Derivation of Delay Equation Climate Models using Projection Methods

Swinda K.J. Falkena, Courtney Quinn, Jan Sieber, Jason Frank, Henk A. Dijkstra

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Introduction

- Conceptual climate models
  - Study physical mechanisms of climate variability
- Differential delay models
  - Infinite-dimensional, but can be formulated in terms of a single variable
  - Mostly introduced in an ad-hoc manner
- Projection methods can place derivation on stronger mathematical foundation
  - Mori-Zwanzig Formalism

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- Projection methods can place derivation on stronger mathematical foundation
  - Mori-Zwanzig Formalism
- El Niño Southern Oscillation (ENSO)

Delay Model of ENSO (Suarez and Schopf (1988))

$$\frac{\mathrm{d}T_e}{\mathrm{d}t} = T_e(t) - T_e^3(t) - \alpha T_e(t-\delta)$$

El Niño Southern Oscillation (ENSO)

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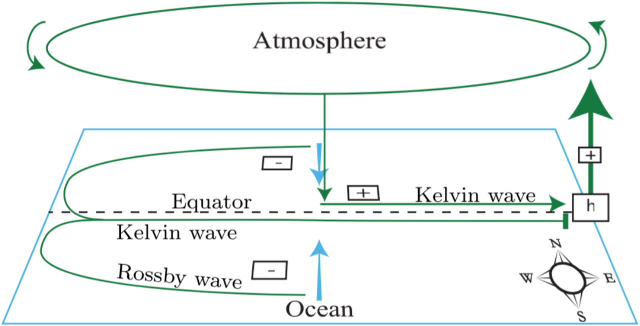

[1] Keane, Krauskopf, Postlethwaite, 2017.

Mori-Zwanzig Formalism

Linear example for $\phi = (\hat{\phi}, \tilde{\phi}) : \mathbb{R} \to \mathbb{R}^n$ continuously differentiable:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat{\phi}\\ \tilde{\phi}\end{pmatrix} = \begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\hat{\phi}\\ \tilde{\phi}\end{pmatrix}, \qquad \begin{pmatrix}\hat{\phi}(0)\\ \tilde{\phi}(0)\end{pmatrix} = \begin{pmatrix}\hat{x}\\ \tilde{x}\end{pmatrix}.$$

Goal: Equation for *resolved* variables $\hat{\phi} \in \mathbb{R}^m$ only,
the *unresolved* variables are $\tilde{\phi} \in \mathbb{R}^{n-m}$.

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Markovian Noise Memory

Mori-Zwanzig Formalism

In general, consider a **system of ODEs**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = R(\phi(t)), \quad \phi(0) = x,$$

$\phi(t) \in \mathbb{R}^n$ continuously differentiable, $R : \mathbb{R}^n \to \mathbb{R}^n$.

Evolution of an observable $u(x,t) := g(\phi(x,t))$ along a solution satisfies the PDE

$$\frac{\partial}{\partial t}u(x,t) = \mathcal{L}u(x,t), \quad u(x,0) = g(x),$$

where $[\mathcal{L}u](x) = \sum_{i=1}^{n} R_i(x)\partial_{x_i}u(x)$ is the **Liouville operator**.

A **projection** P onto a set of resolved variables $\hat{\phi}$, with complement $Q = I - P$.

Mori-Zwanzig Formalism

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Generalized Langevin Equation[2]

$$\frac{\partial}{\partial t}\phi_i(x,t) = R_i(\hat{\phi}(x,t)) + F_i(x,t) + \int_0^t K_i(\hat{\phi}(x,t-s),s)\mathrm{d}s,$$

with

$$F_i(x,t) = [\mathrm{e}^{tQ\mathcal{L}}Q\mathcal{L}g](x), \qquad K_i(\hat{x},t) = [P\mathcal{L}F_i](x,t).$$

$F_i(x,t)$ solves the **orthogonal dynamics equation**:

$$\frac{\mathrm{d}}{\mathrm{d}t}F_i(x,t) = Q\mathcal{L}F_i(x,t), \qquad F_i(x,0) = Q\mathcal{L}x_i.$$

[2] For derivation see Chorin, Hald, Kupferman, 2002.

ENSO Model

Two-Strip Model (rewritten)[3]

$$\partial_t h_c + \epsilon_0 h_c + \partial_x h_c = \mu\left(1 - \frac{\theta}{1+y_n^2}\right) g(x) T_e(x_E, t)$$

$$\partial_t h_n + \epsilon_0 h_n - \frac{1}{y_n^2}\partial_x h_n = -\mu\frac{\theta}{y_n^2} g(x) T_e(x_E, t)$$

$$\partial_t T_e + c_T T_e - c_h\left(h_c + \frac{1}{1+y_n^2} h_n\right) = 0$$

T_e	Temperature at equator
h_e	Thermocline at equator
h_n	Thermocline at $y = y_n$

where $h_c(x,t) = h_e(x,t) - \frac{1}{1+y_n^2} h_n(x,t)$. The boundary conditions are:

$$h_c(0,t) = \left(r_W - \frac{1}{1+y_n^2}\right) h_n(0,t), \quad h_c(1,t) = \left(\frac{1}{r_E} - \frac{1}{1+y_n^2}\right) h_n(1,t).$$

