The focusing NLS equation with non-zero boundary conditions and the nonlinear stage of modulational instability

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with thanks to the National Science Foundation

SIAM Analysis of PDEs, Scottsdale, AZ, December 7–10, 2015

NLS and MI

- **Nonlinear Schrödinger equation** (NLS): ($\nu = \mp 1$: focusing/defocusing)
$$iq_t + q_{xx} - 2\nu(|q|^2 - q_o^2)q = 0\,.$$
- Background solution: $q(x,t) = q_o$.
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- Linearized NLS: if $q(x,t) = q_o + v(x,t)$, with $v(x,t) = O(\varepsilon)$, then
$$iv_t + v_{xx} - 2\nu q_o^2(v + v^*) = O(\varepsilon)\,.$$
- Solve with Fourier transforms: $v(x,t) = \dfrac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{i\zeta x}\hat{v}(\zeta,t)\,\mathrm{d}\zeta\,,$

$$\hat{v}(\zeta,t) = \big[\cos(\gamma t) - (2q_o^2 - \zeta^2)/(i\gamma)\,\sin(\gamma t)\big]\,\hat{v}(\zeta,0) + (2iq_o^2/\gamma)\sin(\gamma t)\hat{v}^*(\zeta,0)\,,$$

$$\gamma(\zeta) = \zeta\sqrt{\zeta^2 + 4\nu q_o^2}.$$

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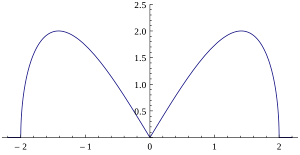
$$\gamma(\zeta) = \zeta\sqrt{\zeta^2 + 4\nu q_o^2}.$$

- MI: if $\nu = -1$, wavenumbers $\zeta <\in (-2q_o, q_o)$ are linearly unstable!
- Growth rate: $|\mathrm{Im}\,\gamma(\zeta)| = |\zeta|\sqrt{4q_o^2 - \zeta^2}$.



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- Here: nonlinear stage of MI via IST.

  One could think that IST for focusing NLS w/ NZBC is pointless b/c of MI.
  But, in fact, MI is not an impediment to IST.
  In fact, IST is the only tool to study the nonlinear stage of MI!

IST for focusing NLS with NZBC: Preliminaries

- Focusing NLS: $iq_t + q_{xx} + 2(|q|^2 - q_o^2)q = 0$:

  Lax pair: $\phi_x = X\,\phi$ & $\phi_t = T\,\phi\,,$

  $X = ik\sigma_3 + Q$, $T = -i(2k^2 + q_o^2 + Q^2 + Q_x)\sigma_3 - 2kQ$, Q&σ_3 as before.

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- NZBC: $q(x,t) \to q_\pm$ as $x \to \pm\infty$, with $|q_\pm| = q_o > 0$.

  Extra term in NLS added s.t. BC $q_\pm$ are independent of time: let $\tilde{q} = q\,\mathrm{e}^{-2iq_o^2 t}$.

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- Asymptotic scattering problem:

  $$\phi_x = X_\pm\,\phi, \quad X_\pm = ik\sigma_3 + Q_\pm = \lim_{x\to\pm\infty} X.$$

- The eigenvalues of $X_\pm$ are $\pm i\lambda$, with $\lambda^2 = k^2 + q_o^2$.

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- Branch points: $k \in \mathbb{C}$ s.t. $\lambda(k) = 0$, i.e., $k = \pm iq_o$.
- Branch cut: $i[-q_o, q_o]$.

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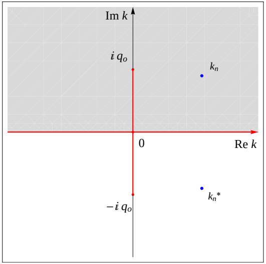
- Branch cut: $i[-q_o, q_o]$.
- Standard approach: two-sheeted Riemann surface defined by $\lambda(k)$, uniformization variable $z = k + \lambda$.
- Then $k \in \mathbb{C}_{\mathrm{I}} \mapsto |z|^2 > q_o$, $k \in \mathbb{C}_{\mathrm{II}} \mapsto |z|^2 < q_o$.
  Moreover, $k = \frac{1}{2}(z - q_o^2/z)$, $\lambda = \frac{1}{2}(z + q_o^2/z)$.

