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Random Attractors and How They Help Understand Climate Change and Variability

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ENS

Please visit these sites for more info.

http://www.atmos.ucla.edu/tcd/, http://www.environnement.ens.fr/
and https://www.researchgate.net/profile/Michael_Ghil

Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Nonequilibrium climate sensitivity
- Pull vs. snap: a tale of two (kinds of) attractors
- Conclusions and references
  - natural variability and anthropogenic forcing: the "grand unification"
  - selected bibliography

Climate and Its Sensitivity

Let's say CO_2 doubles:

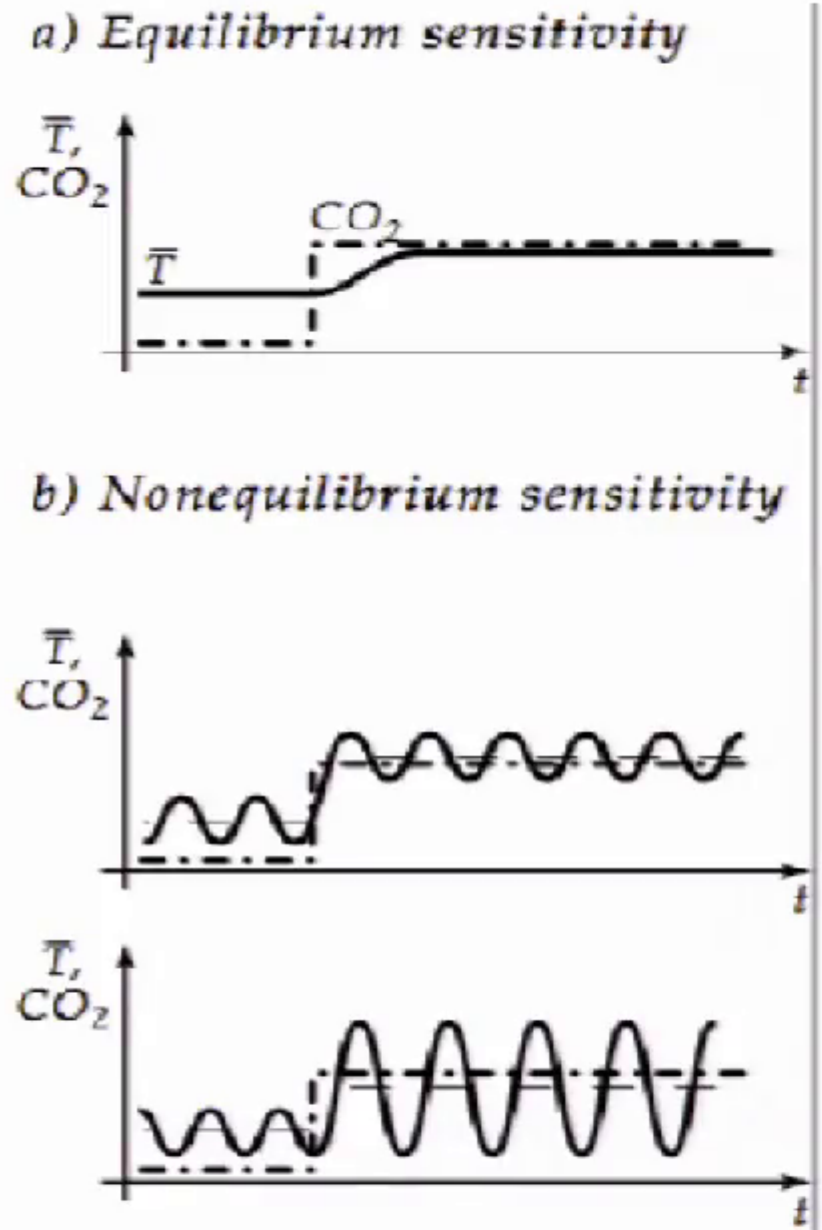
How will "climate" change?

