# Sparse Dynamics and Sensor Placement in Complex Systems


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Cray 1,
160 Megaflops, 1976

Moore's Law


MIRA Blue Gene/Q
8.1 Petaflops, 2013

Corollary

iPhone
~1 Gigaflop, 2013

## Brute Force is not enough!

Dimensionality Reduction (patterns exist)

Sparsity (efficient measurement)

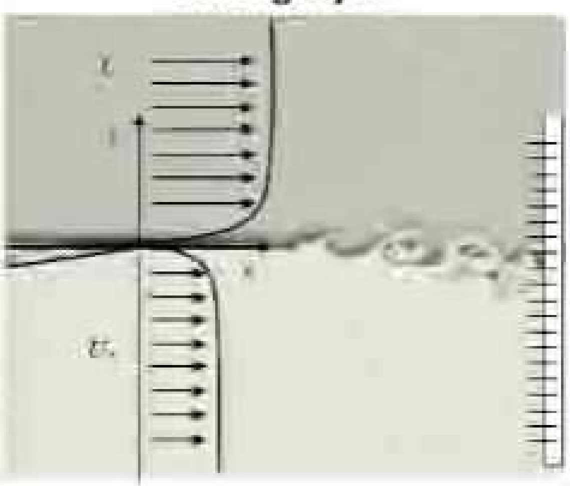
Feedback Control

## Actuator/sensor placement is NP-Hard problem

# Example 2: (Optimal) Sparse Sensors

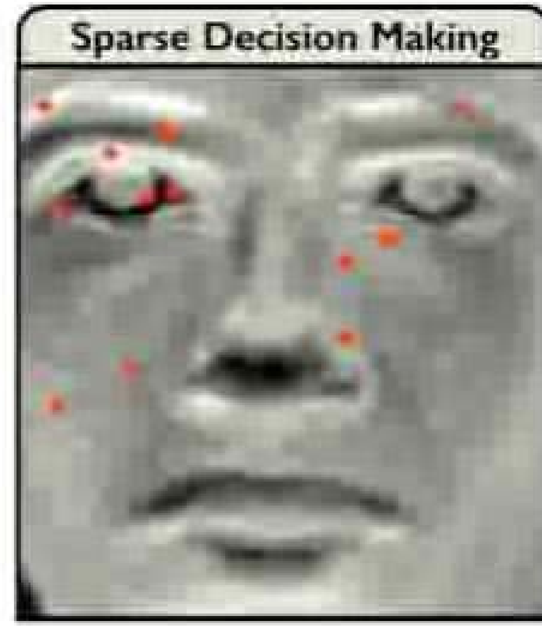
Given a fixed budget of sensors, where should they be placed to optimally inform decision-making?

**mixing layer**



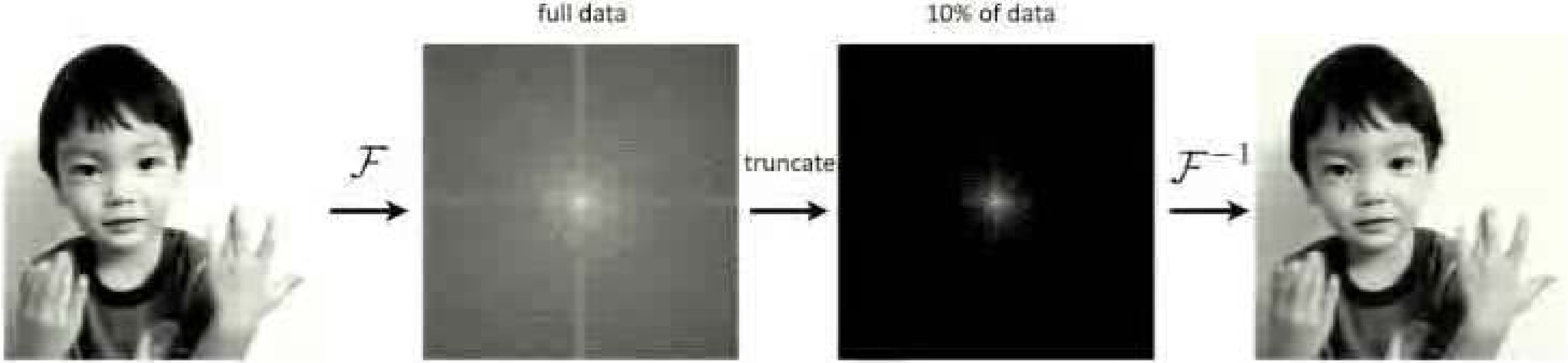
**unsteady aerodynamics**


sensors mapped by Brad Dickerson




- **Brunton, <u>Brunton</u>, Proctor, Kutz,** ***in review in IEEE PAMI*****, 2013.**
- **Brunton, <u>Brunton</u>, Proctor, Kutz,** ***provisional patent filed*****, 2013.**