IST without Riemann surface, Jost eigenfunctions

- Define $\lambda(k)$ as a single-valued function $\forall k \in \mathbb{C}$, with a jump discontinuity across $i[-q_o, q_o]$.
- On $k \in i[-q_o, q_o]$, we define $\lambda(k)$ to be continuous from the right, i.e.,
  $\lambda(ik_i) = \lim_{k_r \to 0^+} \lambda(k_r + ik_i)$.



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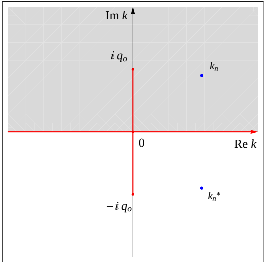
- Eigenvector matrices:

  $E_\pm(k) = I + i/(k+\lambda)\,\sigma_3 Q_\pm$

  s.t. $X_\pm E_\pm = E_\pm\, i\lambda\sigma_3$.



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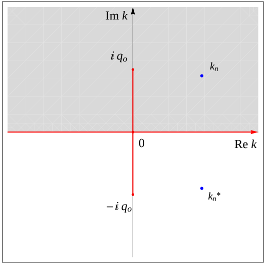

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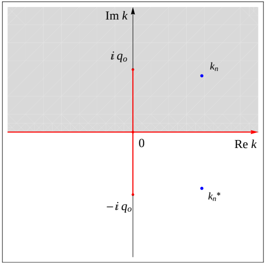

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- Jost eigenfunctions $\phi_\pm$: simultaneous solutions of both parts of Lax pair s.t.

$$\phi_\pm(x,t,k) = E_\pm(k)\,\mathrm{e}^{i\theta(x,t,k)\,\sigma_3} + o(1), \quad x \to \pm\infty, \quad k \in \Sigma.$$

phase function:

$$\theta(x,t,k) = \lambda\,(x - 2kt).$$

- Rigorously: define $\phi_\pm$ via Neumann series for Volterra integral equations.

Scattering matrix/analyticity/symmetries/discrete spectrum

- Scattering matrix: $\det\phi_\pm = \det E_\pm = 1+q_o^2/z^2 \neq 0$, so

$$\phi_-(x,t,k) = \phi_+(x,t,k)\,A(k) \qquad k \in \Sigma \setminus \{\pm i q_o\}\,.$$

Note $A(k)$ and the norming constants are independent of time.

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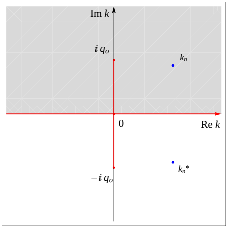
  $$\begin{aligned} &\phi_{+,1}, \phi_{-,2},\ a_{22}: && \mathbb{C}^+ \setminus i[0, q_o], \\ &\phi_{-,1}, \phi_{+,2},\ a_{11}: && \mathbb{C}^- \setminus i[-q_o, 0]. \end{aligned}$$



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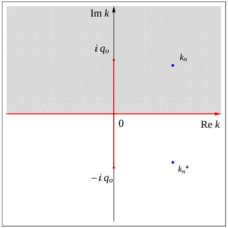
- Symmetry: $(k,\lambda) \mapsto (k^*,\lambda^*)$, which yields

  $$a_{11}(k) = a_{22}^*(k^*), \quad a_{12}(k) = -a_{21}^*(k^*), \quad k \in \Sigma,$$

  plus Schwartz extension when applicable.



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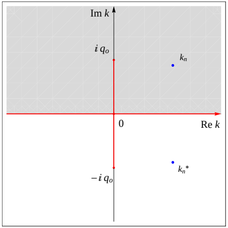

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- Discrete spectrum: $k_1,\ldots,k_N \in \mathbb{C}^+ \setminus i[0,q_o]$ s.t. $a_{22}(k_n) = 0$:

$$\phi_{+,1}(x,t,z_n) = b_n\,\phi_{-,2}(x,t,z_n)\,. \qquad \text{(bound states)}$$

- Symmetries $\Rightarrow$ discrete eigenvalues appear in symmetric pairs k_n & k_n^*. (as in the IVP),

plus corresponding symmetries for the norming constants.