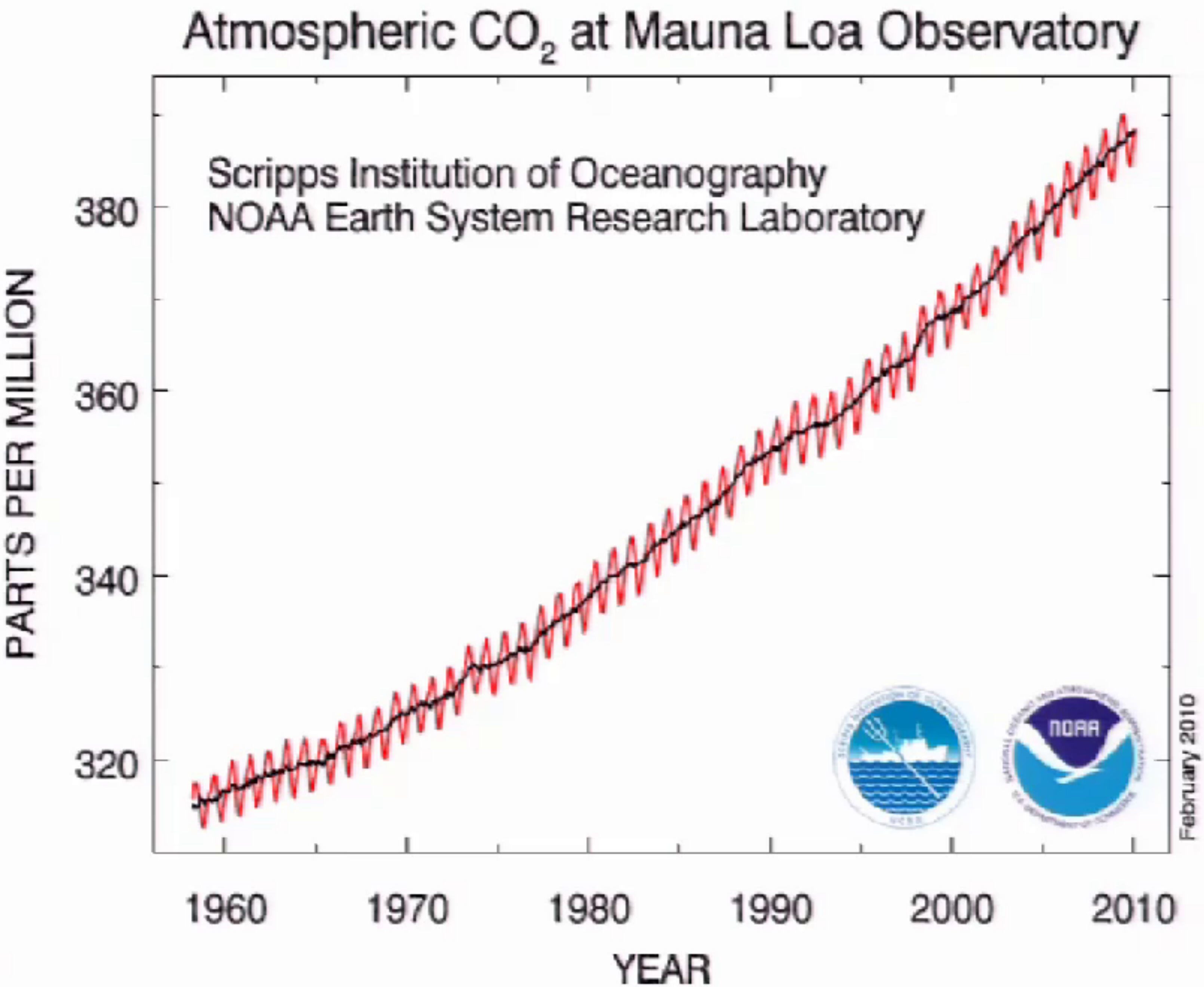
1. Climate is in stable equilibrium (fixed point); if so, mean temperature will just shift gradually to its new equilibrium value.

2. Climate is purely periodic; if so, mean temperature will (maybe) shift gradually to its new equilibrium value. But how will the period, amplitude and phase of the limit cycle change?