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$$\frac{\mathrm{d}T_e}{dt}(x,t) = -c_T(x)T_e(x,t) + c_h(x)\Big(\mathrm{e}^{-(\epsilon_0+\partial_x)t}h_c(x,0) + \frac{1}{1+y_n^2}\mathrm{e}^{-(\epsilon_0-\frac{1}{y_n^2}\partial_x)t}h_n(x,0)\Big) + \int_0^t c_h(x)\Big(B_0\mathrm{e}^{-(\epsilon_0+\partial_x)(t-s)} - B_1\mathrm{e}^{-(\epsilon_0-\frac{1}{y_n^2}\partial_x)(t-s)}\Big) \cdot g(x)T_e(x_E,s)\mathrm{d}s$$

See derivation in arXiv:1902.03198.

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Is this a delay equation?

Characteristics

Memory-Term

$$\int_0^t c_h(x_E)\left[\left(B_0 \mathrm{e}^{-(\epsilon_0+\partial_x)(t-s)} - B_1 \mathrm{e}^{-(\epsilon_0-\frac{1}{y_n^2}\partial_x)(t-s)}\right) \cdot g(x)\right]_{x_E} T_e(x_E, s)\mathrm{d}s$$

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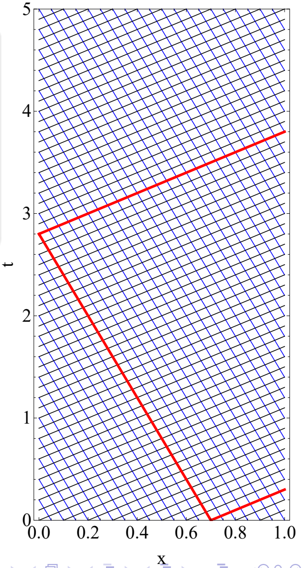
- Interested in east of the basin: $x = x_E$
- Follow signal along characteristics to eastern boundary
  - $\partial_t f = -\epsilon_0 f - \partial_x f \quad \rightarrow x - x_0 = t - t_0$
  - $\partial_t f = -\epsilon_0 f + \frac{1}{y_n^2}\partial_x f \rightarrow x - x_0 = \frac{-1}{y_n^2}(t - t_0)$
- No reflection at eastern boundary
  - Energy loss at western boundary: A_{rW}



Characteristics

Memory-Term

$$c_h(x_E)A_0\Big(B_0 e^{-\epsilon_0(1-x_w)}T_e(x_E, t-(1-x_w))$$
$$-B_1 A_{rW} e^{-\epsilon_0(1+y_n^2 x_w)}T_e(x_E, t-(1+y_n^2 x_w))\Big)$$

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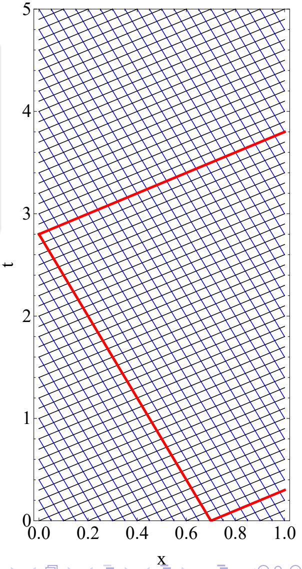
- No reflection at eastern boundary
  - Energy loss at western boundary: A_{rW}
- Wind forcing acts locally: $g(x) = A_0 \delta_{x_w}(x)$



Linear Delay Equation

$$\frac{\mathrm{d}T_e^E}{\mathrm{d}t} = -c_T(x_E)T_e^E(t) + c_h(x_E)A_0\Big(B_0\mathrm{e}^{-\epsilon_0(1-x_w)}T_e^E(t-(1-x_w)) - B_1A_{rW}\mathrm{e}^{-\epsilon_0(1+y_n^2x_w)}T_e^E(t-(1+y_n^2x_w))\Big)$$

Note the noise term vanishes by assuming no reflection at the eastern boundary.

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Note the noise term vanishes by assuming no reflection at the eastern boundary.

Since $1 - x_w \ll 1 + y_n^2 x_w$ we assume $T_e^E(t-(1-x_w)) \approx T_e^E(t)$.

Delay Model ENSO

$$\frac{dT_e^E}{dt} = c_S T_e^E(t) - c_L T_e^E(t-d)$$

Nonlinear ENSO Model

Nonlinear Temperature Equation[4]

$$\partial_t T_e + c_T(x) T_e - \boxed{c_h^*(x)(1 - \beta T_e^2)} \left(h_c + \frac{1}{1 + y_n^2} h_n \right) = 0$$

[4] Based on Dijkstra, Neelin, 1995.