Inverse problem: Sectionally meromorphic matrices

- Formulate inverse problem as a matrix Riemann-Hilbert problem (RHP).
- Sectionally meromorphic matrices:

$$M(x,t,k) = \begin{cases} \big(\phi_{+,1}\,,\phi_{-,2}/a_{22}\big)\,\mathrm{e}^{-i\theta\sigma_3}, & k \in \mathbb{C}^+ \setminus i[0,q_o], \\ \big(\phi_{-,1}/a_{11}\,,\,\phi_{+,2}\big)\,\mathrm{e}^{-i\theta\sigma_3}, & k \in \mathbb{C}^- \setminus i[-q_o,0]. \end{cases}$$

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- Asymptotics: $M \to I$ as $k \to \infty$.
- Next, need a jump condition for the RHP.
- For $k \in \mathbb{R}$, use scattering relation $\phi_- = \phi_+ A$ (as usual):

$$M^+ = M^- V \qquad k \in \mathbb{R}\,,$$

  Jump matrix:

$$V(x,t,k) = I - \mathrm{e}^{i\theta\sigma_3} \begin{pmatrix} 0 & -\tilde{\rho} \\ \rho & \rho\tilde{\rho} \end{pmatrix} \mathrm{e}^{-i\theta\sigma_3}, \quad k \in \mathbb{R}.$$

  Reflection coefficients:

$$\rho(k) = a_{21}/a_{11}, \qquad \tilde{\rho}(k) = a_{12}/a_{22} = -\rho^*(k).$$

- Notation:
  - subscripts $\pm$: normalization as $x \to \pm\infty$.
  - superscripts $\pm$: projection from the left/right of Σ.

Inverse problem: RHP and reconstruction formula

- To obtain the jump condition for the RHP for $k \in i[-q_o, q_o]$, one must relate the limits of the analytic eigenfunctions to the left and the right of Σ:

$$V(x,t,k) = \frac{i}{k-\lambda}\begin{pmatrix} -\tilde{\rho}\,\mathrm{e}^{2i\theta} & 1-\rho\tilde{\rho} \\ 1 & \rho\,\mathrm{e}^{-2i\theta} \end{pmatrix} \mathrm{diag}(q_+^*, q_+), \qquad k \in i[0, q_o],$$

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- As usual, the RHP reduces to a closed linear system of algebraic-integral equations:
  - subtract the asymptotic behavior as $k \to \infty$ and the pole contributions at the discrete eigenvalues, apply Cauchy projectors, use Plemelj's formulae,
  - evaluate regular columns at discrete spectrum and use residue conditions.

- Reconstruction formula: Compute the asymptotics of $M(x,t,k)$ as $k \to \infty$ and compare with asymptotics of $\phi_\pm(x,t,k)$:
$$q(x,t) = -2i \lim_{k\to\infty} [kM_{12}(x,t,k)].$$

- Can also obtain trace formulae and the so-called "theta" condition [which yields $\arg(q_+/q_-)$ from discrete eigenvalues and reflection coefficient].

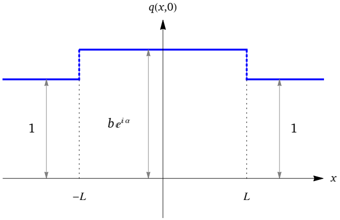
- Reflectionless potentials: determinantal solution form.

- Rich family of soliton solutions: Kuznetsov-Ma, Peregrine, Akhmediev, Watanabe-Tajiri…

Explore MI with IST

- Test: piecewise constant, box-like IC,

$$q(x,0) = \begin{cases} 1 & |x| > L\,, \\ b\,e^{i\alpha} & |x| < L\,. \end{cases}$$



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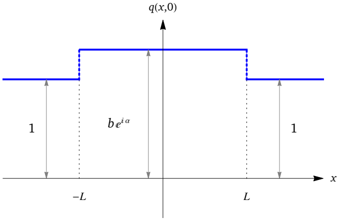

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- Can compute solutions in each sub-domain, then impose continuity at $x = \pm L$ to obtain Jost eigenfunctions $\forall x \in \mathbb{R}$.
- Can compute full scattering matrix analytically; look for discrete eigenvals.

