# The Certified Reduced-Basis Method for Darcy Flows in Porous Media

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## *Reservoir simulation in petroleum industry*

- Uses :
  - well placement, optimization of production scenarios,
  - History-matching,
  - Sensitivity analysis,
  - Quantification of uncertainties.
- Difficulties :
  - Long simulation times (grid's size, time iterations...),
  - Simulations should be rerun for different input parameters.
- A significant part of the simulation time is spent in the resolution of the pressure equation.
  - Aim of this study : build a reduced solution for this problem,
  - Among all possible methods : Reduced Basis (RB).

## *Outline*


- Two-phase flow model
  - Problem statement
  - Parameterization
  - Finite volume discretization
- Reduced model for the pressure equation
  - Variational RB formulation
  - A posteriori error
- Results
  - First numerical results
  - Reduction of the computational complexity
- Conclusions

## Problem statement

- Equations in $\Omega \subset \mathbb{R}^2$

$$\phi \partial_t S + \operatorname{div}\left(f_w(S)\mathbf{v}\right) = g, \qquad (1)$$

$$\operatorname{div}\left(\mathbf{v}\right) = f, \qquad (2)$$

$$\mathbf{v} + \lambda_T(S)\mathcal{K}\nabla P = 0, \qquad (3)$$

where

$$\lambda_T(S) = \lambda_w(S) + \lambda_o(1 - S),$$

$$\lambda_\alpha(S) = \frac{kr_\alpha(S)}{\mu_\alpha}, \ \alpha \in \{w, o\},$$

$$kr_w(S) = \left(\frac{S - S_{w,i}}{1 - S_{w,i} - S_{o,r}}\right)^2,$$

$$kr_o(S) = \left(\frac{1 - S - S_{o,r}}{1 - S_{w,i} - S_{o,r}}\right)^2,$$

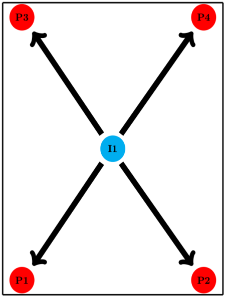
$$f_w(S) = \frac{\lambda_w(S)}{\lambda_T(S)}.$$

- Slip boundary conditions

$$\mathbf{v} \cdot \mathbf{n}_{|\partial\Omega} = 0. \qquad (4)$$

- Initial condition

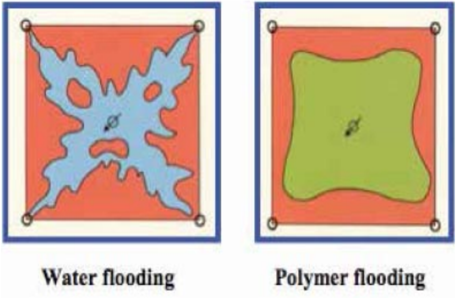
$$S(\cdot, 0) = 0.2 \qquad (5)$$



## *Parameterization*

Goal : assess numerically the potential of the enhanced oil recovery by polymer flooding.

Variations of $\mu_w$ due to operational conditions $\mu := \mu_w$, $\mu \in \mathscr{P} = [1, 30]$.



# *IMPIMS scheme and finite volume discretization*

- IMPIMS scheme for the time discretization

$$\phi\frac{S_\mu^{n+1}-S_\mu^n}{\Delta t}+\operatorname{div}\left(f_w(S_\mu^{n+1},\mu)\mathbf{v}_\mu^{n+1}\right)=g_\mu^{n+1}, \tag{6}$$

$$\operatorname{div}\left(\mathbf{v}_\mu^{n+1}\right)=f, \tag{7}$$

$$\mathbf{v}_\mu^{n+1}=-\lambda_T(S_\mu^n,\mu)\mathcal{K}\nabla P_\mu^{n+1}. \tag{8}$$

