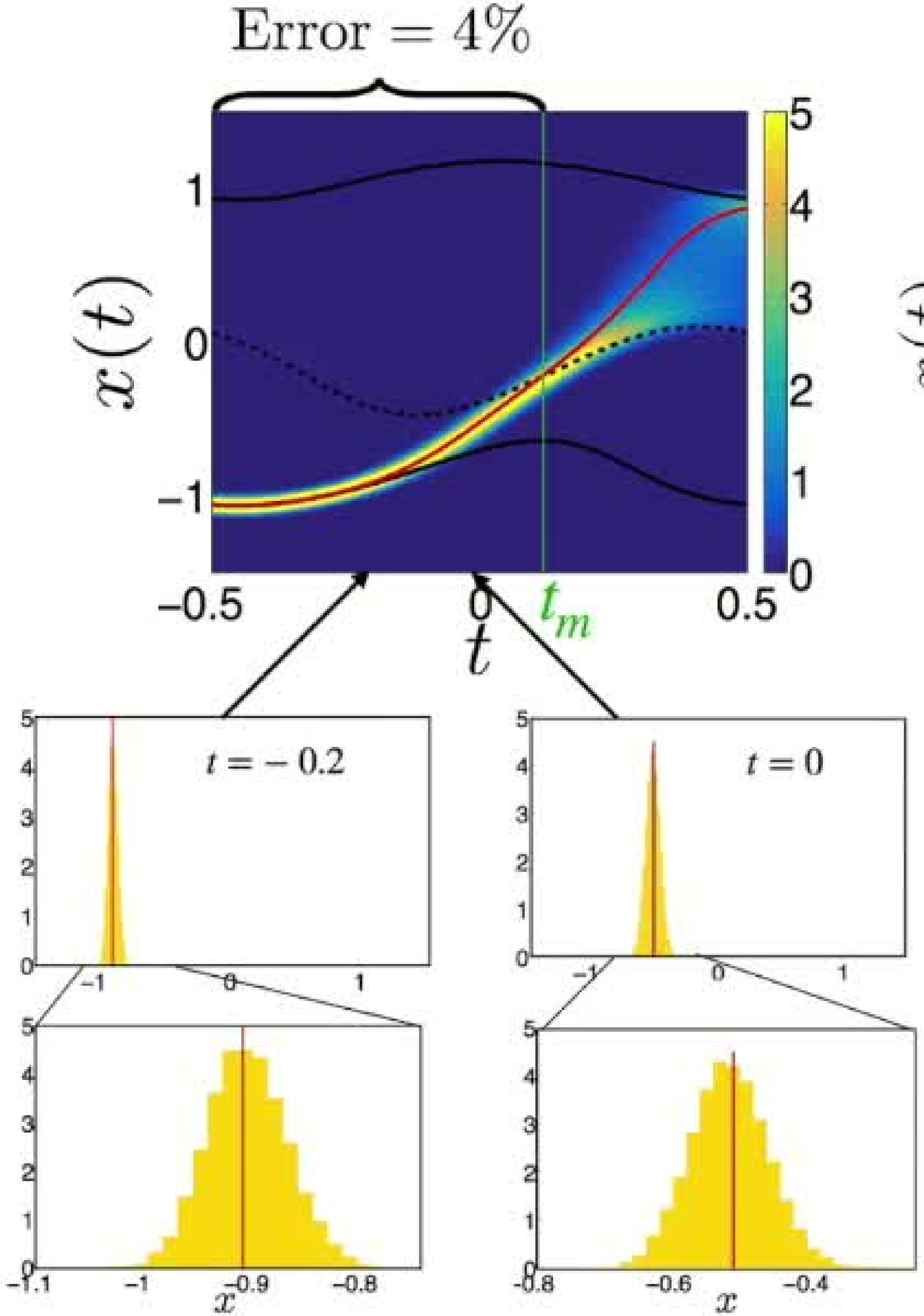
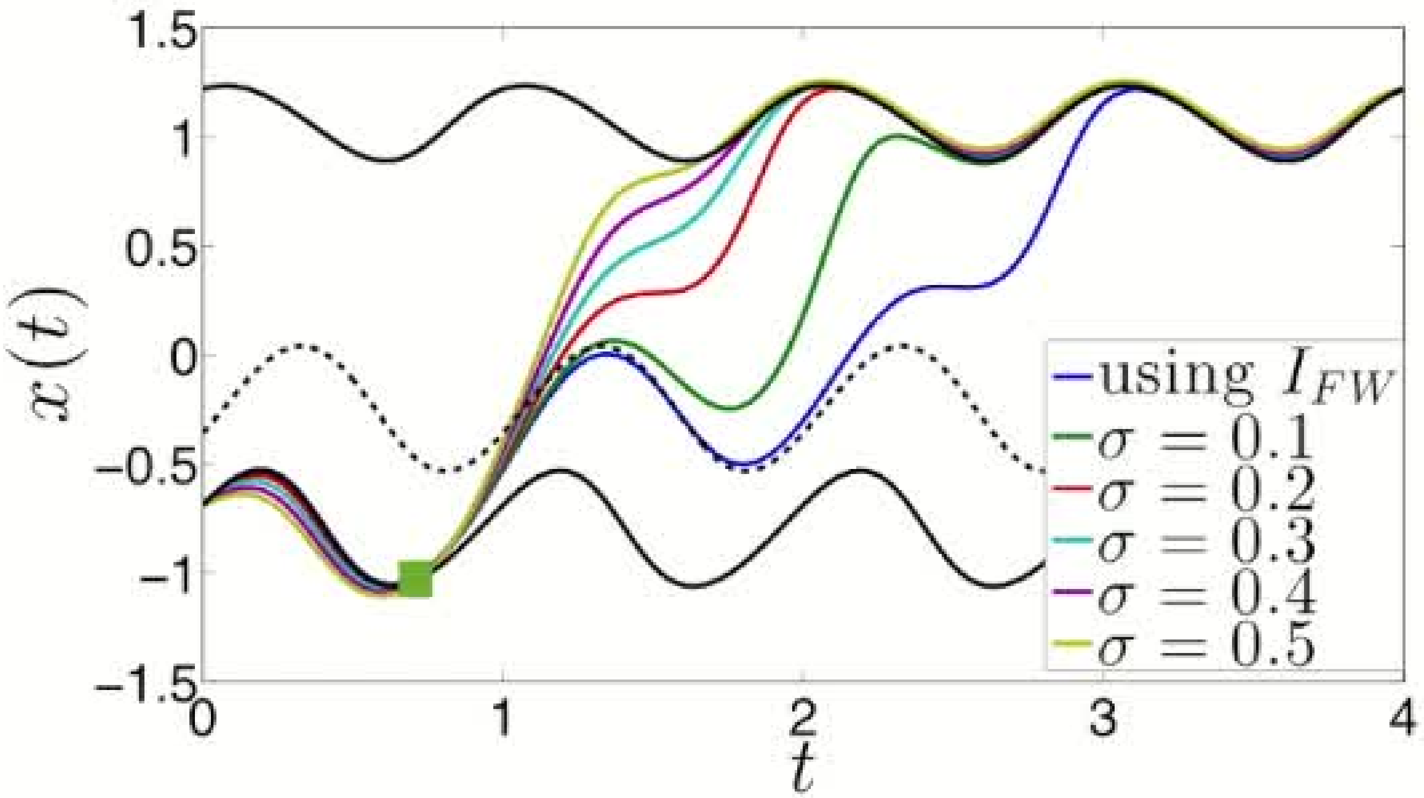