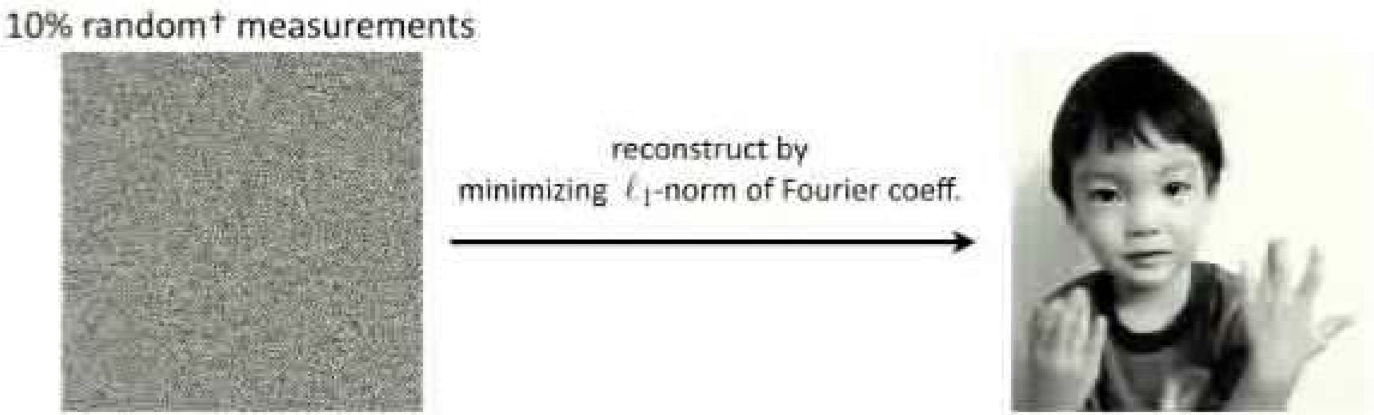
# Compression vs Compressed Sensing




Compression

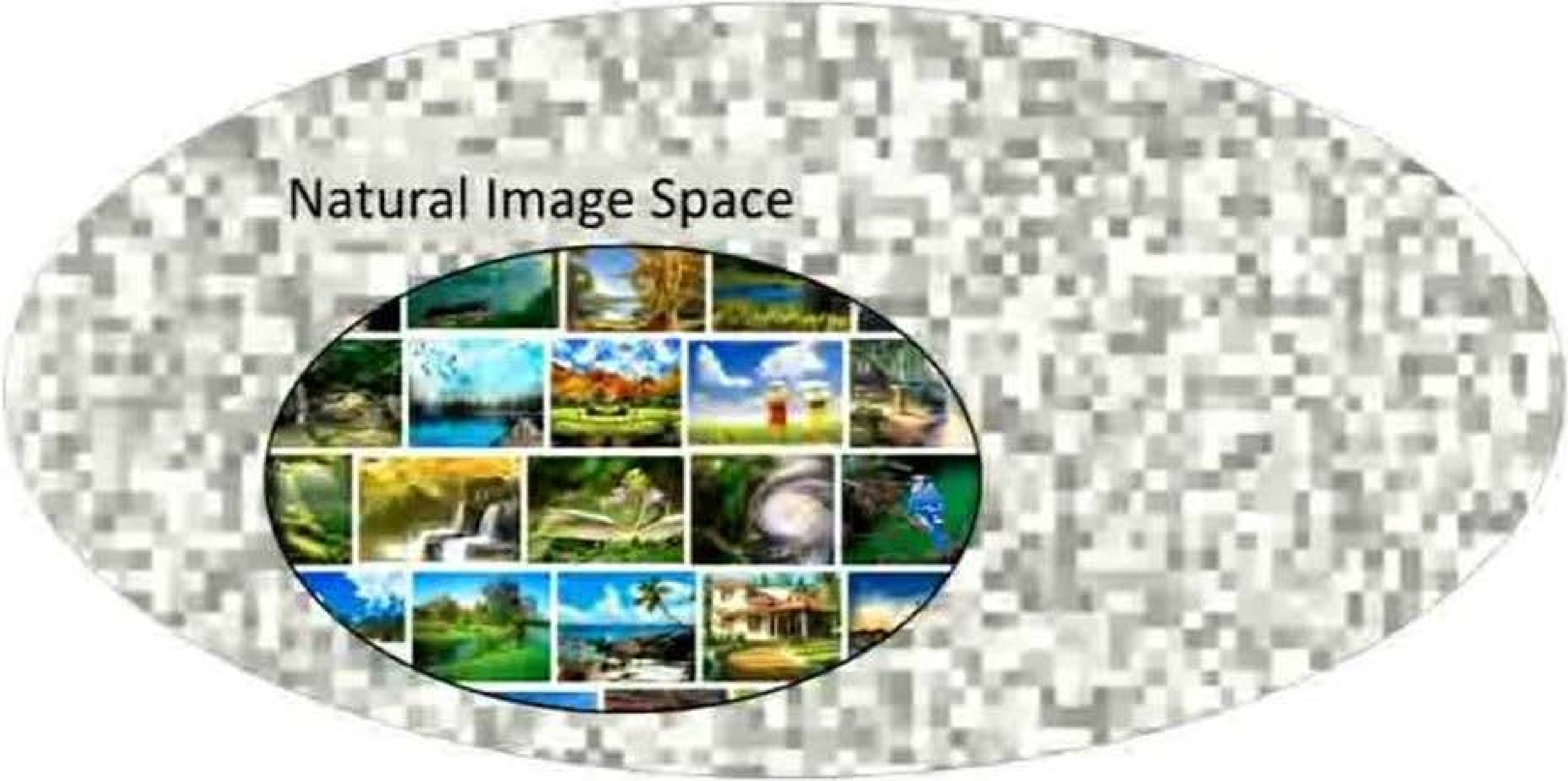
vs

Compressive Sensing

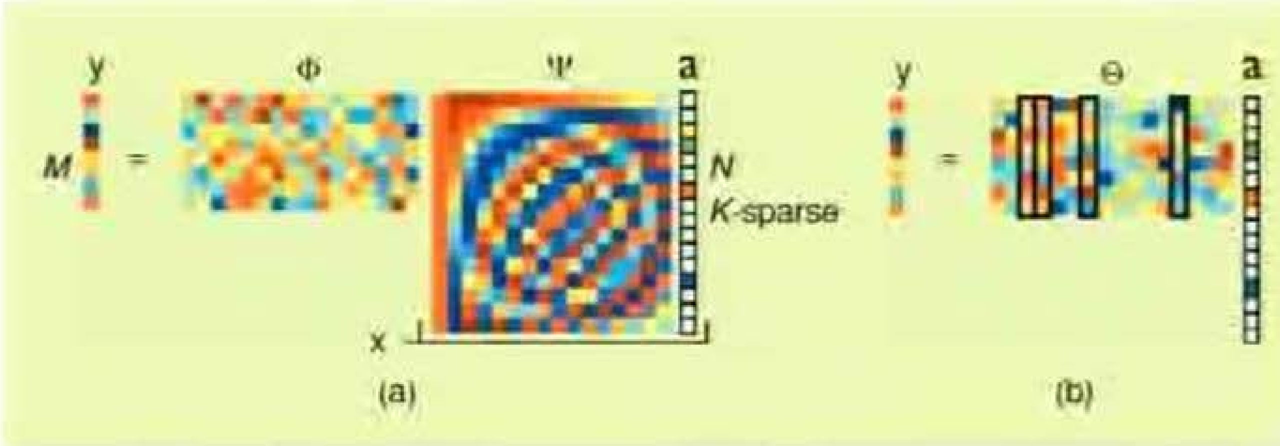



† subject to some specific constraints

# Pixel Space is (larger than) Astronomical



# Reconstruction by Compressive Sensing




from Baraniuk, 2007.

IN: low-dimensional measurements

random measurement matrix

OUT: sparse coefficients

$$y = \Phi x = \Phi \Psi a = \Theta a$$

transform basis

IMPORTANT: measurement matrix must be **incoherent** with respect to the transform basis

**To reconstruct:**

$$\text{minimize } ||a||_1,$$
$$\text{such that } y = \Theta a$$

Proofs by:

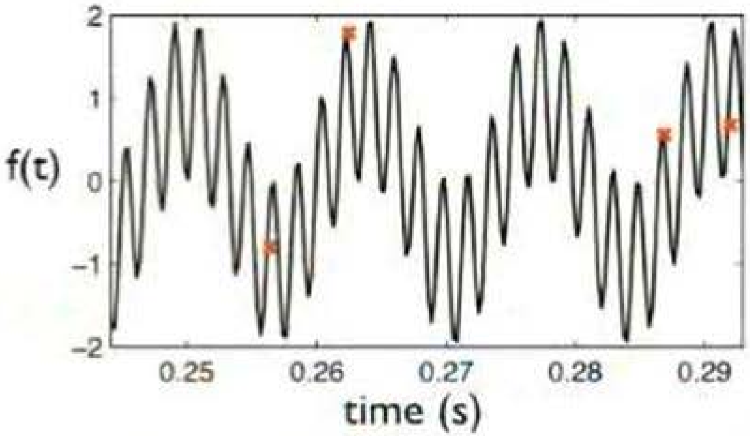
- Candès, Romberg & Tao, 2006.
- Donoho, 2006.

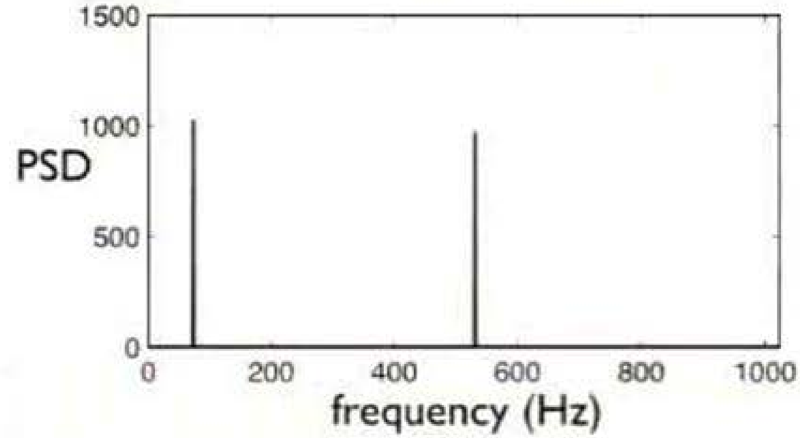
# Beating Nyquist

$$f(t) = \sin(73 \times 2\pi t) + \sin(531 \times 2\pi t)$$

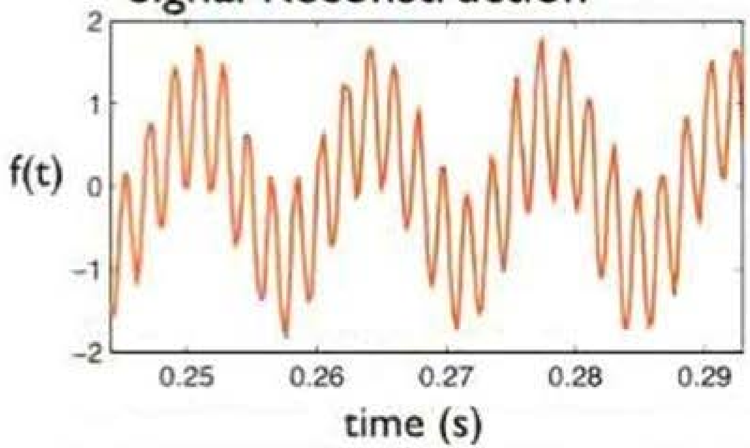
Nyquist: 1062 samples/second

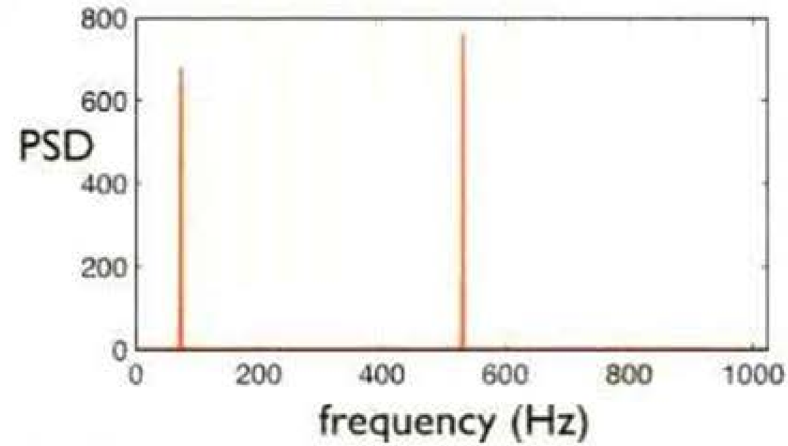
Compressive Sampling: 128 samples/second