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Two approaches:

1. Approximation to Mori-Zwanzig formalism

$$\frac{\mathrm{d} T_e^E}{\mathrm{d} t} = (c_S^* - c_T(x_E)) T_e^E(t) - c_L^* T_e^E(t - d) - \beta c_S^* T_e^E(t)^3 + \beta c_L^* T_e^E(t - d)^3$$

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2. Variation of constants
   - Equations for h_c and h_n are still linear

$$\frac{\mathrm{d} T_e^E}{\mathrm{d} t} = (c_S^* - c_T(x_E)) T_e^E(t) - c_L^* T_e^E(t-d) - \beta c_S^* T_e^E(t)^3 + \beta c_L^* T_e^E(t)^2 T_e^E(t-d)$$

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ENSO Delay Models

Suarez and Schopf Model (S&S)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = T(t) - T^3(t) - \alpha T(t-\delta)$$

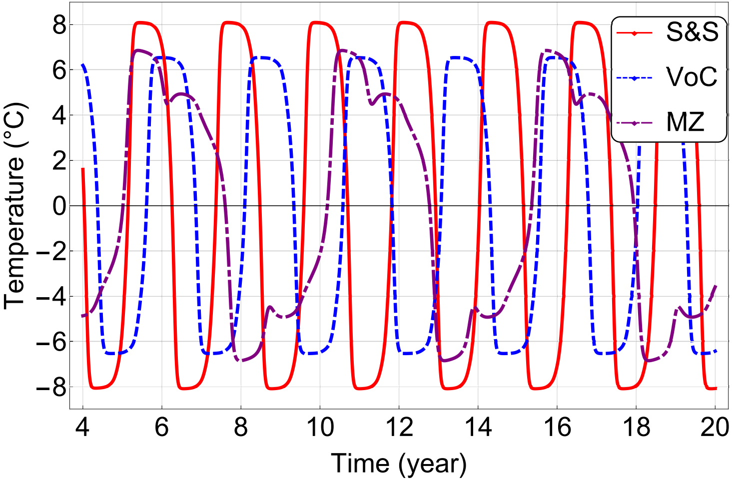
Variation of Constants Model (VoC)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = T(t) - T^3(t) - \alpha T(t-\delta)(1-\gamma T^2(t))$$

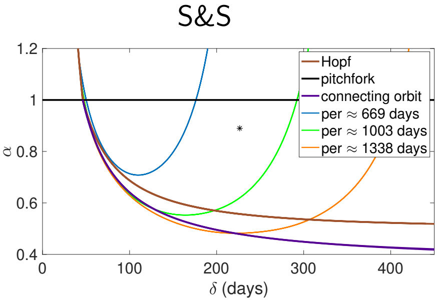
Mori-Zwanzig Model (MZ)

$$\frac{\mathrm{d}T}{\mathrm{d}t} = T(t) - T^3(t) - \alpha T(t-\delta)(1-\gamma T^2(t-\delta))$$

Periodic Solutions



Bifurcation Diagrams



Top: $\gamma = 0$.

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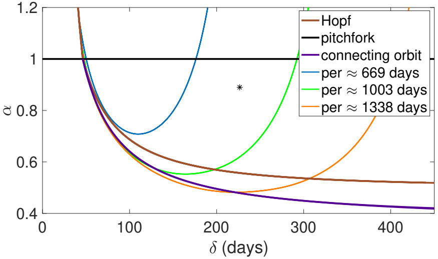
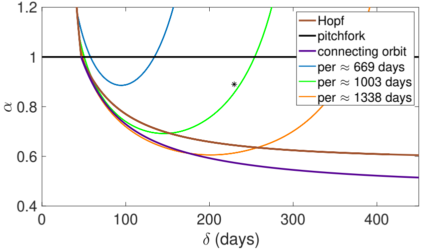
S&S

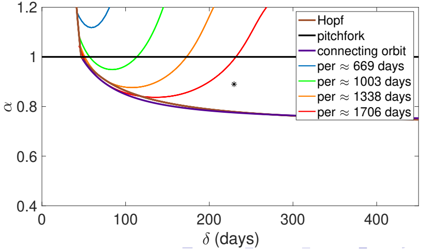


VoC



Top: $\gamma = 0$. Right: $\gamma = 0.49$.

MZ



Summary

- Mori-Zwanzig formalism can be used to derive delay equation models
  - When the equations for the unresolved variables are linear variation of constants is equivalent
- Application to a two-strip ENSO model leads to an improvement in period compared to a previously studied model
- Method can be extended to other wave equations (firstly, to those which are linear in the unresolved variables)
- For nonlinear models better approximation techniques for the orthogonal dynamics are needed

S.K.J. Falkena, C. Quinn, J. Sieber, J. Frank, H.A. Dijkstra, *Derivation of Delay Equation Climate Models Using the Mori-Zwanzig Formalism*, 2019, arXiv:1902.03198, (under review in PRSA).

Thank you!