Comparison with Monte Carlo Simulation

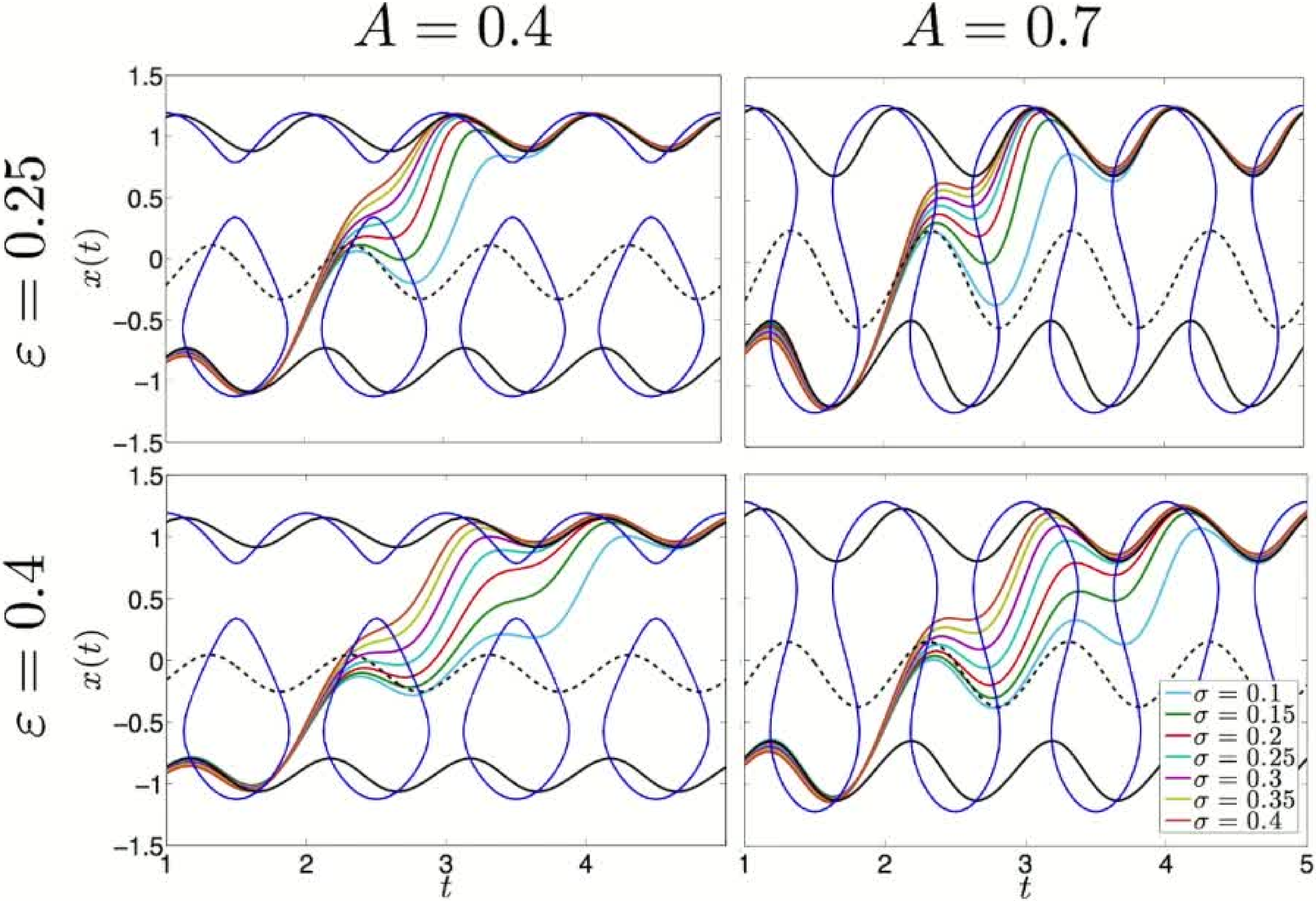




	σ	N	MAPE
$\varepsilon = 0.25$	0.2	49002	5.0%
	0.25	46010	5.6%
	0.3	21183	6.3%
$\varepsilon = 0.4$	0.2	10662	6.0%
	0.25	24882	6.8%
	0.3	48160	8.3%

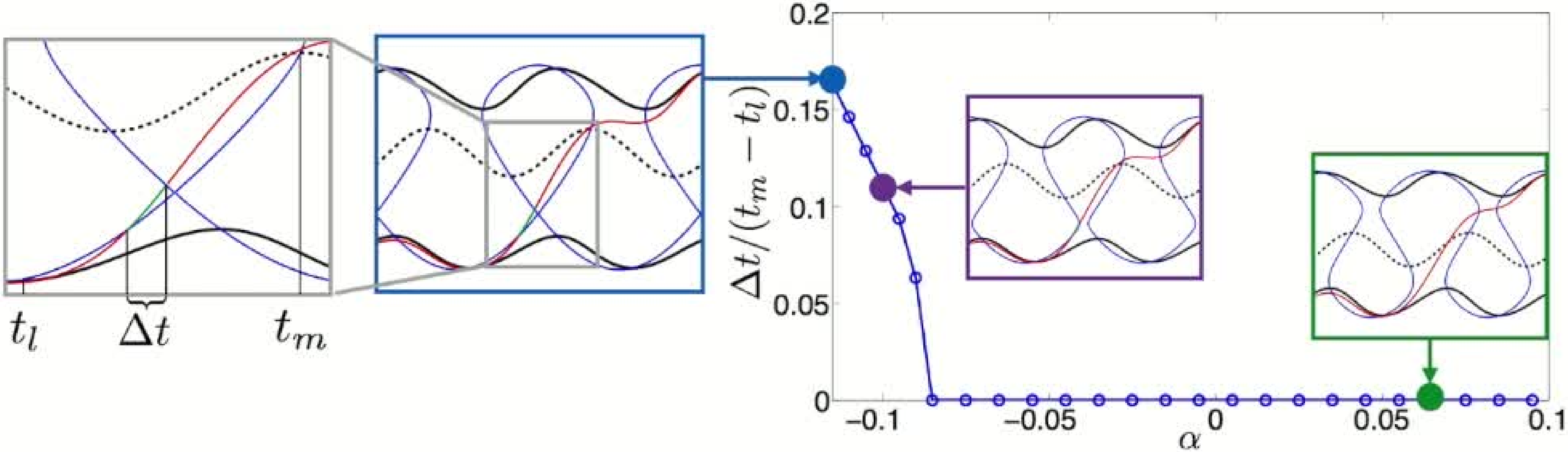
$$MAPE = \frac{100\%}{k} \sum_{i=1}^{k} \left| \frac{(m_i - o_i)}{\bar{m}} \right|$$

Preferred Tipping Phase



Deterministic Predictor of Tipping

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} \underbrace{(\dot{x} - f(x(t),t))^2}_{\text{Flow deviation}} dt + \sigma^2 \int_{t_0}^{t_f} \underbrace{f_x(x(t),t)}_{\text{Floquet exp.}} dt.$$



Nullclines give a strong predictor of tipping events!