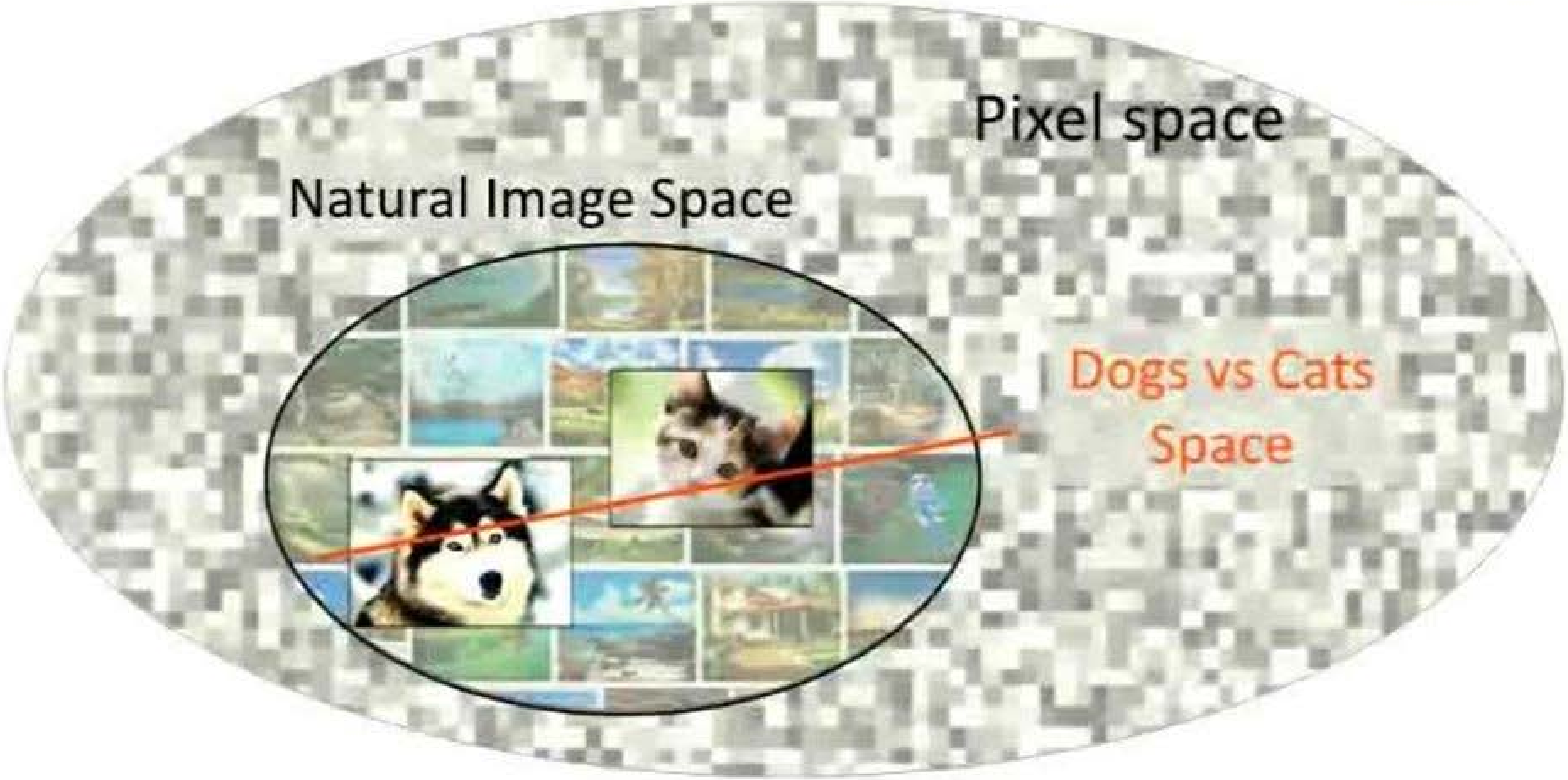




Signal Reconstruction



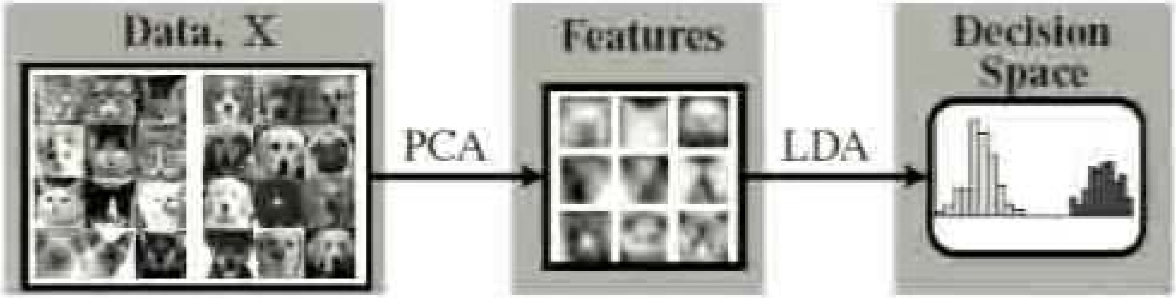
# Enhanced Sparsity




**Enhanced sparsity** for decision-making:
massive reduction in number of measurements
required for classification over reconstruction

Brunton, Brunton, Proctor, Kutz, *in review in IEEE PAMI*, 2013.

# Decision Making (Classification)




**Step 1: Get feature space from data**



$$\mathbf{X} = \Psi \Sigma \mathbf{V}^*$$

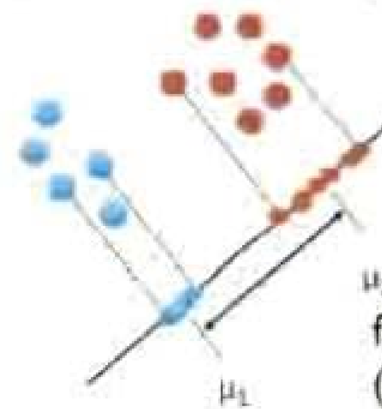
Singular-value Decomposition

$$\Psi_r^T : \mathbb{R}^n \rightarrow \mathbb{R}^r,$$

$$\mathbf{x} \mapsto \mathbf{a}$$

n > 10,000
r ~ 10

**Step 2: Find vector in feature space that optimally separates categories**

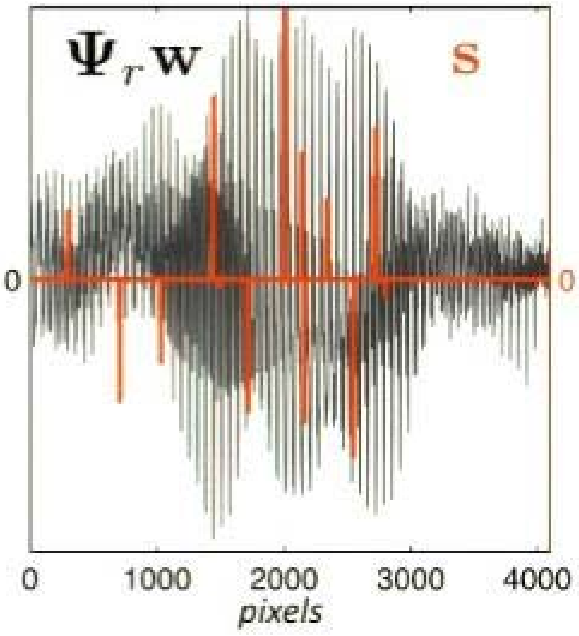


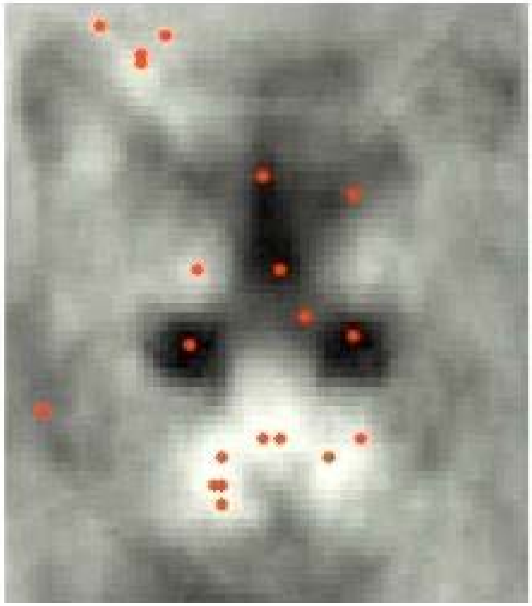

from Kutz text (chap. 17)

$$\mathbf{w}^T : \mathbb{R}^r \rightarrow \mathbb{R}^{c-1},$$

$$\mathbf{a} \mapsto \eta$$

# Sparse Sensor Locations



$\Psi_r \mathbf{w}$

$\mathbf{s}$

from image to decision:

$$\eta = (\Psi_r \mathbf{w})^T \mathbf{x}$$

Image has *n* pixels

$\Psi_r$ feature basis

$\mathbf{w}$ decision vector

$\mathbf{s}$ sparse sensors

To solve for sparse sensor locations,

$$\mathbf{s} = \underset{\mathbf{s}'}{\text{argmin}} \, ||\mathbf{s}'||_1, \quad \text{subject to } \Psi_r^T \mathbf{s}' = \mathbf{w}.$$