3. And how about some "real stuff" now: chaotic + random?

Ghil (in *Encycl. Global Environmental Change*, 2002)

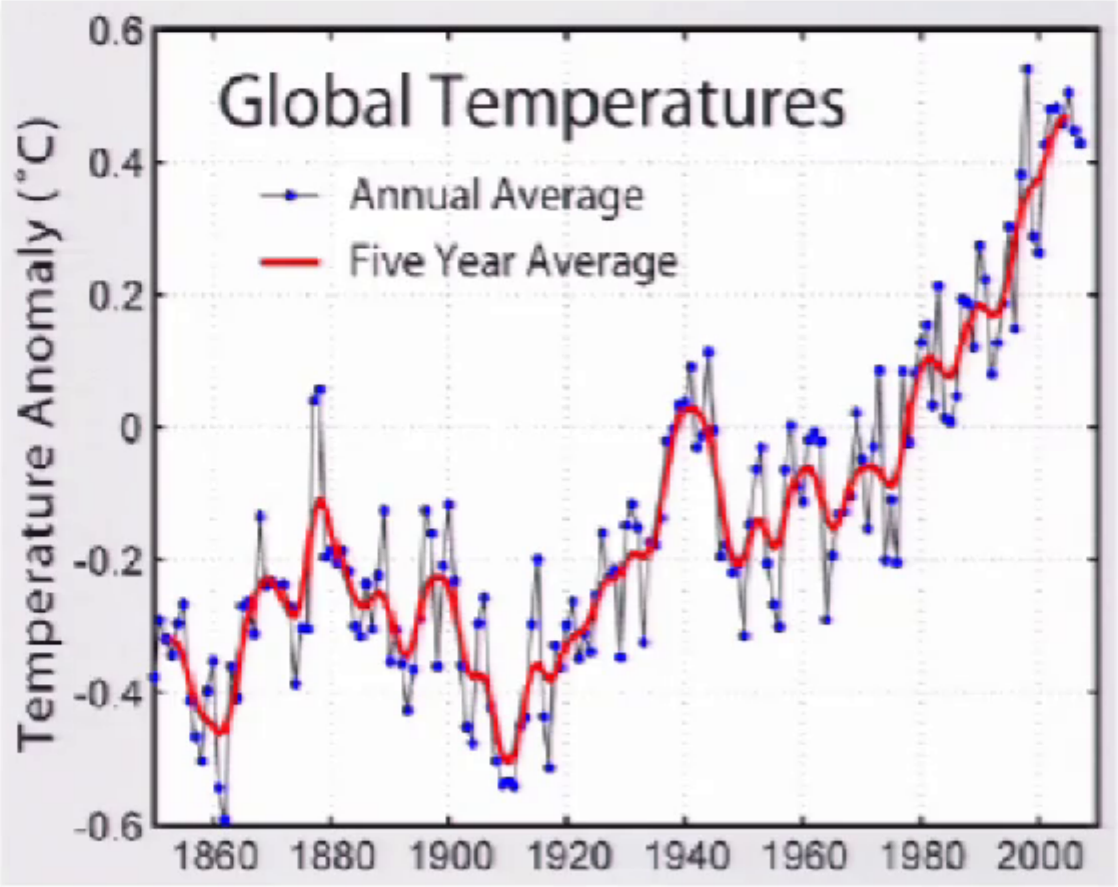



Atmospheric CO_2 at Mauna Loa Observatory
Scripps Institution of Oceanography
NOAA Earth System Research Laboratory
PARTS PER MILLION
380
360
340
320
1960
1970
1980
1990
2000
2010
YEAR
February 2010

Temperatures and GHGs

Greenhouse gases (GHGs) go up,
temperatures go up:

It's gotta do with us, at least a bit,
doesn't it?



Wikicommons, from
Hansen *et al.* (*PNAS*, 2006);
see also http://data.giss.nasa.gov/
gistemp/graphs/

Unfortunately, things aren't all that easy!

What to do?

Try to achieve better interpretation of, and agreement between, models ...

Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

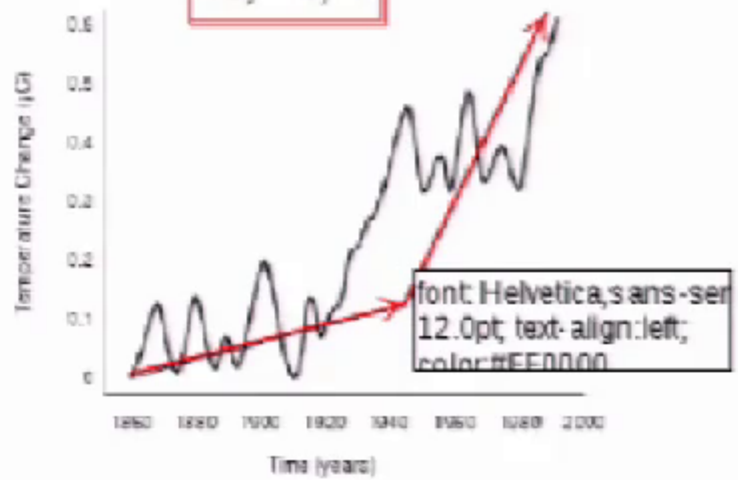
The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$c\frac{dT}{dt} = -kT + Q$$

$k = \sum k_i$ – feedbacks (+ve and -ve)

$Q = \sum Q_j$ – sources & sinks

$Q_j = Q_j(t)$



Linear response to CO_2 vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

$$\frac{dX}{dt} = N(X, t, \mu, \beta)$$

Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend,
is blowing in the wind,
i.e., it depends on the
accuracy and reliability
of the forecast …

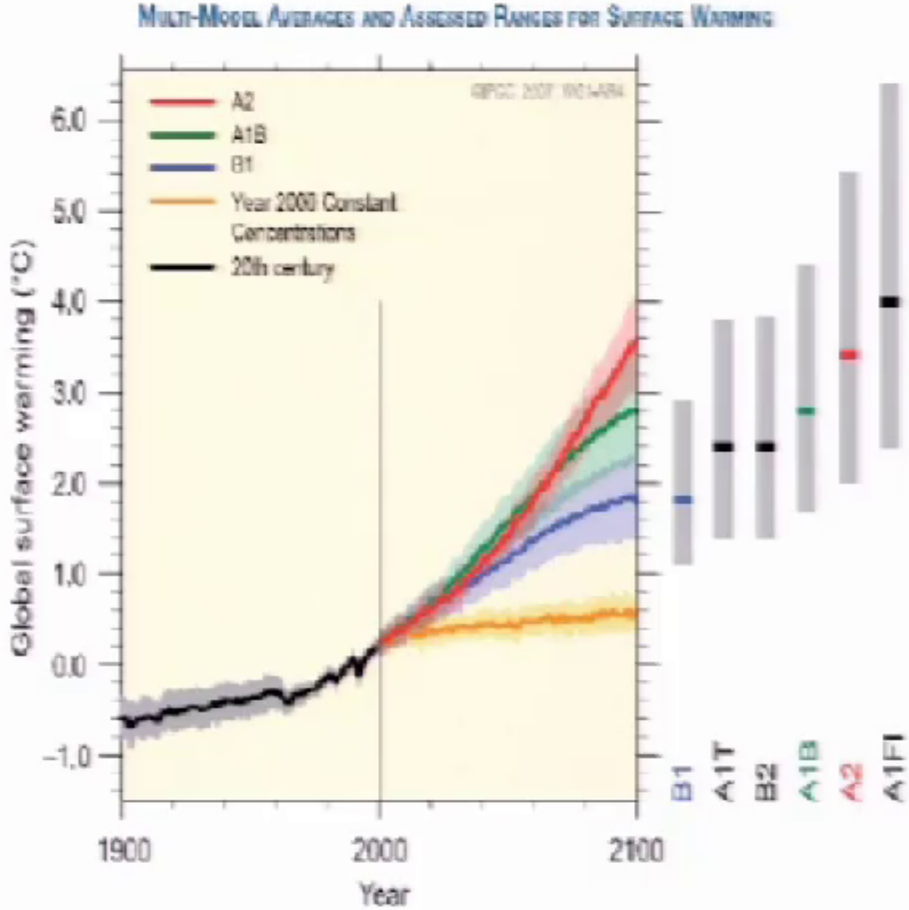


Figure SPM.5. Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the ±1 standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The gray bars at right indicate the best estimate (solid line within each bar) and the likely range assessed for the six SRES marker scenarios. The assessment of the best estimate and likely ranges in the gray bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. {Figures 10.4 and 10.29}