Dynamical System

$$dx_t = \varepsilon^{-1}\left(-V'(x) + A\cos(2\pi t)\right)dt + \sigma dW_t.$$

Euler Lagrange Equations:

$$\begin{cases} \ddot{x} = \varepsilon^{-2}\left(V'(x)V''(x) - A\cos(2\pi t)V''(x)\right) + \varepsilon^{-1}\left(2\pi A\sin(2\pi t) - \sigma^2 V'''(x)\right) \\ x(t_0) = x_0 \\ x(t_f) = x_f \end{cases}$$

Legendre transformation:

$$\Psi = \dot{\varphi} + \varepsilon^{-1}\left(V'(\varphi) - A\cos(2\pi t)\right)$$

Hamiltonian:

$$H(x,\Psi) = \frac{\Psi^2}{2} + A\cos(2\pi t)\Psi + \varepsilon^{-1}\left(A\cos(2\pi t) - V'(x)\right)$$

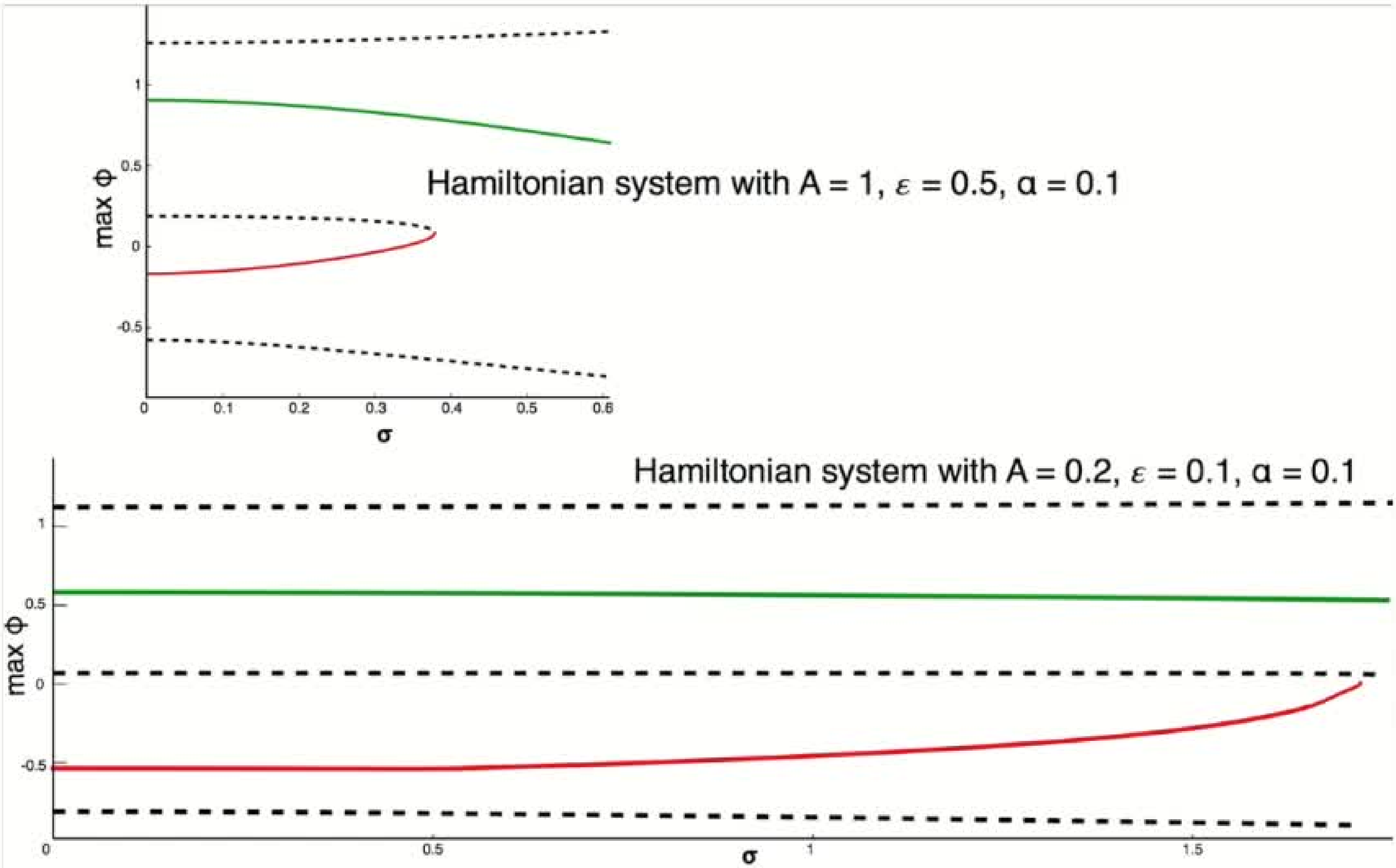
Dynamical system:

$$\begin{cases} \dot{x} = \Psi + \varepsilon^{-1}\left(A\cos(2\pi t) + V'(x)\right) \\ \dot{\Psi} = \varepsilon^{-1}\left(\Psi V''(x) - \sigma^2 V'''(x)\right) \end{cases}.$$

Interesting parameter regimes:

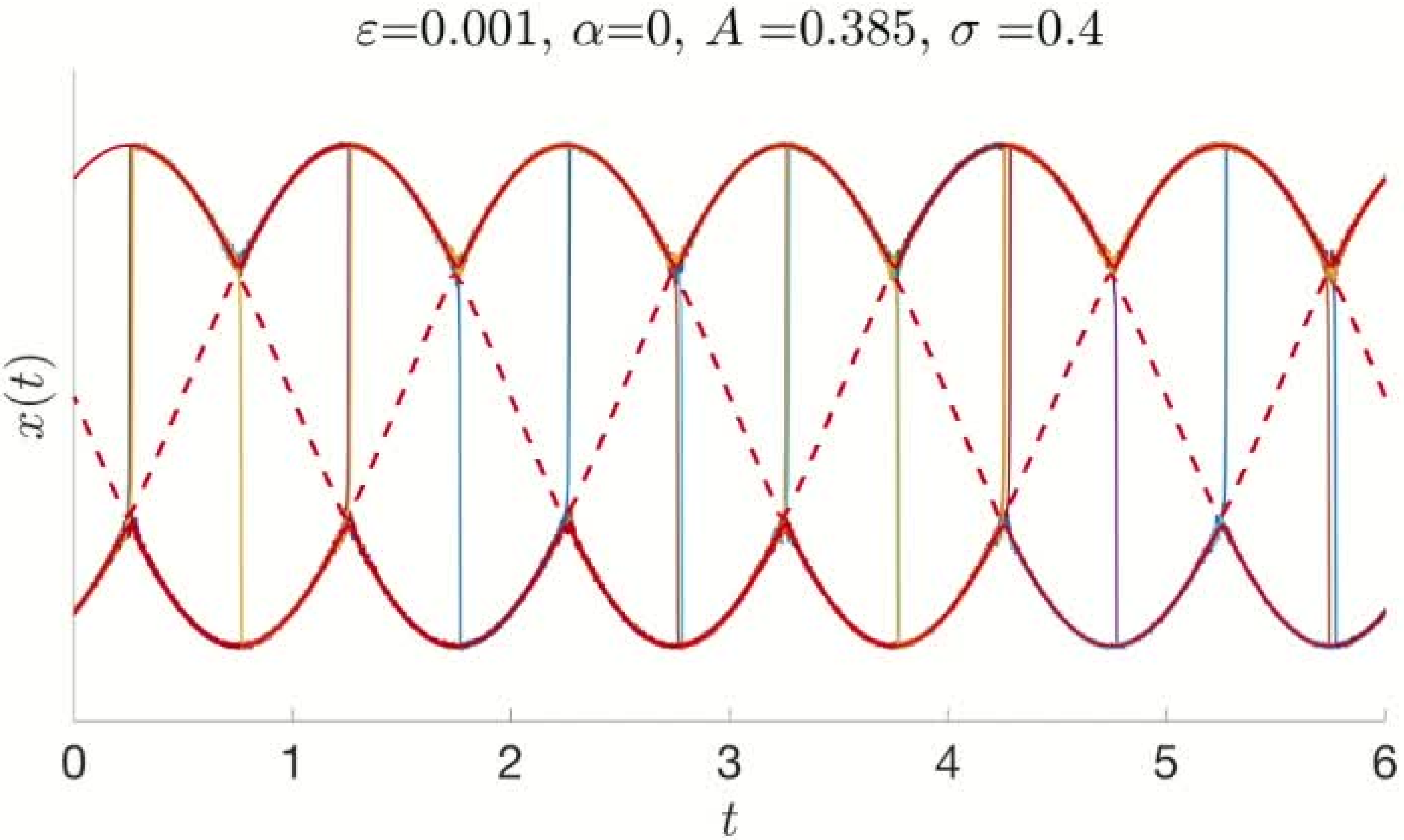
$\varepsilon \ll \sigma^2 \ll 1 \qquad \varepsilon \sim \sigma^2 \ll 1$