- A finite volume method for the space discretization (TPFA). For the pressure, one needs to solve a $\mathcal{N}\times\mathcal{N}$ linear system :

$$\mathbb{A}_\mu\mathbf{P}_\mu^{n+1}=\mathbf{f},$$

where

  - $(\mathbb{A}_\mu\mathbf{P}_\mu^{n+1})_K=\sum_{\sigma=K|L}a_\sigma^n(\mu)(P_{\mu,K}^{n+1}-P_{\mu,L}^{n+1}),$
  - $a_\sigma^n(\mu)=$ harmonic mean of $\lambda_T(S_\mu^n,\mu)\mathcal{K}$ on the edge $\sigma$,
  - $\mathbf{f}_K=\int_K f.$

## *Homogeneous case*

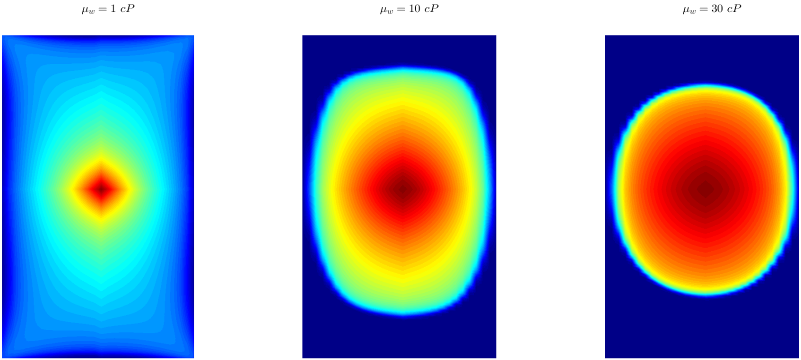



FIGURE: Water saturations for different parameter values at $T = 1,000$ days

## SPE10 case[1] (layer 85)

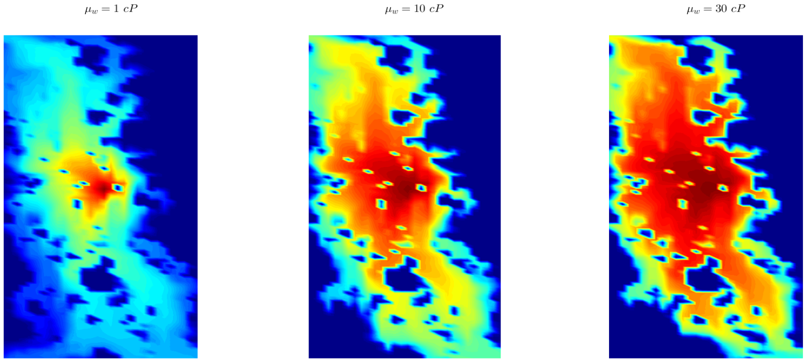



FIGURE: Water saturations for different parameter values at $T = 1,000$ days

1. Tenth SPE comparative solution project : a comparison of upscaling techniques, Christie and Blunt (2001), SPE Reserv. Eval. Eng., 4(4) :308-317.

## *Reduced model for the pressure equation*

The pressure equation reads

$$\begin{cases} \operatorname{div} \mathbf{v}_\mu = f, & \text{in} \quad \Omega, \\ \mathbf{v}_\mu = -a(\mu)\nabla P_\mu, & \text{in} \quad \Omega, \\ \mathbf{v}_\mu \cdot \mathbf{n}_\Omega = 0, & \text{on} \quad \partial\Omega, \\ \displaystyle\int_\Omega P_\mu = 0, & \end{cases} \tag{9}$$

where $a(\mu) = \lambda_T(S_\mu, \mu)\mathcal{K}$.

Difficulties :

- equation (9) is coupled with a transport equation for $S_\mu$.
- (9) is discretized with a cell-centered finite volume scheme.
- $a(\mu)$ does not fulfill the affine parameter dependence assumption.

**Idea 1.**

Let us suppose $S_\mu$ (and so $a(\mu)$) known for all $\mu \in \mathscr{P}$.

# *Discrete Galerkin Framework*

**Idea 2.**

Express our finite volume discretization as a variational formulation.