Source : IPCC (2007),
AR4, WGI, SPM

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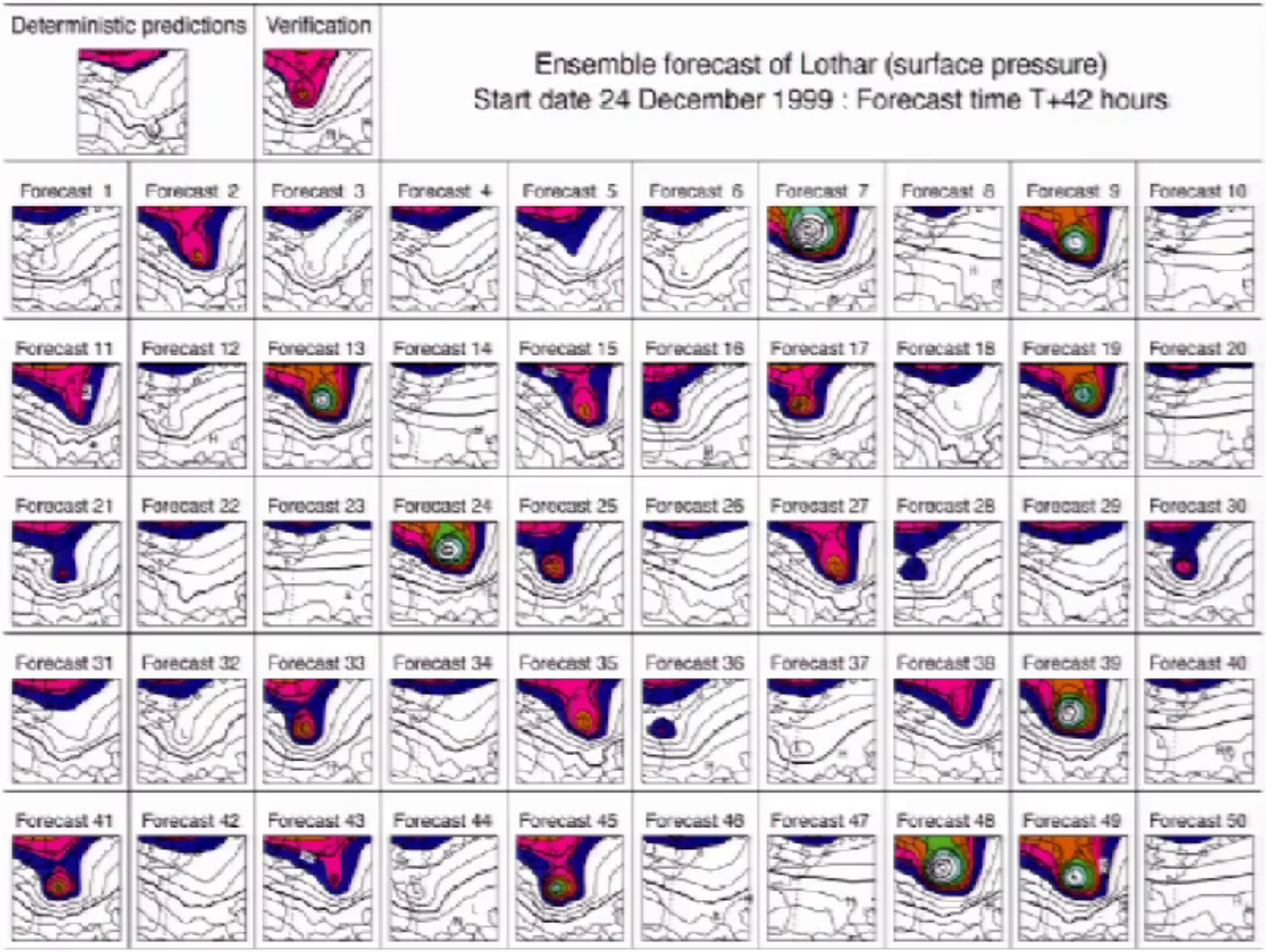