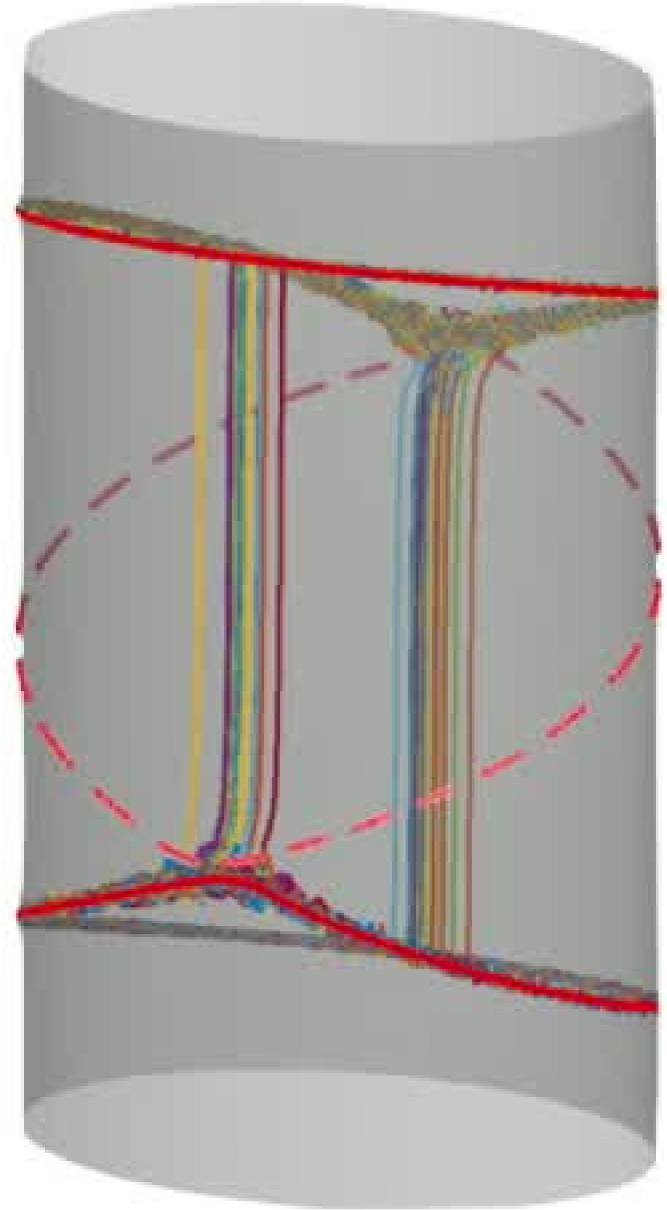
Bifurcations with Noise?



Stochastic Resonance

Predictable tipping every period

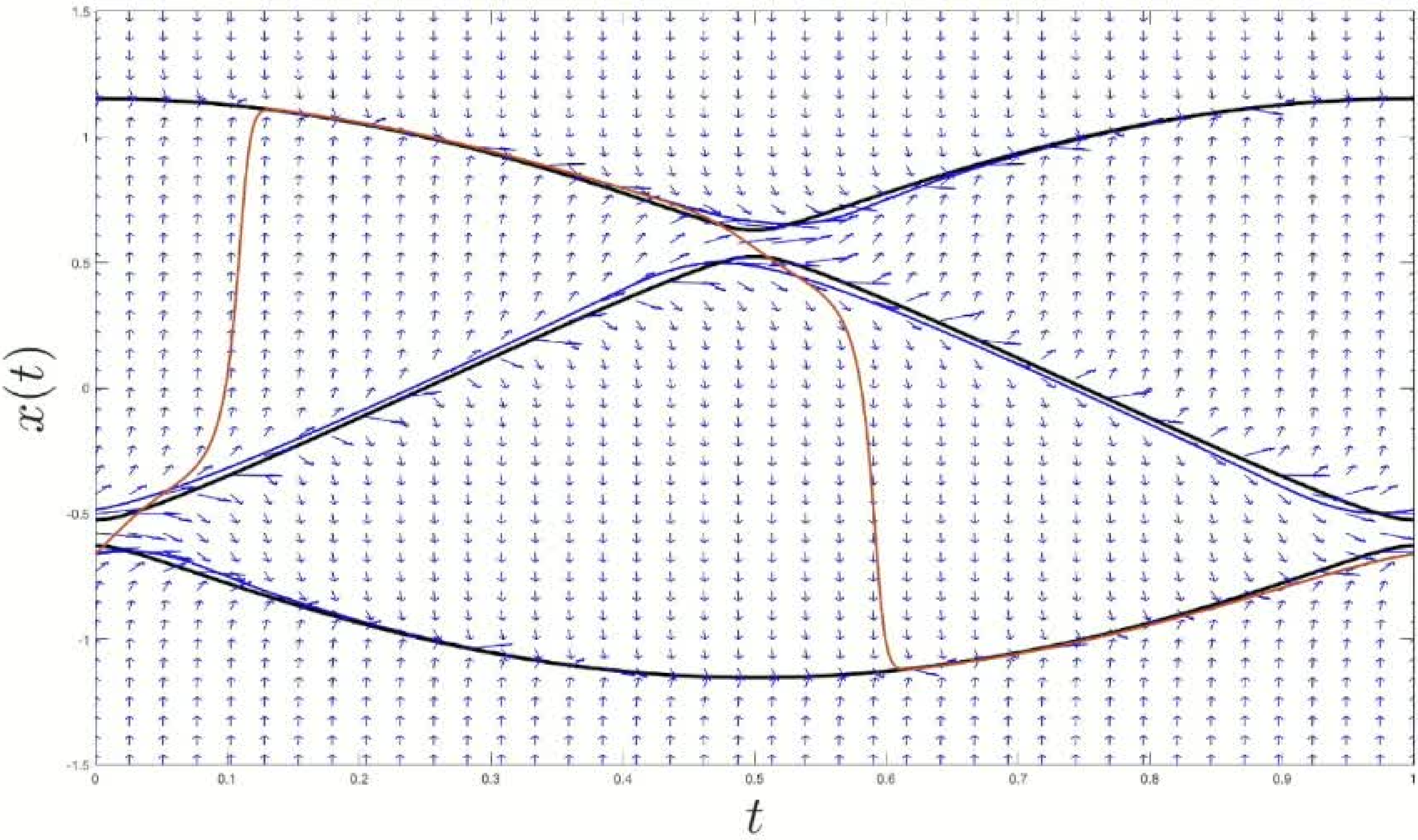




Stochastic Resonance

Predictable tipping every period

A=0.38, α =0, ϵ =0.01, σ =0.1



Stochastic Resonance

$$I_\sigma[x(t)] := \int_{t_0}^{t_f} \left(\dot{x} + \varepsilon^{-1}\bar{V}_x(x(t),t)\right)^2 dt - \frac{\sigma^2}{\varepsilon}\int_{t_0}^{t_f} \bar{V}_{xx}\left(x(t),t\right) dt.$$