Let $Q_{\mathcal{N}}$ be the space of piecewise-constant L2-functions subject to zero-mean condition, i.e.,

$$\frac{1}{\mathcal{N}} \sum_{K} |K| P_K = 0.$$

Given $\mu \in \mathscr{P}$, consider the problem

$$\begin{aligned} &\text{Find } P_\mu^{\mathcal{N}} \in Q_{\mathcal{N}} \text{ such that} \\ &C_{\mathcal{N}}(P_\mu^{\mathcal{N}}, q; \mu) \quad = \quad L_f(q), \quad \forall q \in Q_{\mathcal{N}}, \end{aligned} \tag{10}$$

where

$$C_{\mathcal{N}}(P_\mu^{\mathcal{N}}, q; \mu) = \sum_{\sigma = K|L} a_\sigma(\mu) \left(P_{\mu,K}^{\mathcal{N}} - P_{\mu,L}^{\mathcal{N}}\right)\left(q_K - q_L\right) \text{ and } L_f(q) = \sum_K f_K q_K.$$

The space $Q_{\mathcal{N}}$ is equipped with the energy norm

$$|||p|||_{\mathcal{N},\,\mu}^2 = C_{\mathcal{N}}(p, p; \mu). \tag{11}$$

The reduced problem reads

$$\begin{aligned} &\text{Find } P_\mu^N \in Q_N \text{ such that} \\ &C_{\mathcal{N}}(P_\mu^N, q; \mu) \quad = \quad L_f(q), \quad \forall q \in Q_N. \end{aligned} \qquad (12)$$

The reduced basis method consists in solving a small N x N system

$$\mathbb{A}^N(\mu)\mathbf{P}_\mu^N = \mathbf{f}^N, \qquad (13)$$

with

- $\mathbf{P}_\mu^N \in \mathbb{R}^N$ such that $P_\mu^N = \sum_{n=1}^{N} (\mathbf{P}_\mu^N)_n P_{\mu_n}$,
- $\mathbb{A}^N(\mu) \in \mathbb{R}^{N\times N}$, such that $\mathbb{A}_{m,n}^N(\mu) = \sum_{\sigma=K|L} a_\sigma(\mu)(P_{\mu_m,K}^{\mathcal{N}} - P_{\mu_m,L}^{\mathcal{N}})(P_{\mu_n,K}^{\mathcal{N}} - P_{\mu_n,L}^{\mathcal{N}})$,
- $\mathbf{f}^N \in \mathbb{R}^N$ such that $\mathbf{f}_n^N = \sum_{K\in\mathcal{T}} P_{\mu_n,K}^{\mathcal{N}} \int_K f$.

# *A posteriori error estimate – first attempt*

**Definition 1** (Residual and associated norm).

Let $p \in Q_{\mathcal{N}}$. For all $q \in Q_{\mathcal{N}}$, define the residual

$$\left\langle \mathcal{R}_{\mathcal{N}}(p;\mu), q \right\rangle = C_{\mathcal{N}}(p, q; \mu) - L_f(q),$$

and the associated (discrete) norm

$$\left\| \mathcal{R}_{\mathcal{N}}(p;\mu) \right\|_{\mathcal{N},*,\mu} = \sup_{q \in Q_{\mathcal{N}}} \frac{\left\langle \mathcal{R}_{\mathcal{N}}(p;\mu), q \right\rangle}{|||q|||_{\mathcal{N},\,\mu}}.$$

**Proposition 1** (Energy estimate).

$$|||P_\mu^{\mathcal{N}} - P_\mu^N|||_{\mathcal{N},\,\mu} \leq \Delta_N^{en}(\mu) := \frac{1}{\alpha_{\mathcal{N}}(\mu)} \left\| \mathcal{R}_{\mathcal{N}}(P_\mu^N;\mu) \right\|_{\mathcal{N},*,\mu} \tag{14}$$

with

$$\alpha_{\mathcal{N}}(\mu) = \inf_{q \in Q_{\mathcal{N}} \setminus \{0\}} \frac{C_{\mathcal{N}}(q, q; \mu)}{|||q|||_{\mathcal{N},\,\mu}}.$$