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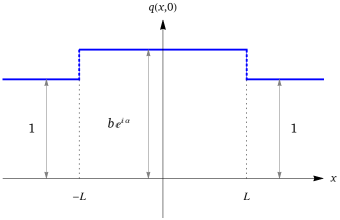

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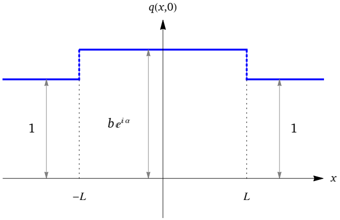
- Can compute full scattering matrix analytically; look for discrete eigenvals.
- Theorem: If $b > 1$ and $\cos\alpha > 1/b$, no threshold for discrete eigenvalues.
(All eigs lie in $i\mathbb{R}^+$; proof uses evaluation of $a_{11}(k)$ on $\partial\mathbb{C}_\mathrm{I}$ plus Rouché's theorem.)
- Corollary: no area theorem is possible for focusing NLS w/NZBC.
(This is like KdV and defocusing NLS w/NZBC, and unlike focusing NLS w/ZBC.)

Explore MI with IST

- Test: piecewise constant, box-like IC,

$$q(x,0) = \begin{cases} 1 & |x| > L\,, \\ b\,e^{i\alpha} & |x| < L\,. \end{cases}$$



- Scattering problem is a first-order system of ODEs with piecewise-constant coefficients:
- Can compute solutions in each sub-domain, then impose continuity at $x = \pm L$ to obtain Jost eigenfunctions $\forall x \in \mathbb{R}$.
- Can compute full scattering matrix analytically; look for discrete eigenvals.
- Theorem: If $b > 1$ and $\cos\alpha > 1/b$, no threshold for discrete eigenvalues. (All eigs lie in $i\mathbb{R}^+$; proof uses evaluation of $a_{11}(k)$ on $\partial\mathbb{C}_\mathrm{I}$ plus Rouché's theorem.)
- Corollary: no area theorem is possible for focusing NLS w/NZBC. (This is like KdV and defocusing NLS w/NZBC, and unlike focusing NLS w/ZBC.)
- Theorem: If $0 < b < 1$ and $\cos\alpha > b$, no discrete eigenvalues exist.
- Therefore solitons cannot be the main medium for MI. (b/c there is a nbhd of the constant solution with no discrete spectrum, whereas all perturbations of the background are linearly unstable)

Small-deviation limit of IST

- Restrict the ICs s.t. $q(x,t) \to q_o$ as $x \to \pm\infty$ (i.e., set $q_\pm = q_o$).
- Also, let $q(x,t) = q_o + v(x,t)$, with $v(x,0) = O(\varepsilon)$ as before.
- Neglecting possible contributions from the continuous spectrum,

$$q(x,t) = q_o - \frac{1}{2\pi} \int_\Sigma e^{2i\theta(x,t,z)}\, a_{12}(z)\, dz + O(\varepsilon^2)\,,$$

$$a_{12}(z) = \frac{1}{q_o^2 + z^2} \int_{\mathbb{R}} e^{-2i\lambda(z)y} (-z^2 v(y,0) + q_o^2 v^*(y,0))\, dy + O(\varepsilon^2)\,.$$

(Here we used the formulation of IST with uniformization.)

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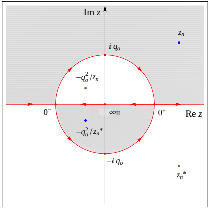

- Recall $z = k + \lambda$, $k = \frac{1}{2}(z - q_o^2/z)$, $\lambda = \frac{1}{2}(z + q_o^2/z)$.
- Perform the change of variable

$$z = \lambda + \sqrt{\lambda^2 - q_o^2} \quad \text{or} \quad z = \lambda - \sqrt{\lambda^2 - q_o^2}$$

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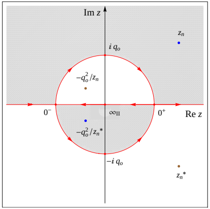

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as appropriate on the various portions of the contour.
- The resulting expression coincides exactly with that from linearization.
- That is, IST nonlinearizes the Fourier transform — as expected.
- But note IST is likely more accurate than linearization, because the latter neglects the possible contributions of the discrete spectrum.

What does this mean for MI?