Theorem:

$$\lim_{\varepsilon\to 0}\min_{x\in\mathcal{A}} \varepsilon I_\sigma[x(t)] = \min_{x\in BV} I_*[x(t)]$$

$$I_*[x(t)] := \begin{cases} 4\#V - \sigma^2 \int_{t_0}^{t_f} \bar{V}_{xx}(x(t),t)\, dt, & \bar{V}_x(x(t),t) = 0 \text{ a.e.} \\ \infty, & \text{o.w.} \end{cases}$$

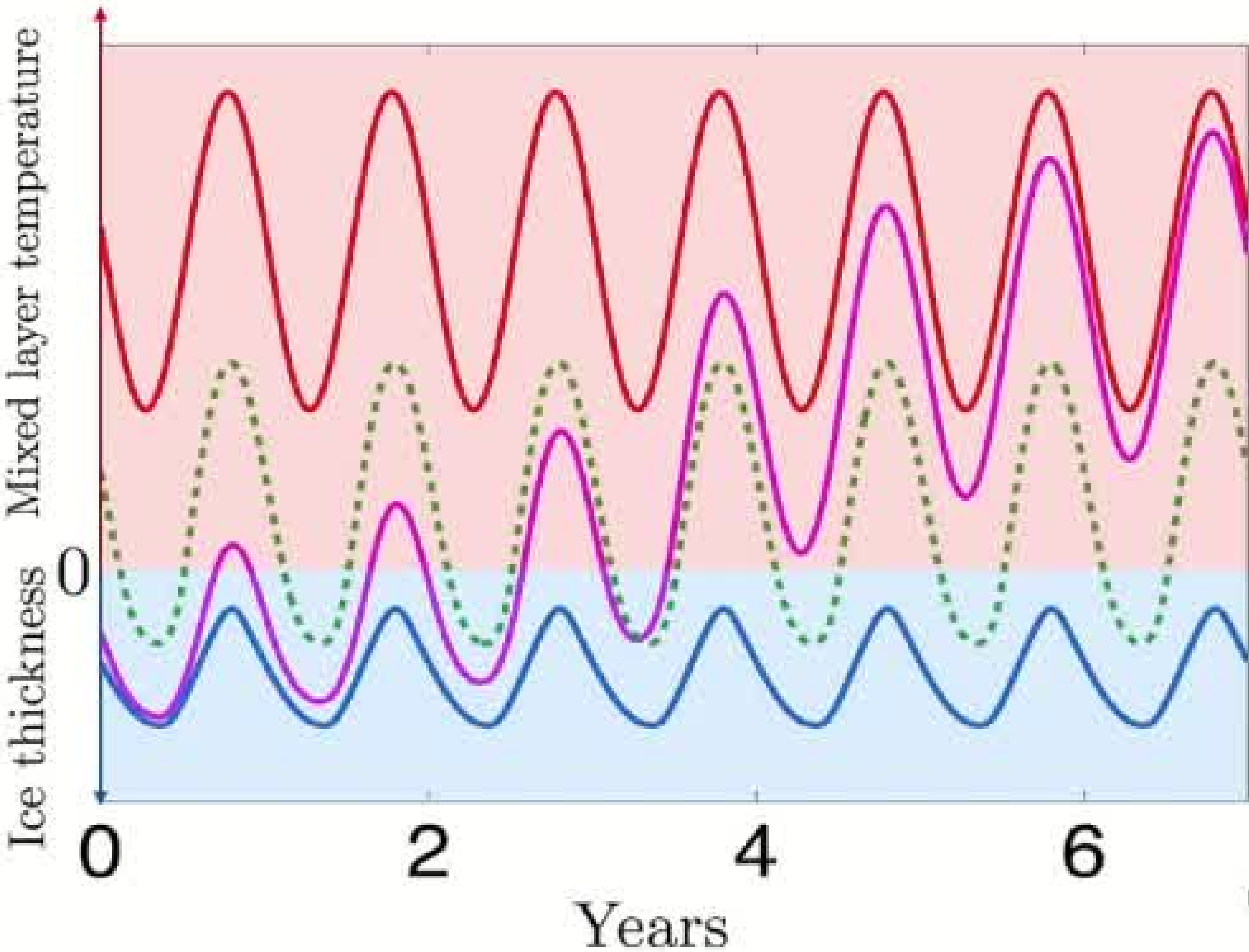
Proof Ideas:

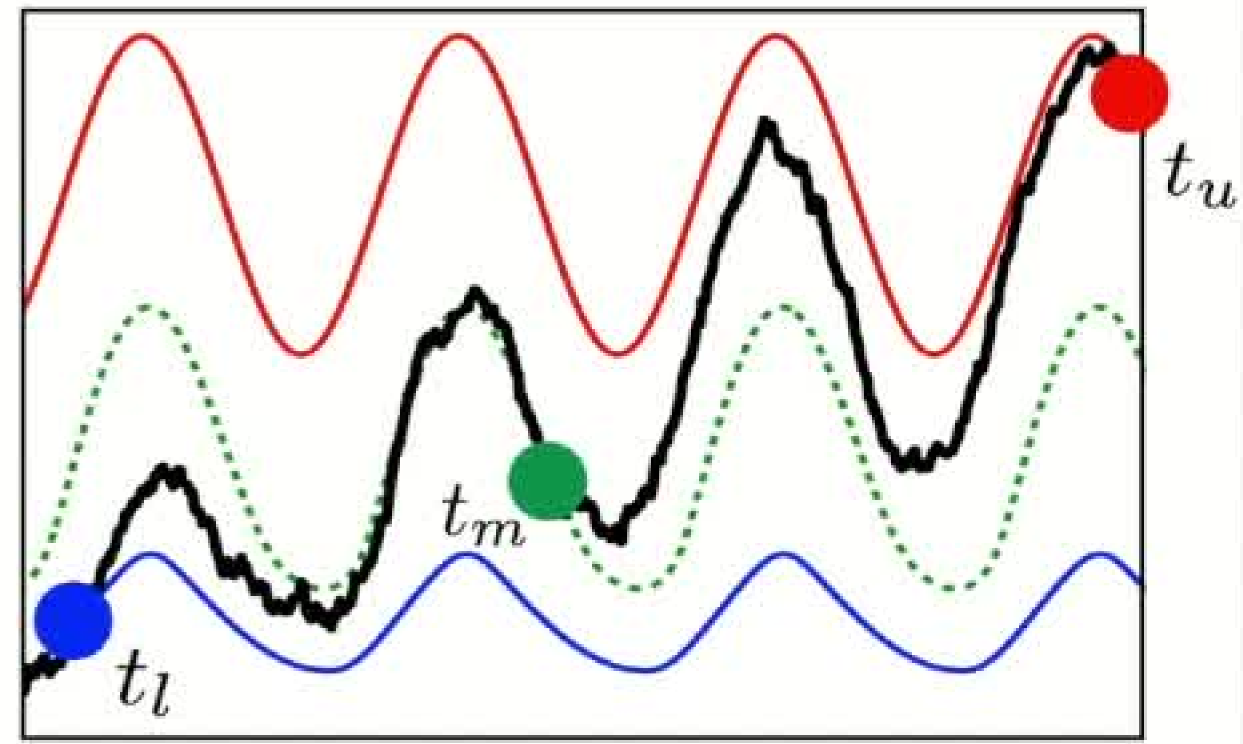
1. Lower bound: $I_\sigma[x(t)] \geq 4\#V - \dfrac{\sigma^2}{\varepsilon}\displaystyle\int_{t_0}^{t_f} \bar{V}_{xx}\left(x(t),t\right) dt.$