## *Practical computation of error bounds*

- Computing the stability factor

$$\alpha_{\mathcal{N}}(\mu) = 1, \ \forall \mu \in \mathscr{P}$$

- Computing the norm of the residual

$$\begin{aligned}\left\|\mathcal{R}_{\mathcal{N}}(P_\mu^N;\mu)\right\|^2_{\mathcal{N},*,\mu} &= (\mathbf{f} - \mathbb{A}_\mu \mathbb{P}^N \mathbf{P}_\mu^N)^t \mathbb{A}_\mu^\dagger (\mathbf{f} - \mathbb{A}_\mu \mathbb{P}^N \mathbf{P}_\mu^N) \\ &= \mathbf{f}^t \mathbb{A}_\mu^\dagger \mathbf{f} - 2\mathbf{f}^t \mathbb{A}_\mu^\dagger \mathbb{A}_\mu \mathbb{P}^N \mathbf{P}_\mu^N + (\mathbb{P}^N \mathbf{P}_\mu^N)^t \mathbb{A}_\mu \mathbb{P}^N \mathbf{P}_\mu^N\end{aligned}$$

**Pros & cons 1.**

✓ trivial computation of $\alpha_{\mathcal{N}}(\mu)$
✗ No affine parameter dependence assumption
✗ $\mathbb{A}_\mu^\dagger$ depends on $\mu$

## *A posteriori error estimate – second attempt*

**Definition 2** (Norm of the residual).

Let $p \in Q_{\mathcal{N}}$, define a new dual norm as

$$\left\| \mathcal{R}_{\mathcal{N}}(p;\mu) \right\|_{\mathcal{N},*,\mu_{ref}} = \sup_{q \in Q_{\mathcal{N}}} \frac{\left\langle \mathcal{R}_{\mathcal{N}}(p;\mu), q \right\rangle}{|||q|||_{\mathcal{N},\,\mu_{ref}}}.$$

**Proposition 2** (Energy estimate).

$$|||P_{\mu}^{\mathcal{N}} - P_{\mu}^{N}|||_{\mathcal{N},\,\mu_{ref}} \leq \Delta_{N}^{en}(\mu) := \frac{1}{\alpha_{\mathcal{N}}(\mu)} \left\| \mathcal{R}_{\mathcal{N}}(P_{\mu}^{N};\mu) \right\|_{\mathcal{N},*,\mu_{ref}} \tag{15}$$

with

$$\alpha_{\mathcal{N}}(\mu) = \inf_{q \in Q_{\mathcal{N}} \setminus \{0\}} \frac{C_{\mathcal{N}}(q,q;\mu)}{|||q|||_{\mathcal{N},\,\mu_{ref}}}.$$

**Pros & cons 2.**

✓ $\mathbb{A}_{\mu}^{\dagger}$ no longer depends on $\mu$

✗ complex evaluation of $\alpha_{\mathcal{N}}(\mu)$

# Numerical results

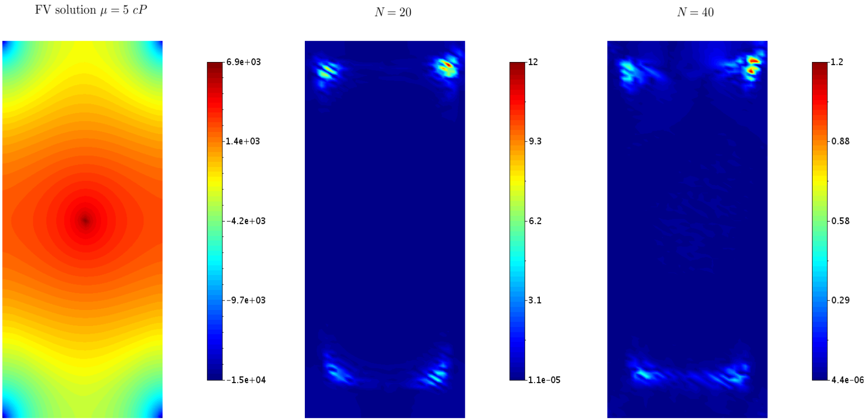



FIGURE: Representative solution and pointwise error for two values of $N$ (homogeneous case).