**s** is mostly zeros; the non-zero elements correspond to sensor locations, where we want to measure.

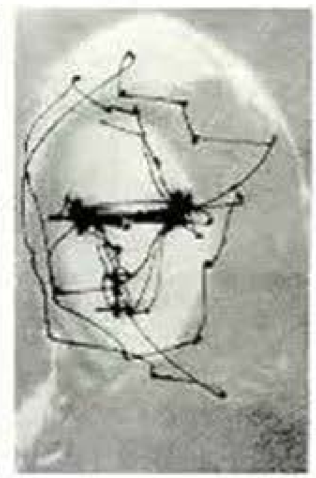
**Brunton, <u>Brunton</u>, Proctor, Kutz, *in review in IEEE PAMI*, 2013.**

# Eye Tracking Results

ensemble of sensor locations

image

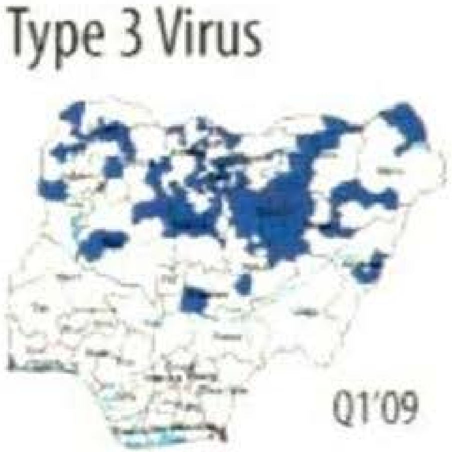
human gaze



Yarbus, 1967

# Not Just Useful for Images

*an Australian DART buoy*




*incidence of polio in Nigeria*

*a NOAA weather station*

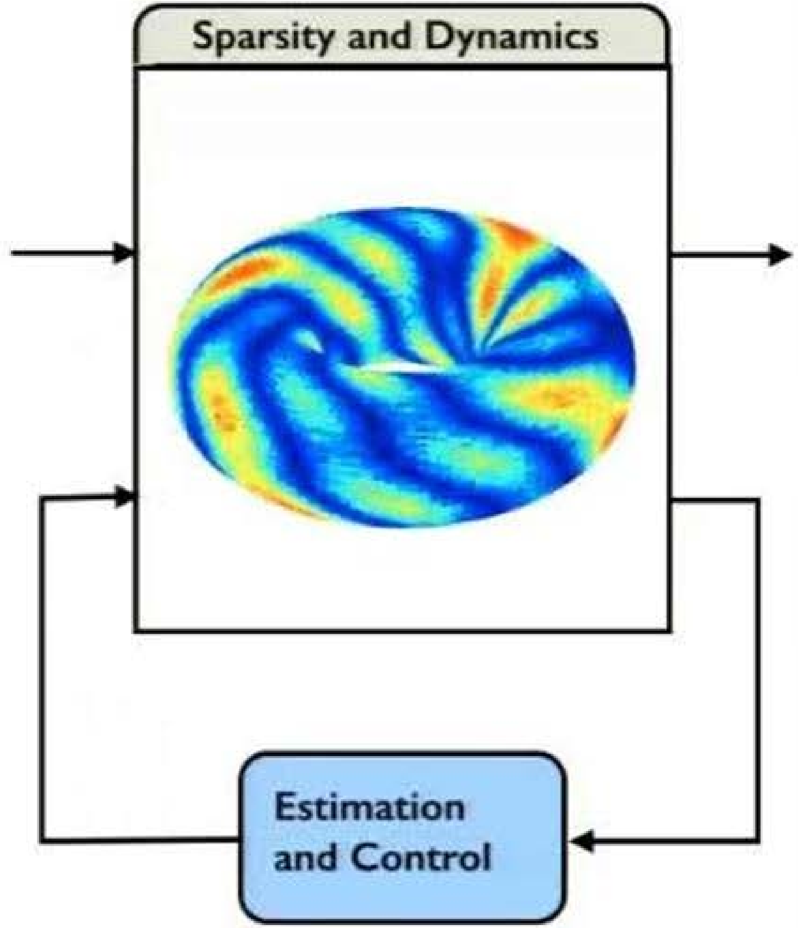
*sensor networks on organisms*

# Example 3: Sparse Sampling Dynamics

Extend sparse sampling
to systems with dynamics

Dynamic mode decomposition (DMD)

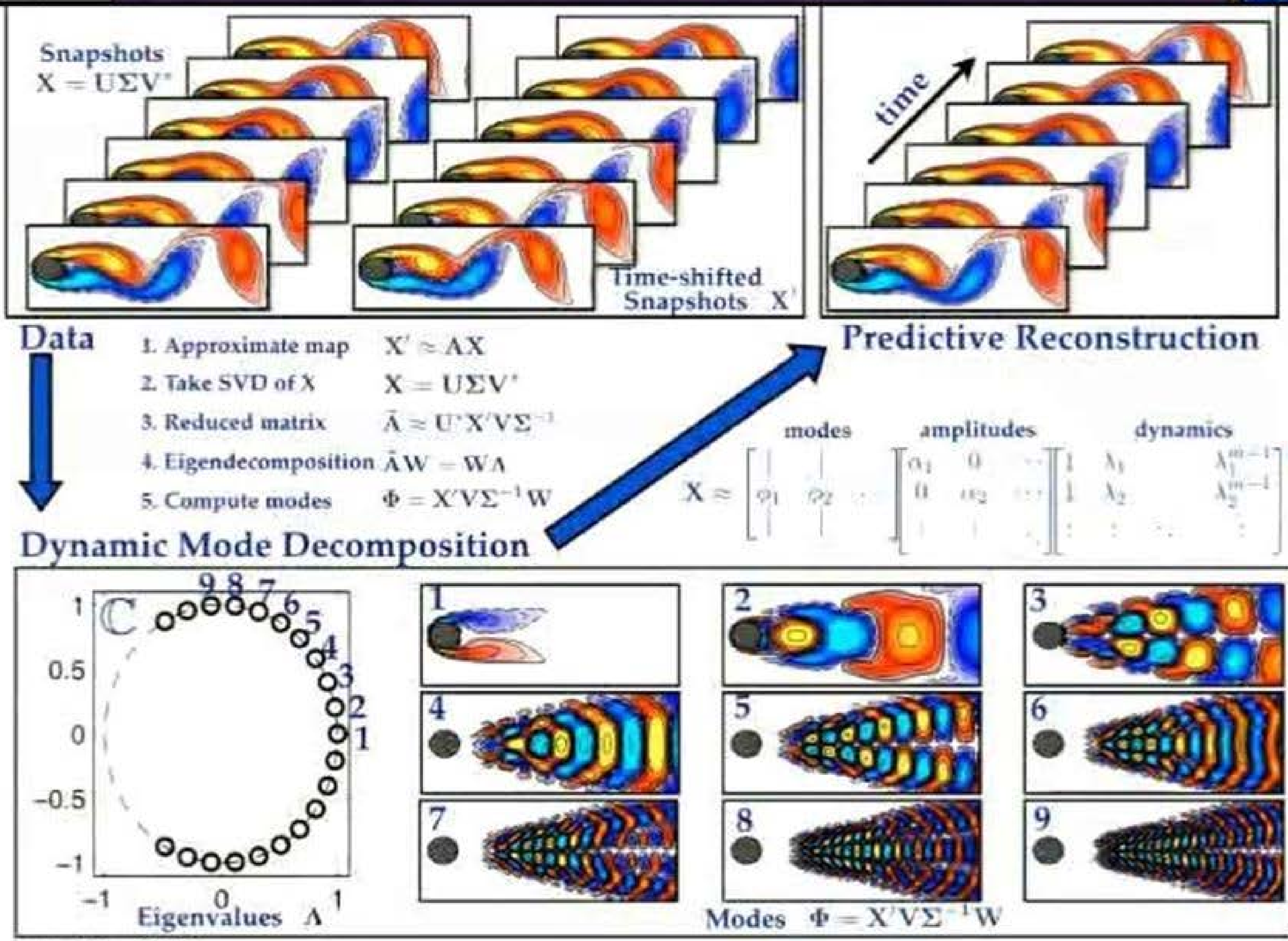
- dimensionality reduction
- equation free





- Brunton, Proctor, Tu, Kutz, *submitted to JCD*, 2013.
- Brunton, Tu, Bright Kutz, *SIADS*, 2014.
- Tu, Rowley, Luchtenburg, Brunton, Kutz, *JCD*, 2014.
- Proctor, Brunton, Kutz, *arxiv*, 2014.