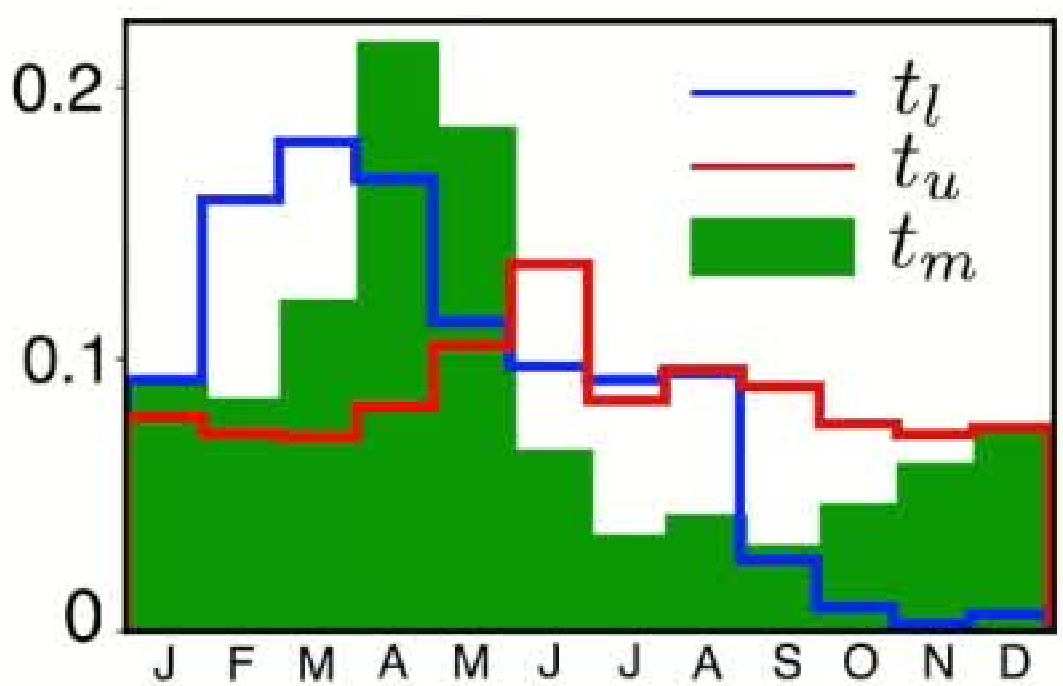
2. Lower bound is achieved "going against the flow".

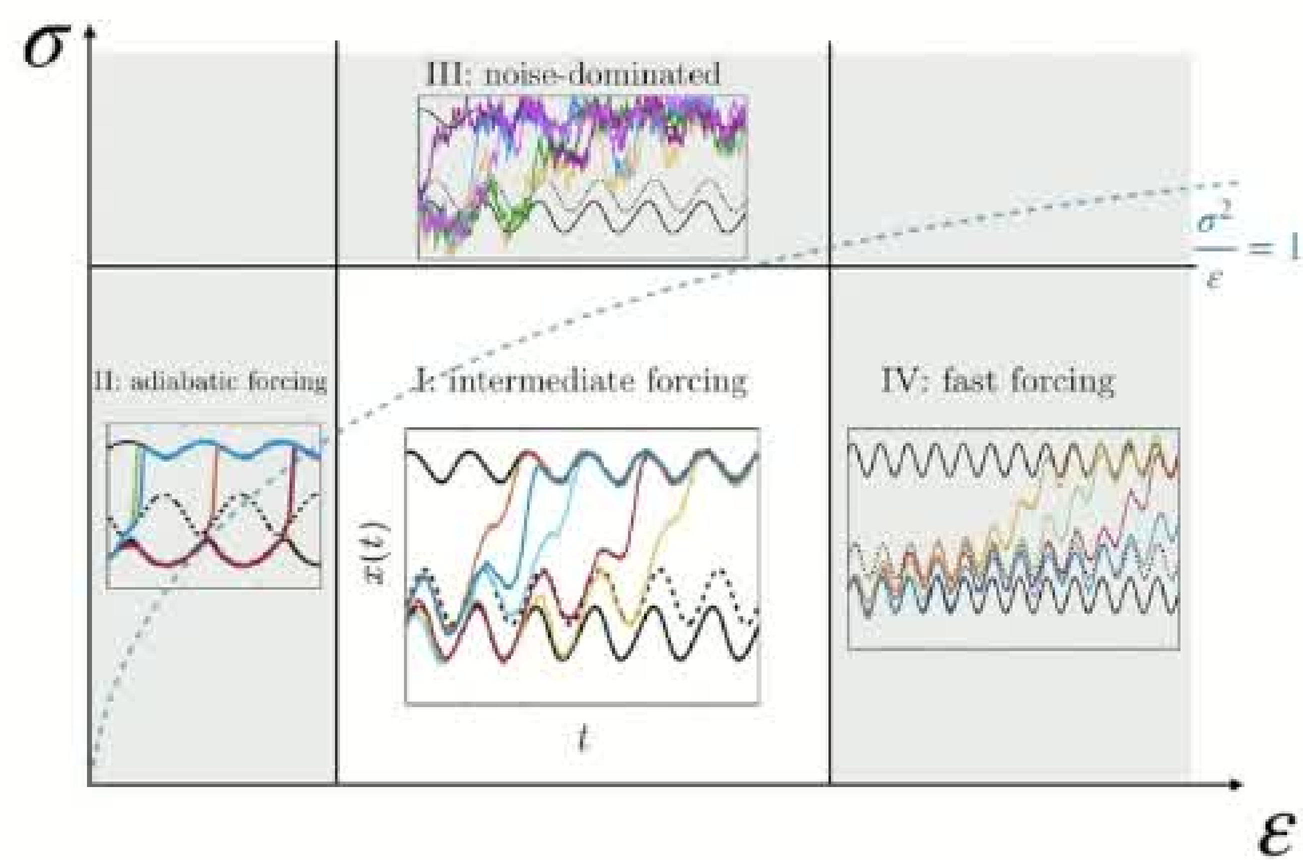
3. Use geometric singular perturbation theory to prove going against flow can be obtained with the slow manifolds: $\overline{V}_x = 0$