- The Jost solutions are nonlinear analogues of Fourier modes.
- Recall: the asymptotic behavior of the Jost solutions as $x \to \pm\infty$ is
$$\phi_\pm(x,t,k) = E_\pm(k)\,\mathrm{e}^{i\theta(x,t,k)\sigma_3} + o(1), \quad \theta(x,t,k) = \lambda(k)\,x - \omega(k)\,t.$$
- The spatial behavior is governed by $\lambda(k) = \sqrt{k^2 + q_o^2}$, and $\lambda(k) \in \mathbb{R}\ \ \forall k \in \mathbb{R} \cup i[-q_o, q_o]$.

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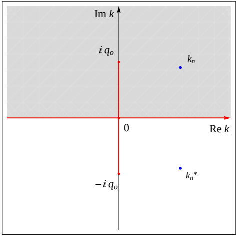
- So, the Jost solutions on the branch cut are precisely the nonlinearization of the unstable Fourier modes.
  (Alternatively, if one defines the Jost solutions with constant BCs, the reflection coefficient depends on time, and on the cut it grows exponentially.
  This is similar to Maxwell-Bloch equations in the unstable case.)

Nonlinear stage of MI and long-time asymptotics

- We have identified the instability mechanism within the context of IST: exponentially growing jumps in the RHP when $k \in i[-q_o, q_o]$.
- Since the solution of NLS remains bounded, this means that IST contains an automatic mechanism for the saturation of the instability.
- Goal: understand this mechanism and characterize the nonlinear stage of MI.



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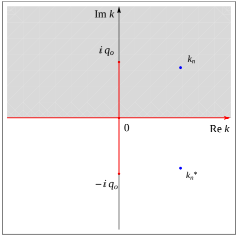

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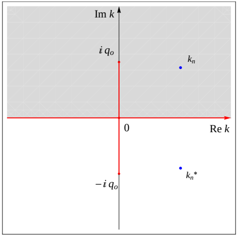

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- Long time asymp. for NLS with ZBC: Ablowitz-Segur, Zakharov-Manakov (1976) (GLM, WKB, similarity solns, multiple scales, etc.)
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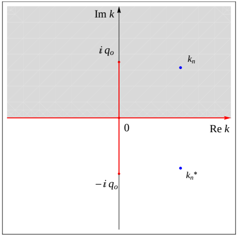

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- RHPs with exponentiallly growing jumps not unprecedented. [Deift-Kamvissis-Kricherbauer-Zhou, 1996; Buckingham-Venakides, 2007; Boutet de Monvel-Kotlyarov-Shepelsky, 2011; Jenkins-McLaughlin, 2014.]

Long-time asymptotics for focusing NLS w/ NZBC

- Recall: the idea behind the Deift-Zhou method is to modify the RHP by appropriate changes of dependent variables and contour deformations to "peel" away the oscillating/growing terms, reducing the problem to:
  - a "model" (or asymptotic) RHP that can be solved exactly, and which yields the leading-order behavior of the solution; plus
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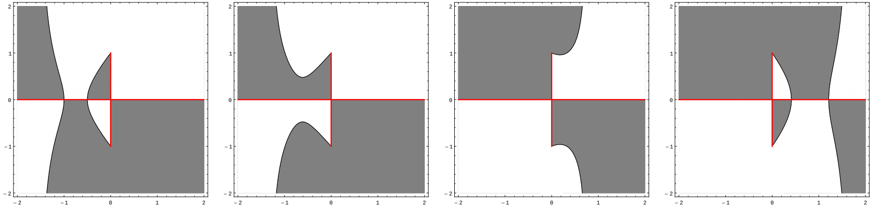
- Stationary points of $\theta(k,\xi)$ as a function of k:
$$k_\pm = \tfrac{1}{8}\big(\xi \pm \sqrt{\xi^2 - 32q_o^2}\big)\,.$$
- Let $\xi_* = 4\sqrt{2}\,q_o$. Two cases:
  $|\xi| > \xi_*$: real stationary points,
  $|\xi| < \xi_*$: complex stationary points.

Long-time asymptotics: outline

Plots: Regions of the k-plane where $\mathrm{Im}[\theta(k,\xi)] > 0$ (gray) or $\mathrm{Im}[\theta(k,\xi)] < 0$ (white) as a function of $\xi = x/t$.