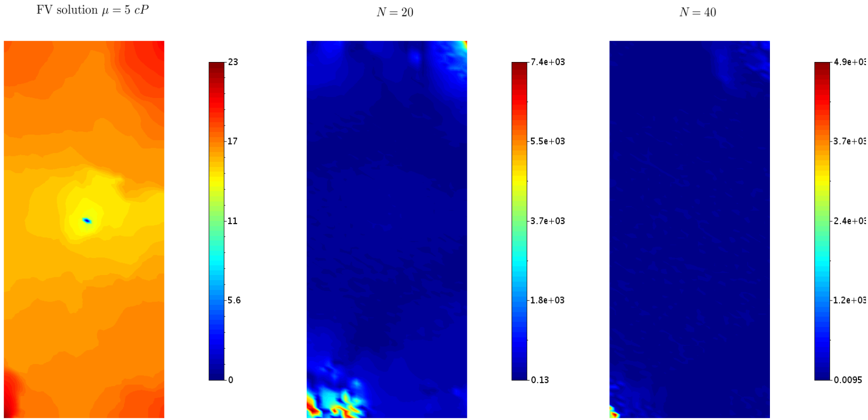



FIGURE: Representative solution and pointwise error for two values of $N$ (SPE10 case).

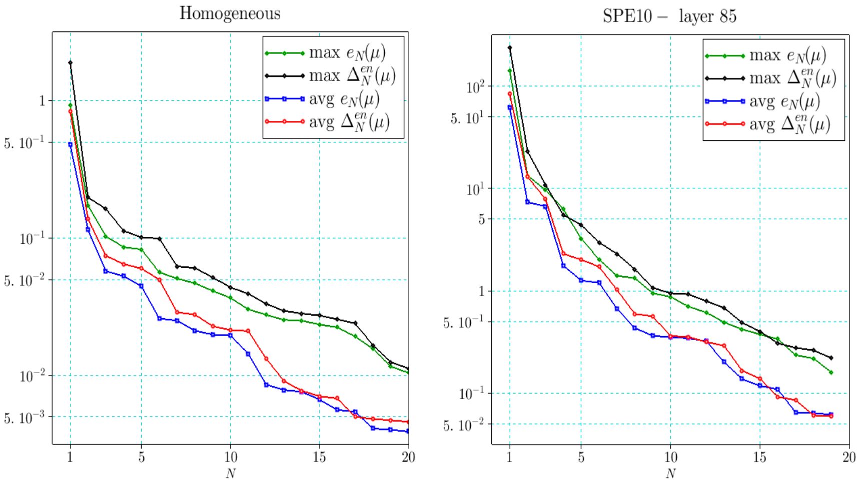



FIGURE: A posteriori error bound. Comparison of the maximum and average exact error $|||P_\mu^{\mathcal{N}} - P_\mu^N|||_\mu$ and the estimator $\Delta_N^{en}(\mu)$ ($\#\Xi_{train} \approx 300$).

# Reduction of the computational complexity

To solve

$$\mathbb{A}^N(\mu)\mathbf{P}^N_\mu = \mathbf{f}^N, \ \forall \mu \in \mathscr{P},$$

one needs to build $\mathbb{A}^N$ whose coefficients are

$$\mathbb{A}^N_{m,n}(\mu) = \sum_{\sigma=K|L} a_\sigma(\mu)(P^{\mathcal{N}}_{\mu_m,K} - P^{\mathcal{N}}_{\mu_m,L})(P^{\mathcal{N}}_{\mu_n,K} - P^{\mathcal{N}}_{\mu_n,L}).$$