Return to Sea Ice

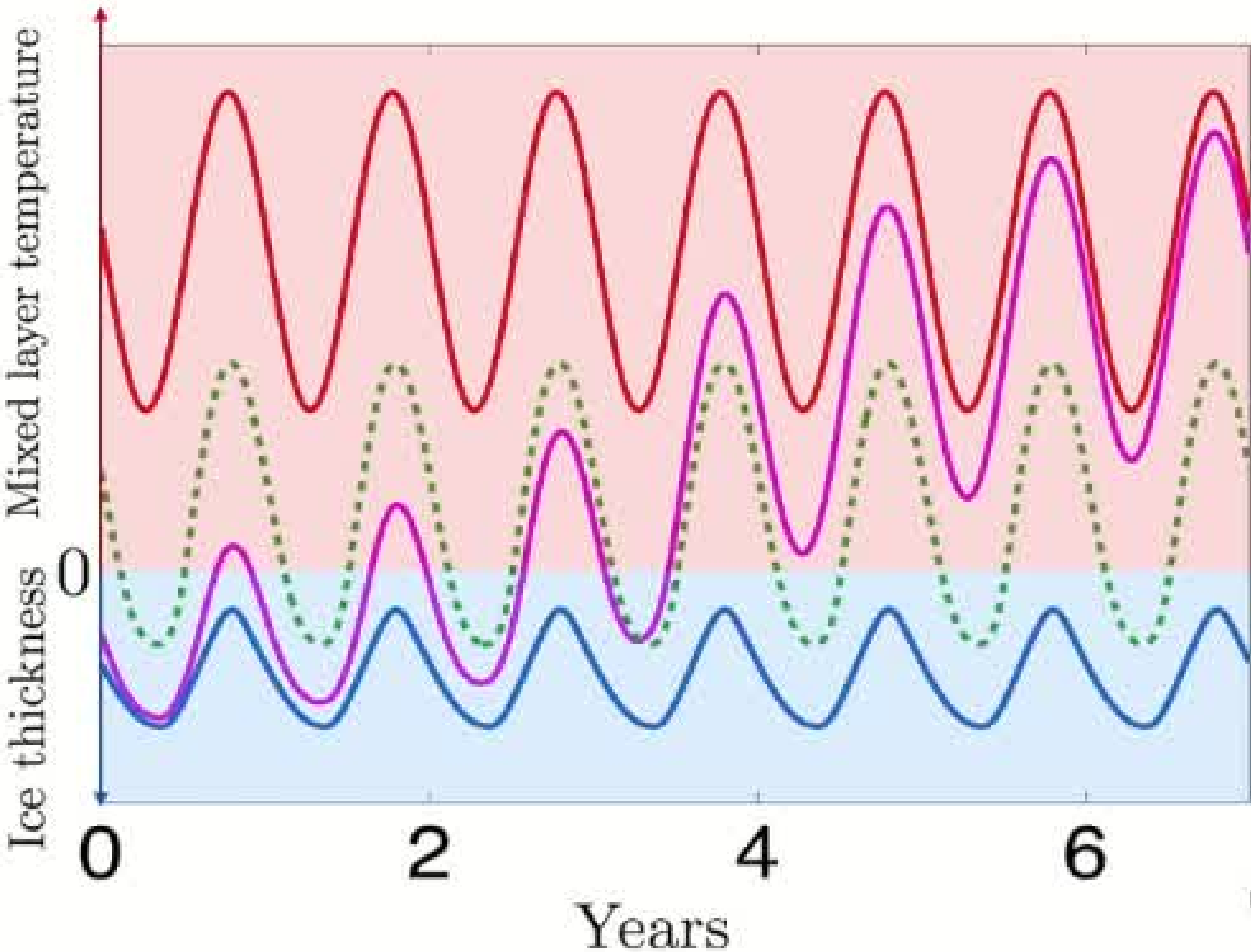


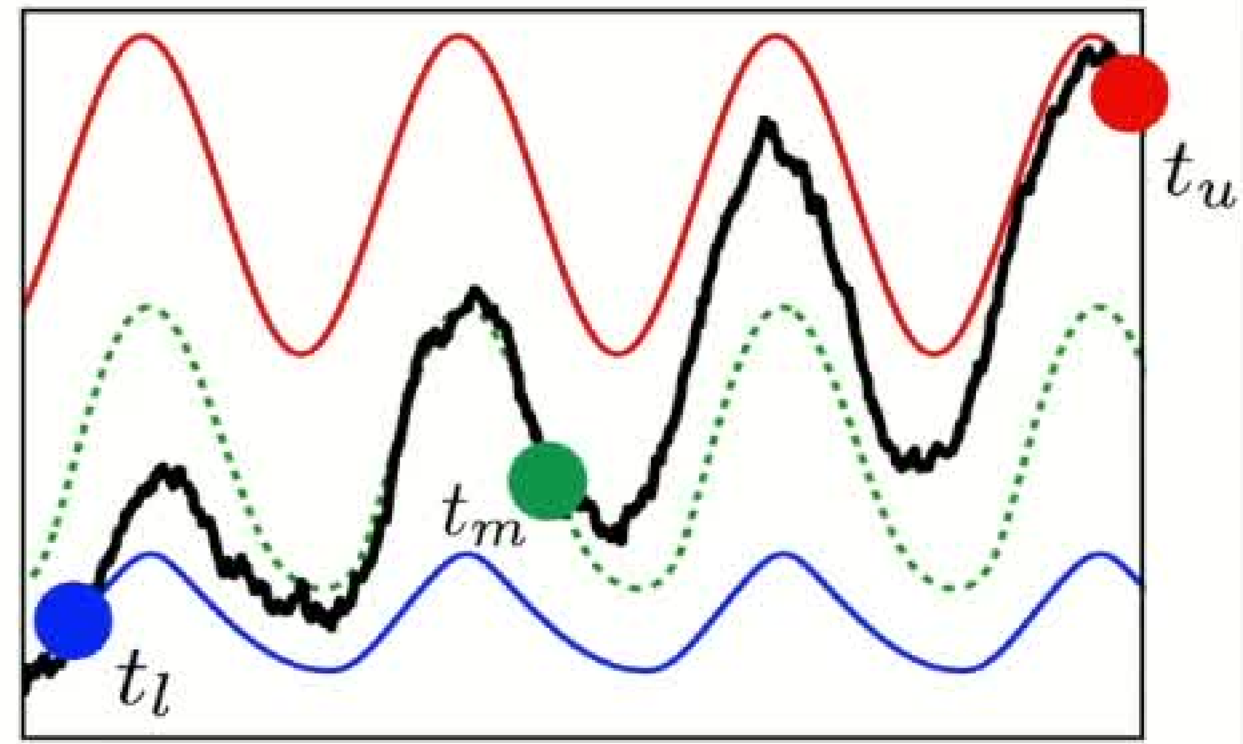


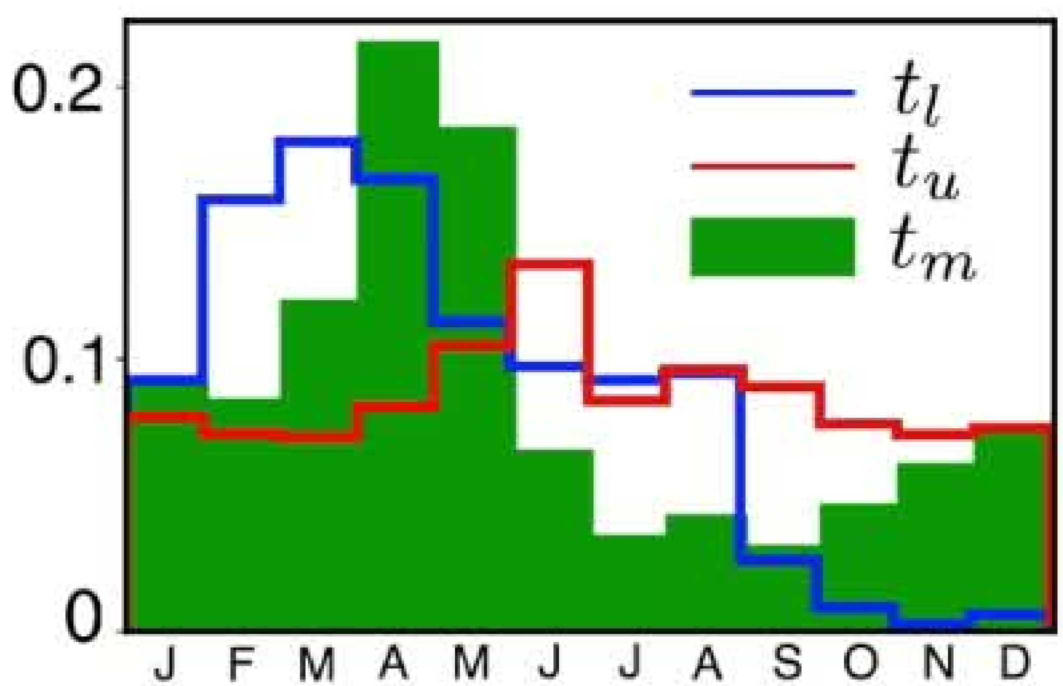


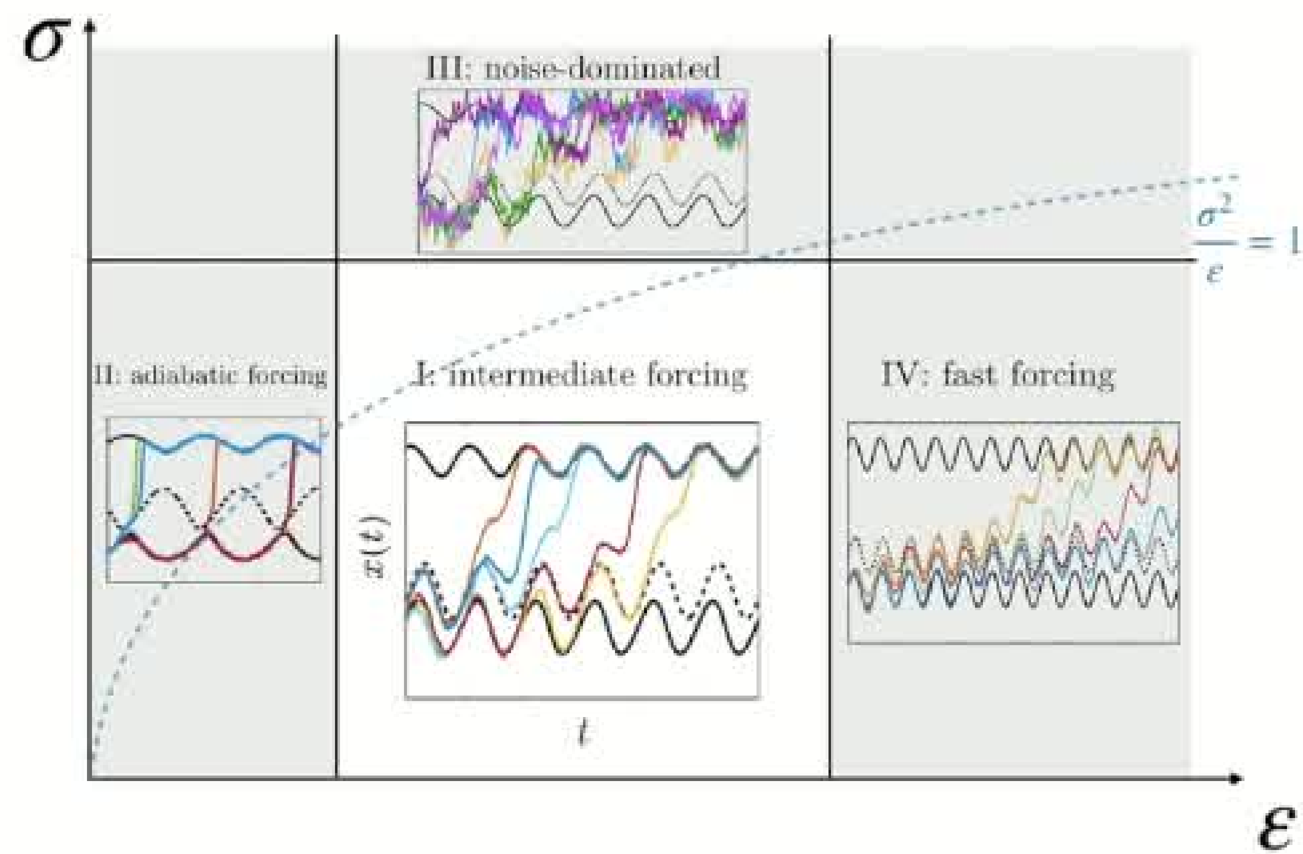


Return to Sea Ice