# Dynamic Mode Decomposition




Rowley *et al.*, *JFM*, 2009.
Schmid, *JFM*, 2010.
Tu, Rowley, Luchtenburg, Brunton, Kutz, *JCD, 2014*

Equation-free method!

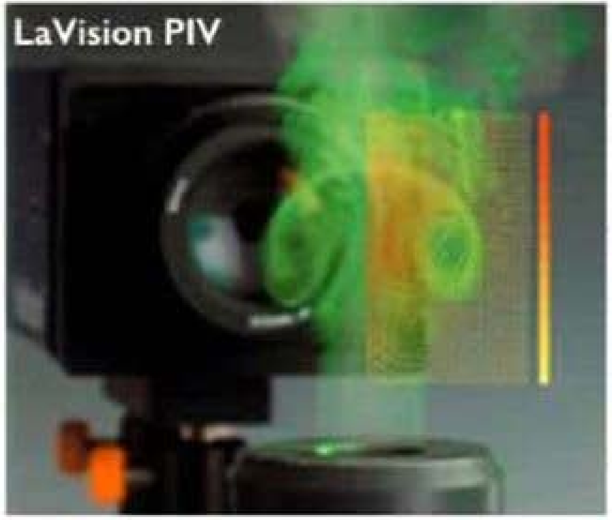
**Extracts spatial-temporal coherent structures**

# Dynamic Mode Decomposition

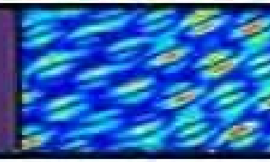
Goal: Reconstruct full-state DMD modes from sparse spatial measurements.

In many applications, sensors are expensive

In particle image velocimetry (PIV), data acquisition is limited by bandwidth.




Brunton, Proctor, Tu, Kutz, *submitted to JCD*, 2013.



# DMD on Sparse dynamical system

**Case 0: Exact DMD on full data**

$$\mathbf{X} = \mathbf{U_X} \boldsymbol{\Sigma}_\mathbf{X} \mathbf{V}_\mathbf{X}^*$$

$$\tilde{\mathbf{A}}_\mathbf{X} = \mathbf{U_X}^* \mathbf{X}' \mathbf{V_X} \boldsymbol{\Sigma}_\mathbf{X}^{-1}$$

$$\tilde{\mathbf{A}}_\mathbf{X} \mathbf{W_X} = \mathbf{W_X} \boldsymbol{\Lambda}_\mathbf{X}$$

$$\boldsymbol{\Phi}_\mathbf{X} = \mathbf{X}' \mathbf{V_X} \boldsymbol{\Sigma}_\mathbf{X}^{-1} \mathbf{W_X}$$

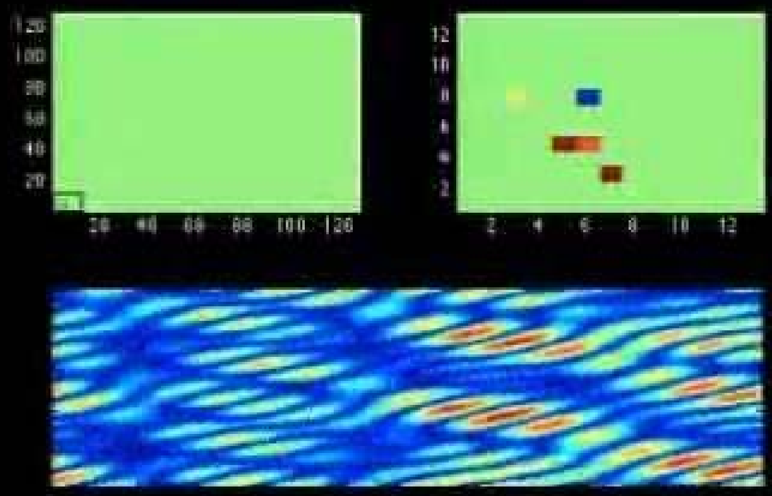
DMD eigenvalues ($\boldsymbol{\Lambda}_\mathbf{X}$)

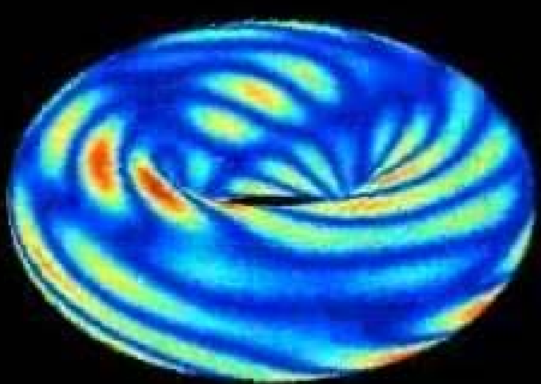
DMD modes ($\boldsymbol{\Phi}_\mathbf{X}$)

full-state snapshots

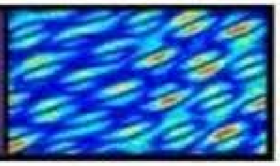
$\mathbf{X}, \mathbf{X}'$ —DMD→ $\boldsymbol{\Lambda}_\mathbf{X}, \boldsymbol{\Phi}_\mathbf{X}$

Tu, Luchtenburg, Rowley, Brunton, Kutz, *submitted to JCD*, 2013.



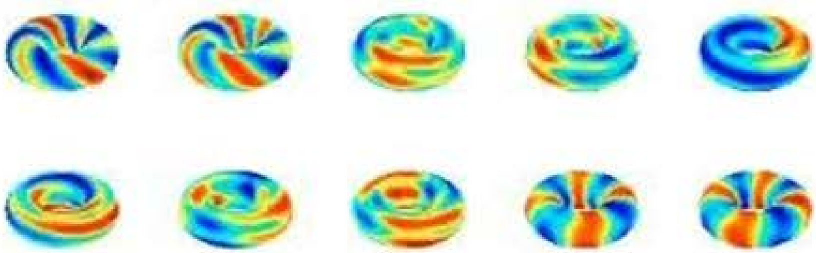
# DMD on Sparse dynamical system



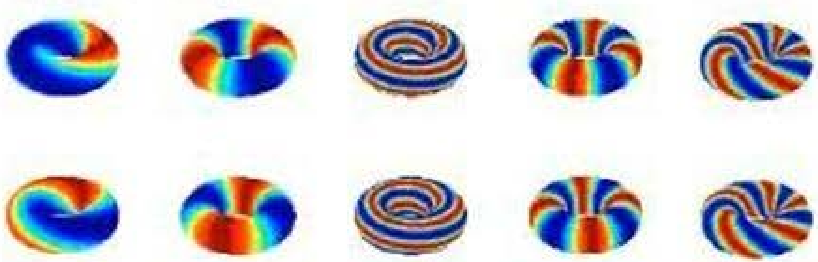
Case 0: Exact DMD on full data

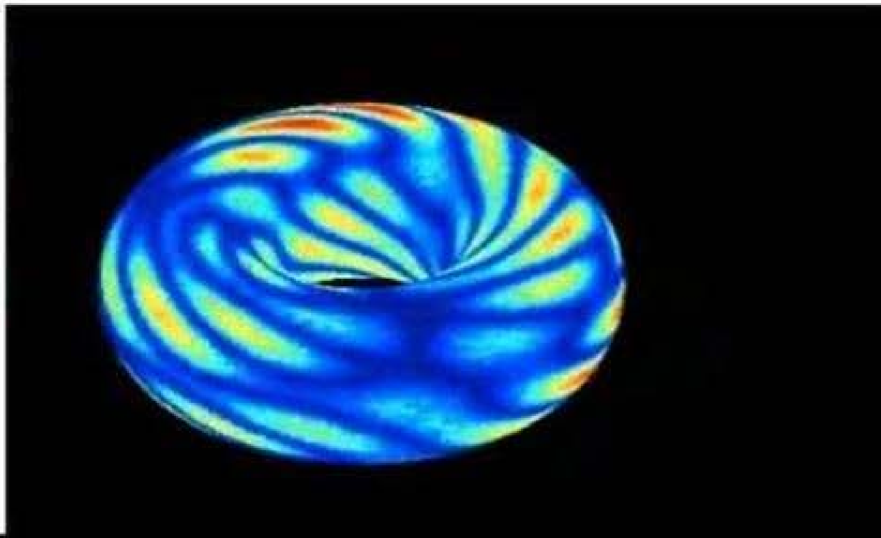

True Spatial Modes:

POD Modes:



DMD Modes:

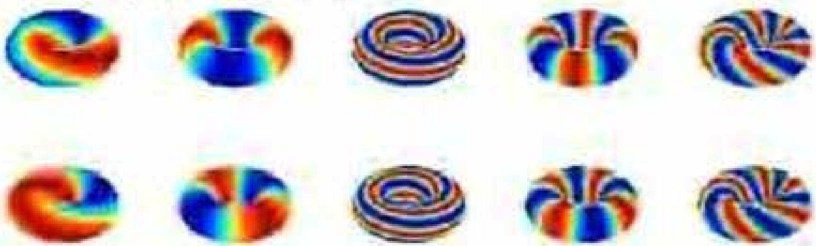




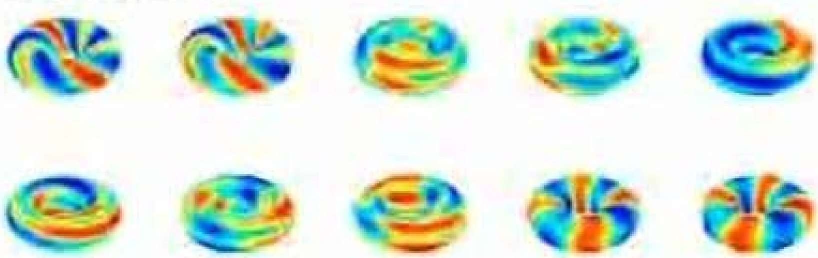
# DMD on Sparse dynamical system

Case 0: Exact DMD on full data

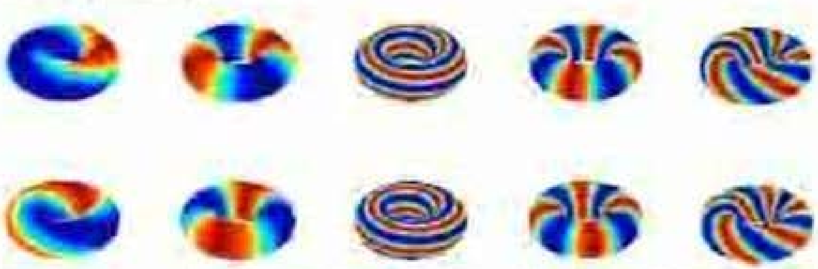
True Spatial Modes:

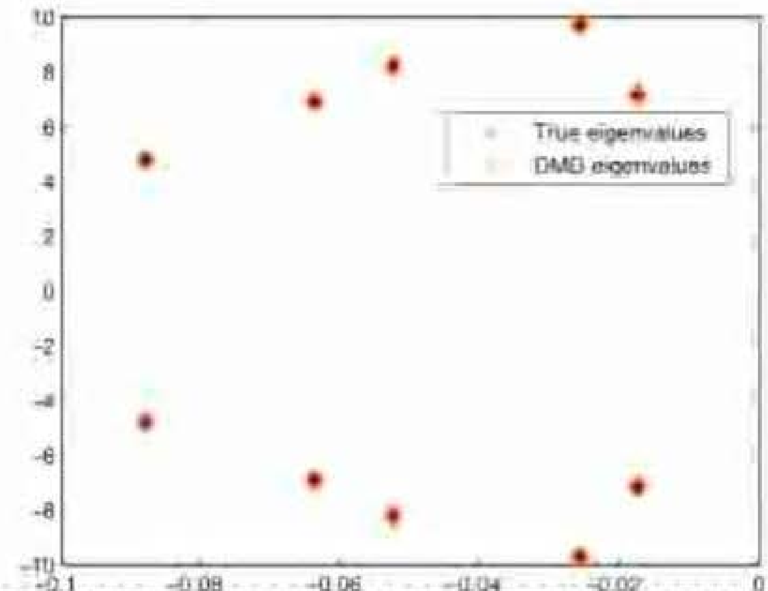


POD Modes:



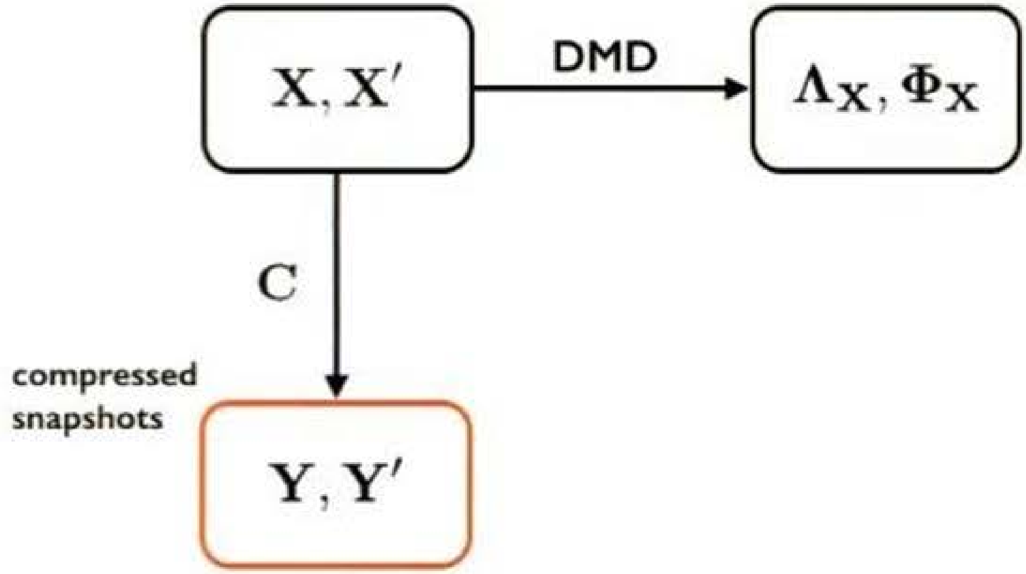
DMD Modes:

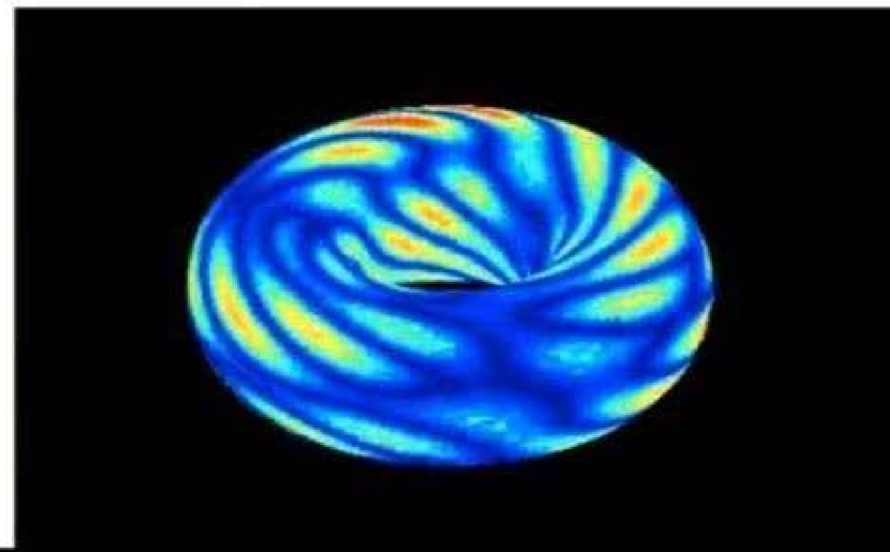


# Compressed Sensing DMD

Case 1: We start with projected data Y, Y'



Brunton, Proctor, Tu, Kutz, *submitted to JCD*, 2013.

# Compressed DMD

Case 2: We start with full-state data X, X'

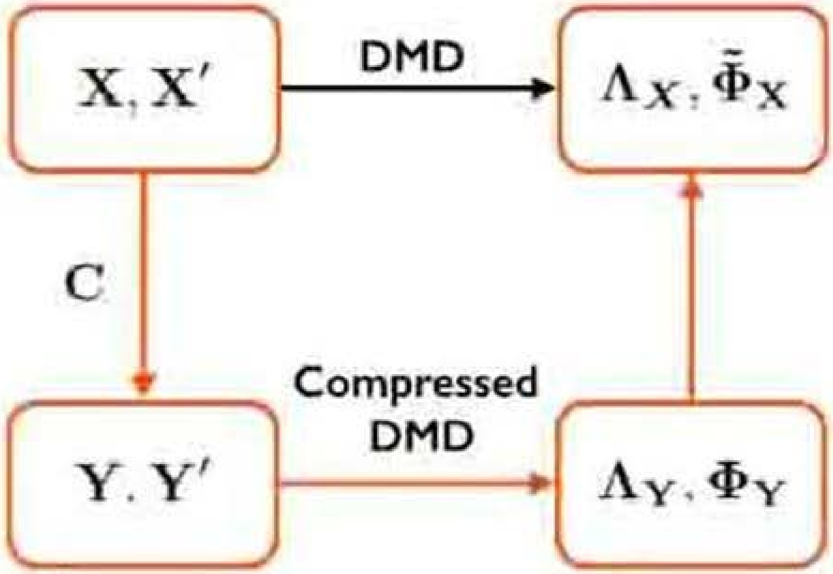
Option 3:

(i) Compress data

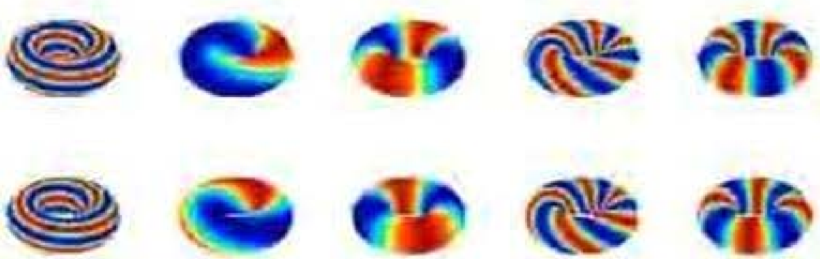
(ii) Compute compressed DMD

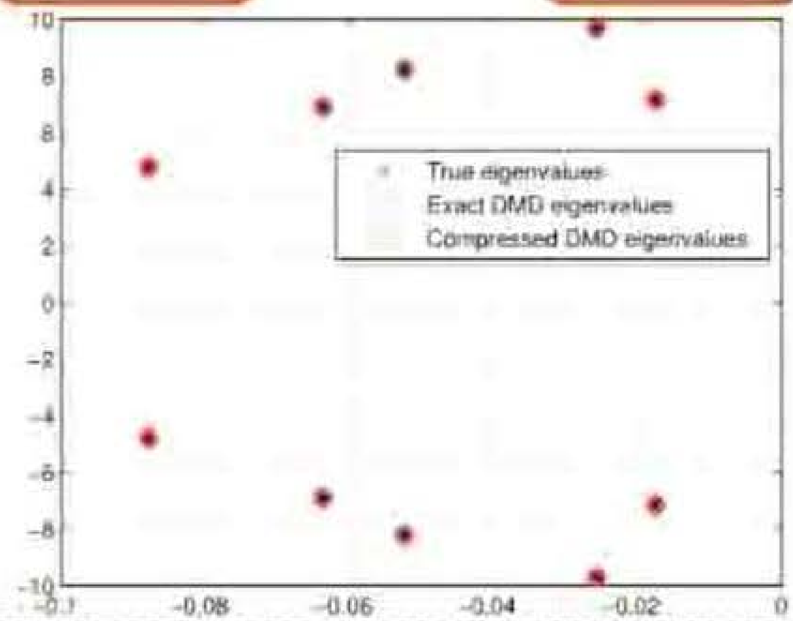
(iii) Reconstruct full-state DMD modes

$$\tilde{\Phi}_{\mathbf{X}} = \mathbf{X}'\mathbf{V}_{\mathbf{Y}}\Sigma_{\mathbf{Y}}^{-1}\mathbf{W}_{\mathbf{Y}}$$



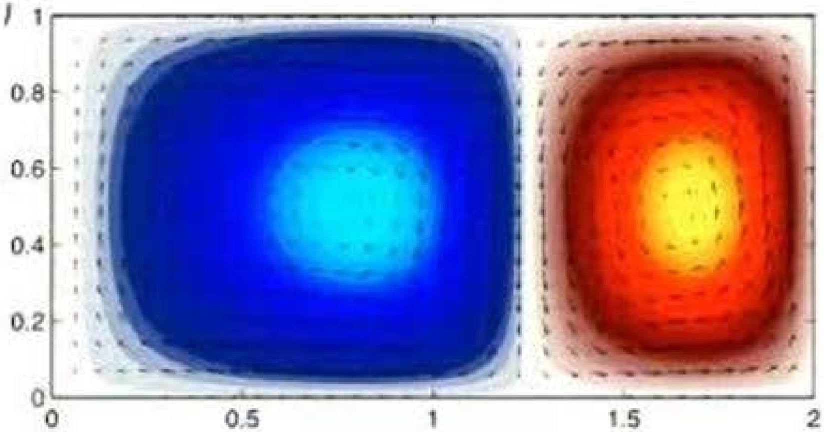

Compressed DMD Modes:



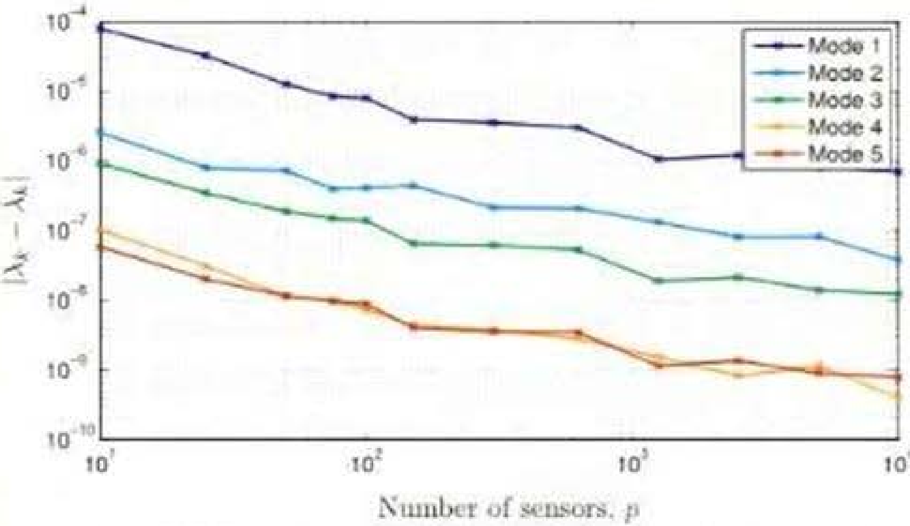


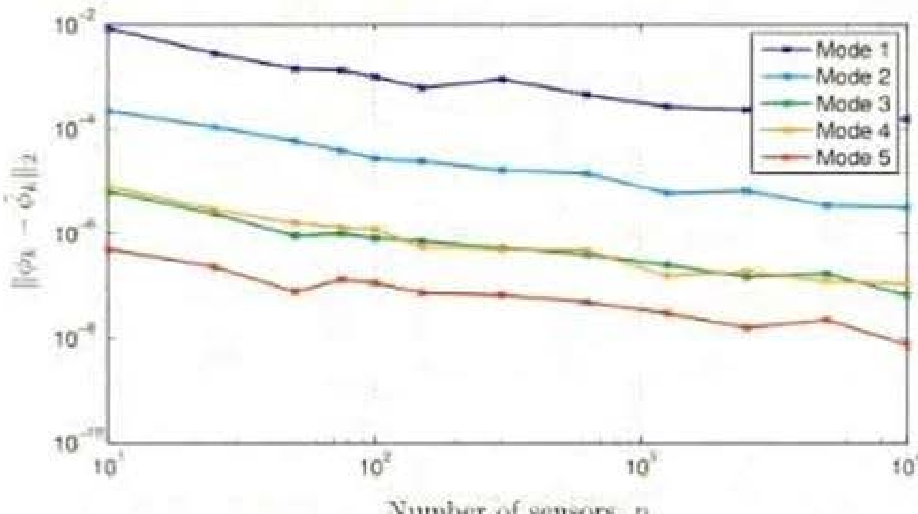

Brunton, Proctor, Tu, Kutz, *submitted to JCD*, 2013.