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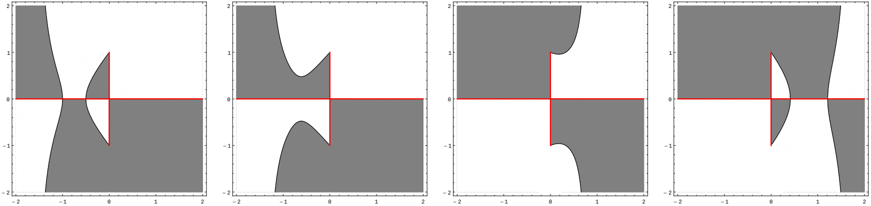

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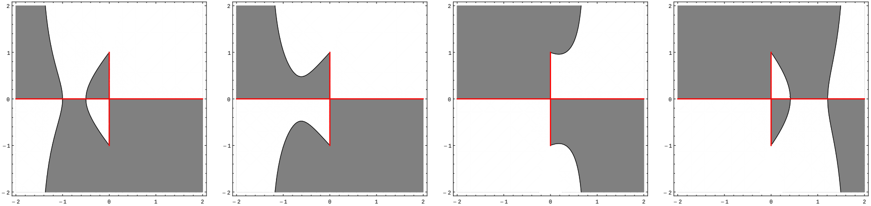

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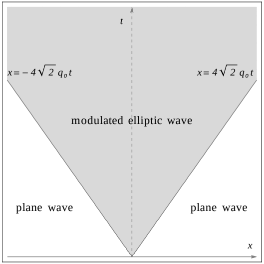

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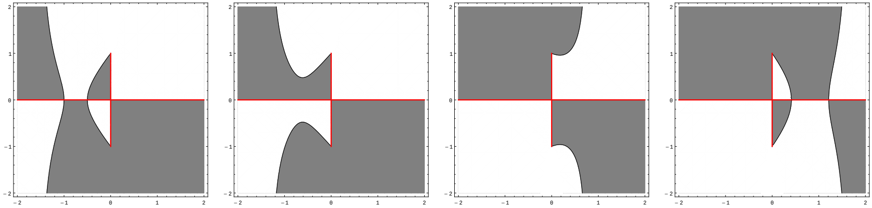

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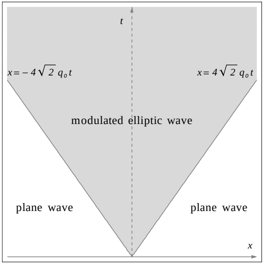
- $|\xi| < \xi_*$ ($k_\pm \in \mathbb{C} \setminus \mathbb{R}$):
  cannot deform away from real k-axis, must introduce an additional branch cut along $[\alpha^*, \alpha]$ and a modified g function.



Long-time asymptotics: main results

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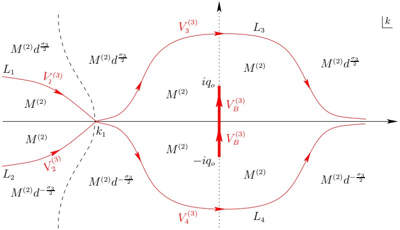

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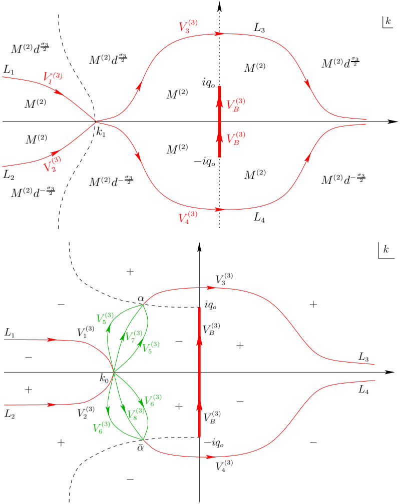