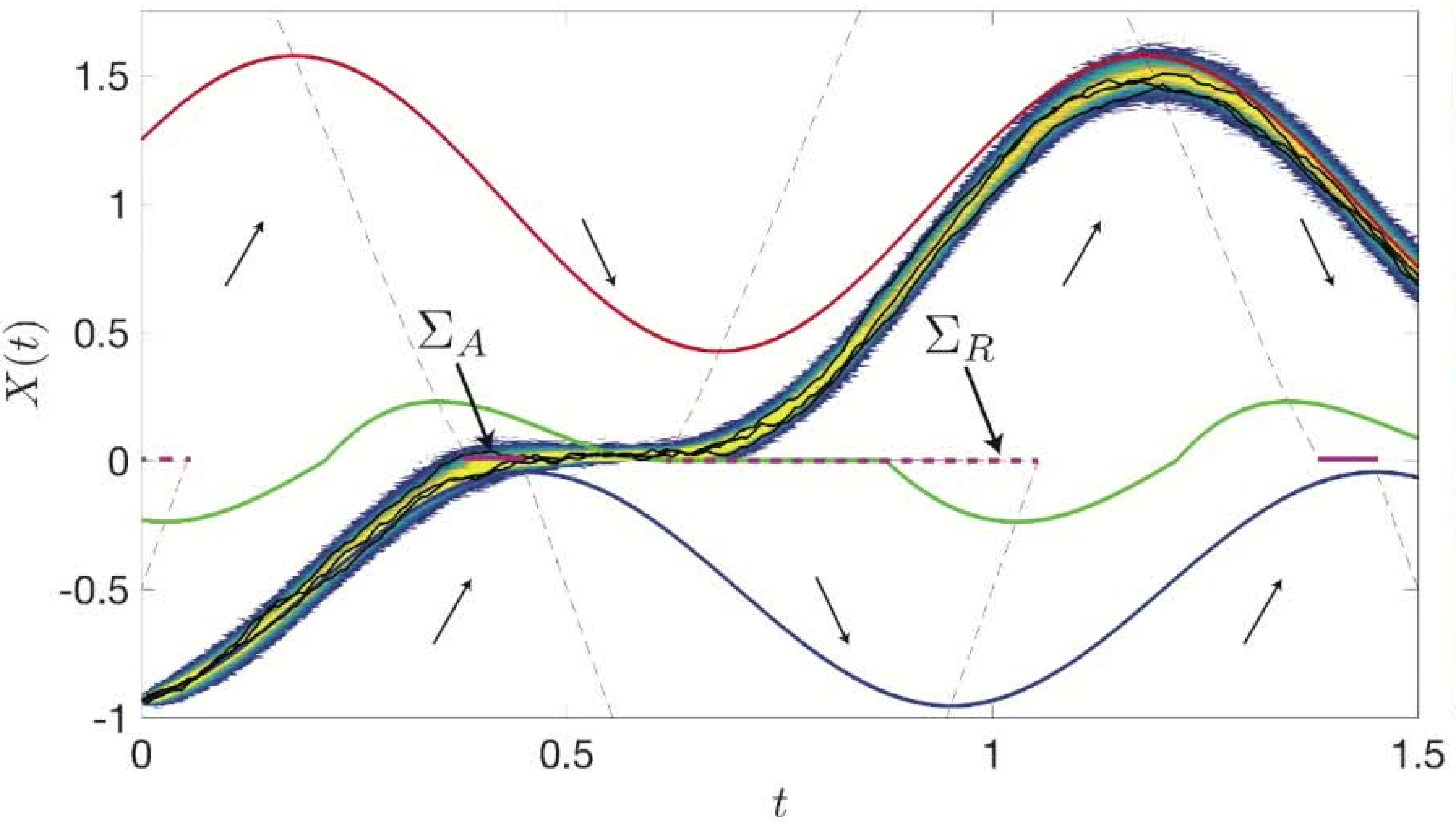








Tipping Across Discontinuity



$$dx = \begin{cases} [-r_+(x-1) + A_+ \cos(2\pi t)]dt + \sigma dW & \text{if } x > 0 \\ [-r_-(x-x_0) + A_- \cos(2\pi(t-p))]dt + \sigma dW & \text{if } x < 0 \end{cases}$$