# Double Gyre Example




**Compressed DMD is extremely accurate, even with very few sensors!**

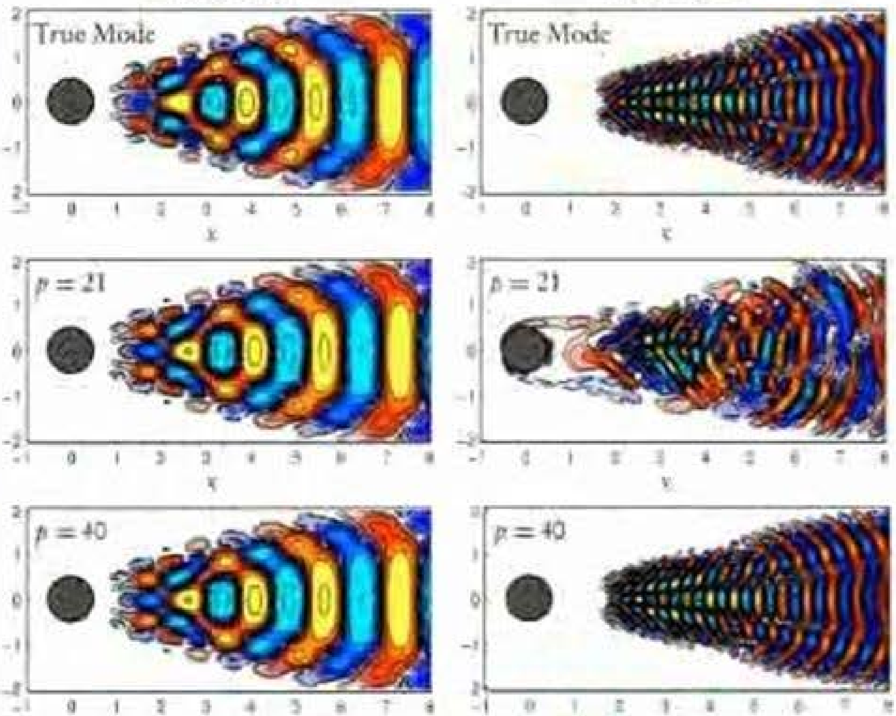


# Cylinder Wake Example

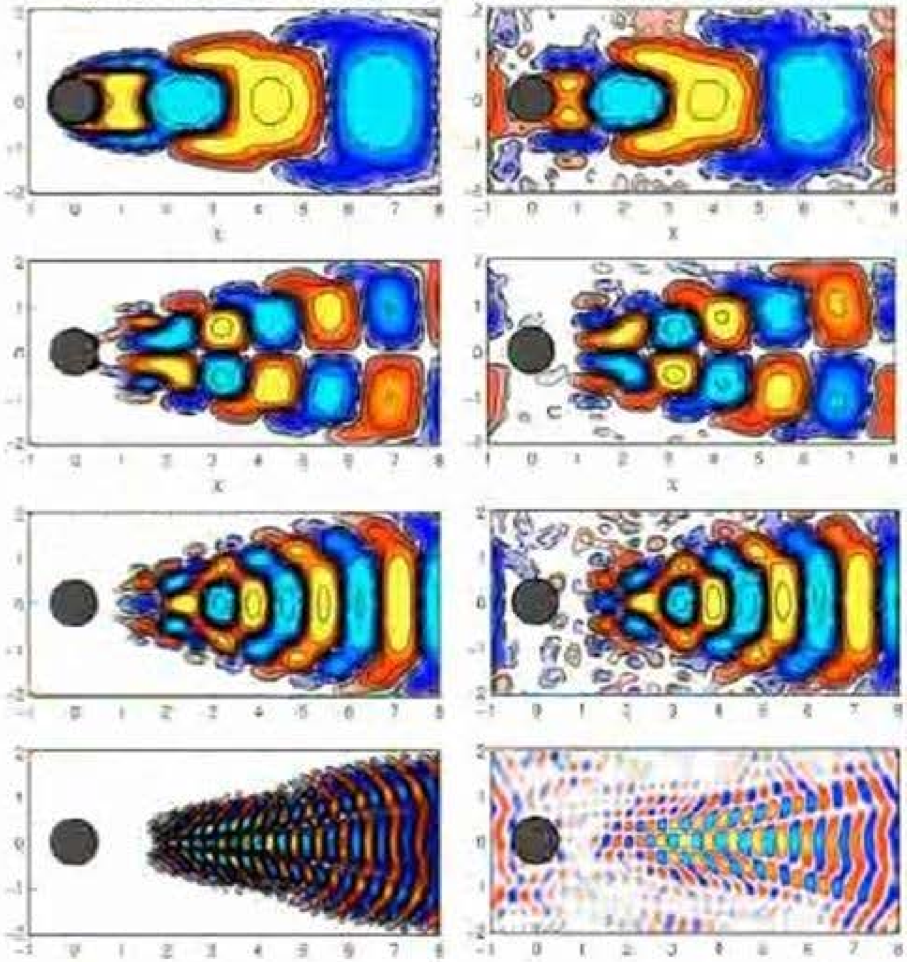
## Compressed DMD Comparison

**Mode 1** | **Mode 2**




## Compressive Sampling DMD Comparison

**Resolved Modes** | **CS Modes**



# Why Compressed DMD Works!

Fact 1: DMD is invariant to right Unitary transformations

Result: Swapping columns of X, X' does not change DMD at all

Fact 2: DMD is (mostly) invariant to left Unitary transformations

$$\mathbf{Y} = \mathbf{C}\mathbf{X}$$

$$\mathbf{Y} = \mathbf{C}\mathbf{U}_\mathbf{X}\mathbf{\Sigma}_\mathbf{X}\mathbf{V}_\mathbf{X}^*$$

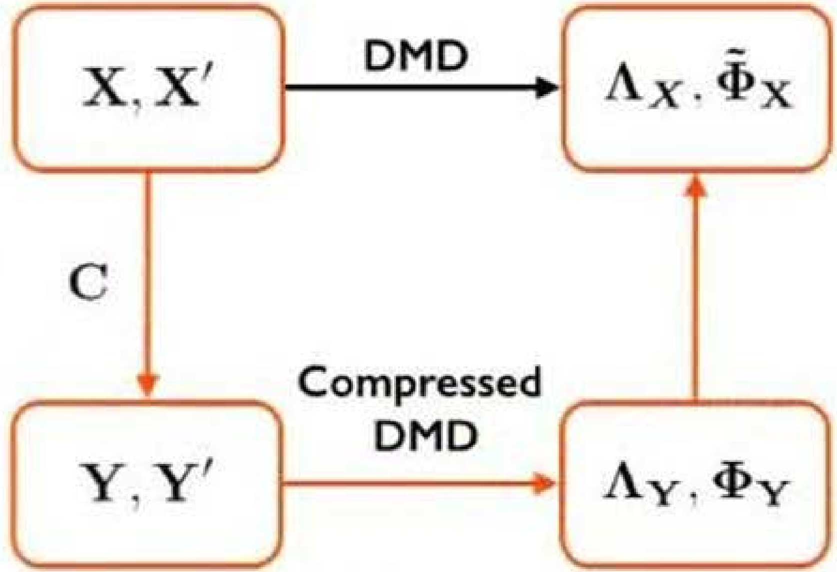
$$\mathbf{\Lambda}_\mathbf{Y} = \mathbf{\Lambda}_\mathbf{X}$$

$$\mathbf{\Phi}_\mathbf{Y} = \mathbf{C}\mathbf{\Phi}_\mathbf{X}$$

Result: DMD of X,X' related to DMD of FFT(X,X') = (S,S').

$$\mathbf{X} = \mathbf{\Psi}\mathbf{S}$$

Brunton, Proctor, Tu, Kutz, *submitted to JCD*, 2013.



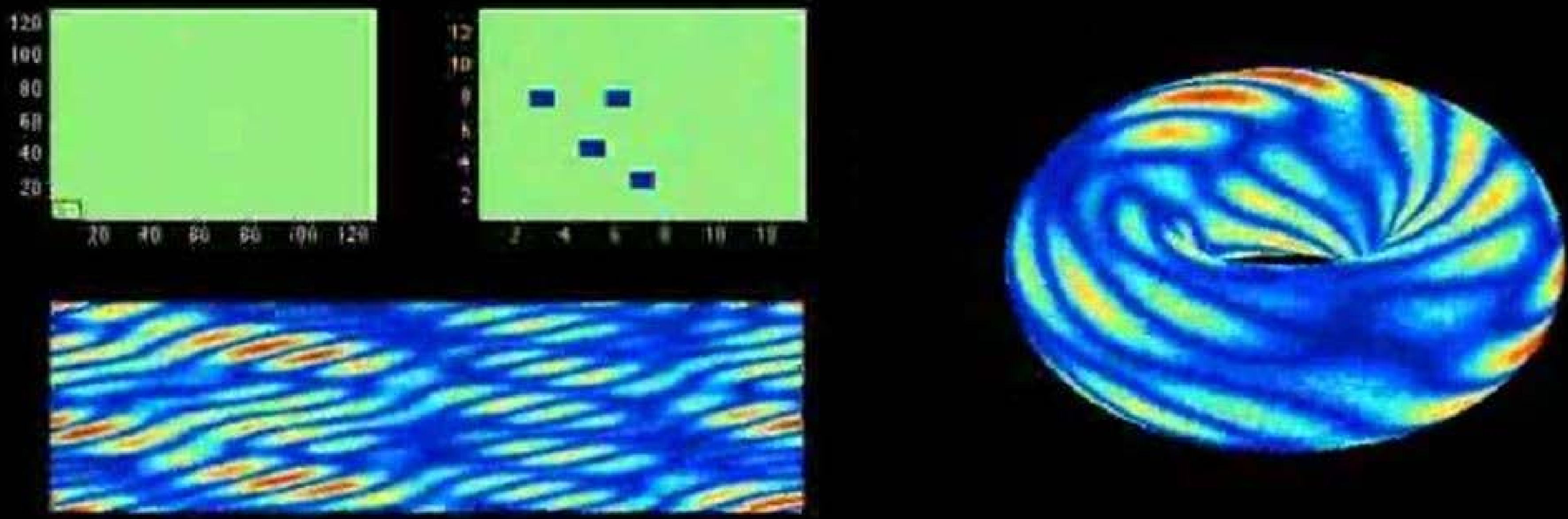

Now, let's say that X is sparse in the basis $\mathbf{\Psi}$

Measurements C that are incoherent with respect to $\mathbf{\Psi}$ result in a product that satisfy the *restricted isometry principle.*

$$\mathbf{Y} = \mathbf{C}\mathbf{\Psi}\mathbf{S}$$

Therefore, $\mathbf{C}\mathbf{\Psi}$ acts as an isometry on sparse vectors S

# Questions?

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