**Idea 3.**

Replace $a(\mu)$ with a collateral affine expansion $a_M(\mu) = \sum_{m=1}^{M} \Theta_m(\mu)\zeta_m$ using the Empirical Interpolation Method [a] to obtain

$$\mathbb{A}^{N,M}_{m,n}(\mu) = \sum_{m=1}^{M} \left( \sum_{\sigma=K|L} \zeta_{m,\sigma}(P^{\mathcal{N}}_{\mu_m,K} - P^{\mathcal{N}}_{\mu_m,L})(P^{\mathcal{N}}_{\mu_n,K} - P^{\mathcal{N}}_{\mu_n,L}) \right) \Theta_m(\mu). \tag{16}$$

a. An 'empirical interpolation' method : application to efficient reduced-basis discretization of partial differential equations, Barrault, Maday, Nguyen and Patera (2004), C. R. Acad. Sci. Paris, Ser. I, 339(9) :667-672.

## *EIM-RB a posteriori error estimate*

Define the residual

$$\left\langle \mathcal{R}_M(P_\mu^{N,M};\mu), q \right\rangle = C_\mathcal{N}\left(P_\mu^{N,M}, q; a_M(\mu)\right) + L_f(q), \ \forall q \in Q_\mathcal{N}.$$

**Proposition 3.**

The following a posteriori error estimate holds

$$|||P_\mu^\mathcal{N} - P_\mu^{N,M}|||_{\mathcal{N},\mu_{ref}} \leq \frac{1}{\alpha_\mathcal{N}(\mu)}\left(\left\|\mathcal{R}_M(P_\mu^{N,M};\mu)\right\|_{\mathcal{N},*,\mu_{ref}} + \gamma(\mu)\delta_a(\mu)\right) \quad (17)$$

where

$$\delta_a(\mu) = \|a(\mu) - a_M(\mu)\|_{L^\infty(\Omega)}, \ \gamma(\mu) = \sup_{q\in Q_\mathcal{N}} \frac{C_\mathcal{N}\left(P_\mu^{N,M}, q; 1\right)}{|||q|||_{\mathcal{N},\,\mu_{ref}}}$$

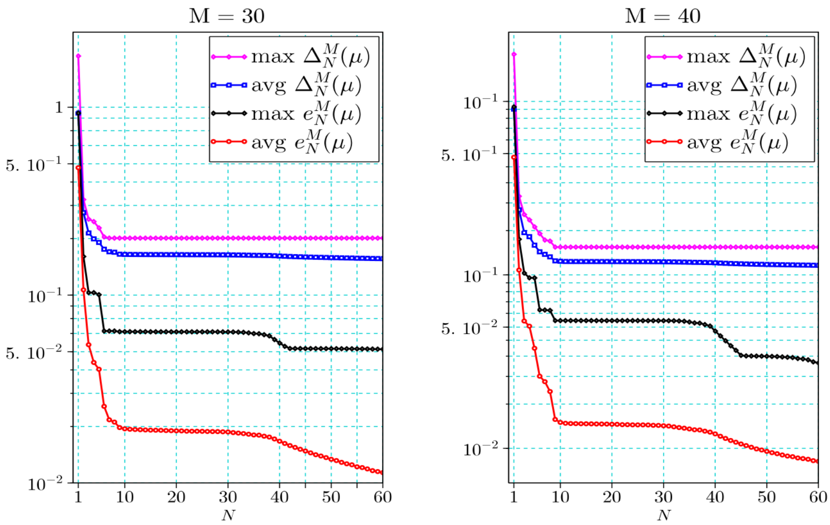



FIGURE: Convergence of the collateral reduced-basis approximation for the homogeneous example.

## *Conclusions and future works*

- Results
  - Reduction of the pressure problem (elliptic) with non-affine dependence in the parameter.
  - Certification of the RB model (a posteriori error estimate in energy norm).
  - Efficient implementation of the RB method, i.e. the reduced problems can be constructed in complexity independent of $\mathcal{N}$.
  - Certification of the RB-EIM model.

- Future works
  - Improvement of the efficiency of the a posteriori error estimate for the highly heterogenous case.
  - Adapt this procedure to reduce the collection of all pressure equations.
  - Consider more complex parameterizations (multi-dimensional parameter).

Thank you for your attention

Any questions ?