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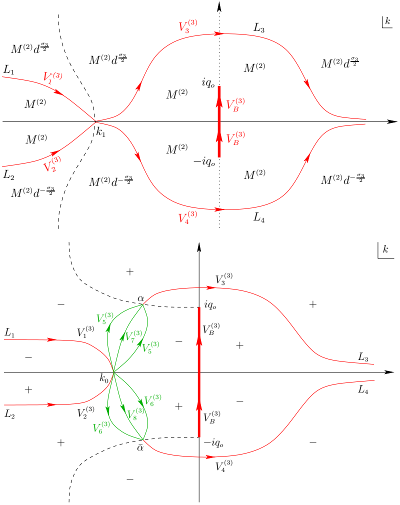
$$q(x,t) = \frac{\Theta(S + w_\infty)\Theta(\frac{1}{2})}{\Theta(S - \frac{1}{2})\Theta(w_\infty)} \, \mathrm{e}^{2i(g_\infty - G_\infty t)} + O(1/t^{1/2}),$$

$\Theta(z) = \theta_3(\pi z, \mathrm{e}^{i\pi\tau})$,
$S(x,t) = (C/2K(m))\,(x - 2\alpha_{\mathrm{re}} t - X)$,
$C = \sqrt{\alpha_{\mathrm{re}}^2 + (q_o + \alpha_{\mathrm{im}})^2}$,
$m = 4q_o\alpha_{\mathrm{im}}/C^2$,
$\alpha = \alpha_{\mathrm{re}} + i\alpha_{\mathrm{im}}$ determined in terms of ξ via a single implicit equation,
τ, w_∞, g_∞, G_∞, X determined explicitly in terms of α and the reflection coefficient.



Long-time asymptotics: genus-1 region

- Reduction to slowly modulated elliptic solution in the genus-1 region:

$$|q_{\mathrm{asymp}}(x,t)|^2 = (q_o + \alpha_{\mathrm{im}})^2 - 4q_o\alpha_{\mathrm{im}}\,\mathrm{sn}^2\big[C(x - 2\alpha_{\mathrm{re}}t - X); m\big],$$

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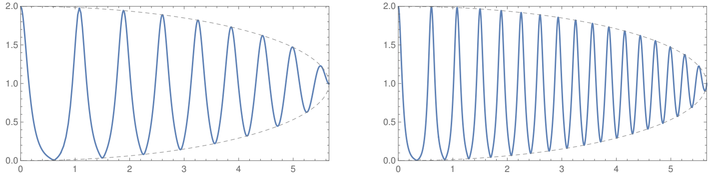

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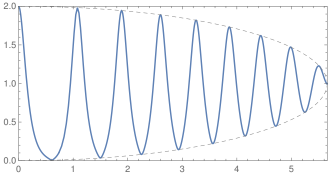
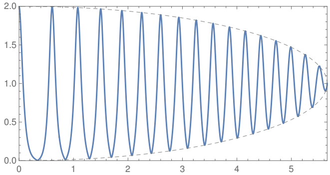


$t = 5$



$t = 10$

- Note the envelope is stationary in the ξt-frame.
- On the other hand, one can show that the oscillations become stationary in the xt-frame as $t\to\infty$!
- In fact, for fixed x, all the peaks become **sech** solitons as $t\to\infty$!

Universal nature of the nonlinear stage of MI

- Kamchatnov (2000): a modulated elliptic solution using Whitham's equations.
  El et al. (1993): motion of the branch points for the modulated solution.
  But those results were not rigorous, had no phase or translation parameters, had no connection to ICs, no error estimates.

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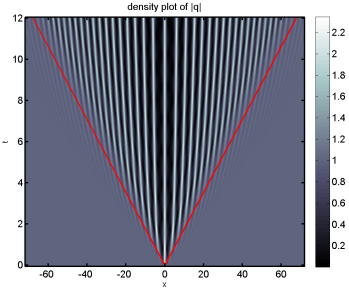
  There is an intermediate range of times for which one sees the asymptotic behavior but no catastrophic roundoff.

  Right:
  Density plot from numerical simulations of NLS with a small Gaussian perturbation of the constant background.
  Red lines:
  analytically predicted boundaries $x = \pm 4\sqrt{2} q_o t$.

  [numerics by Sitai Li]



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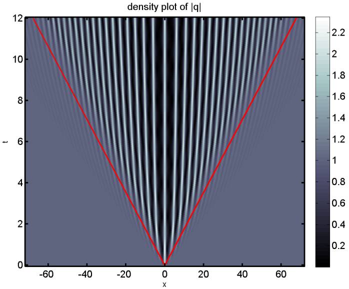

- Experiments?

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Thank you for your attention!