Courtesy Tim Palmer, 2009

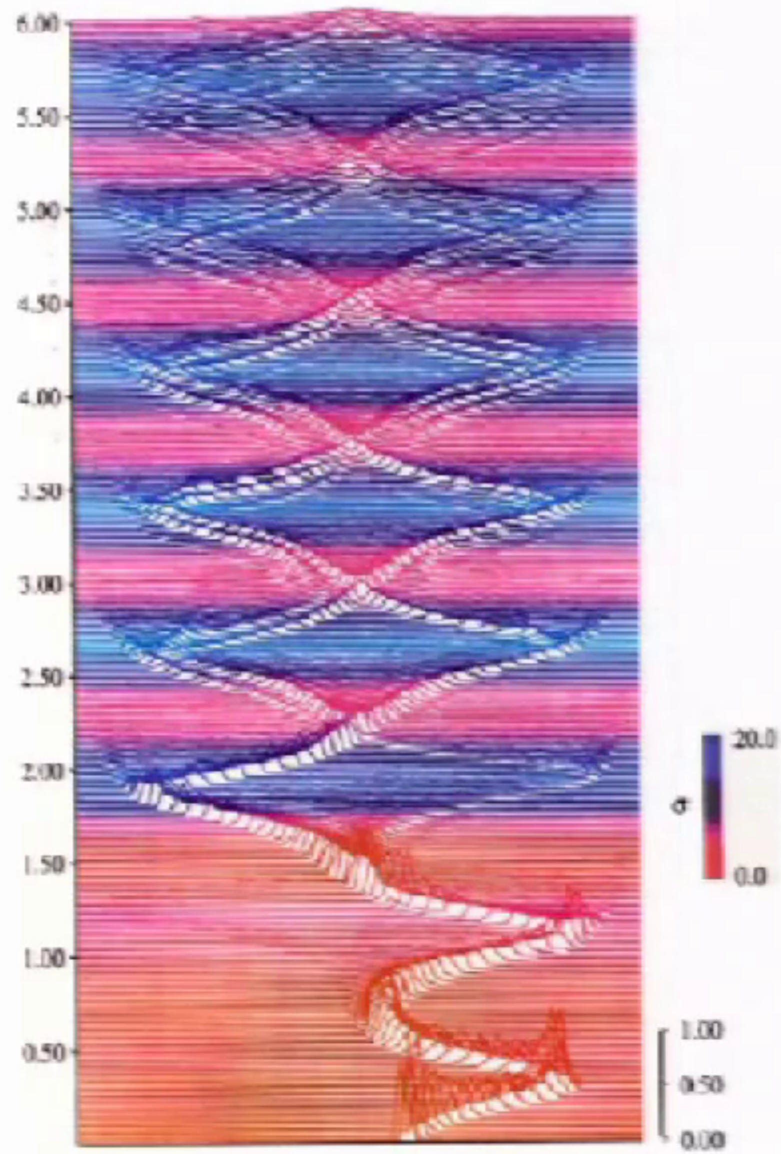
Exponential divergence vs. "coarse graining"

The classical view of dynamical systems theory is:

positive Lyapunov exponent ➔ trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out



L. A. Smith (*Encycl. Atmos. Sci.*, 2003)

So what's it gonna be like, by 2100?

Table SPM.2. *Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend.* {Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9}

Phenomenon[a] and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend[b]	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	*Very likely*[c]	*Likely*[d]	*Virtually certain*[d]
Warmer and more frequent hot days and nights over most land areas	*Very likely*[e]	*Likely* (nights)[d]	*Virtually certain*[d]
Warm spells/heat waves. Frequency increases over most land areas	*Likely*	*More likely than not*[f]	*Very likely*
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	*Likely*	*More likely than not*[f]	*Very likely*
Area affected by droughts increases	*Likely* in many regions since 1970s	*More likely than not*	*Likely*
Intense tropical cyclone activity increases	*Likely* in some regions since 1970	*More likely than not*[f]	*Likely*
Increased incidence of extreme high sea level (excludes tsunamis)[g]	*Likely*	*More likely than not*[f,h]	*Likely*[i]

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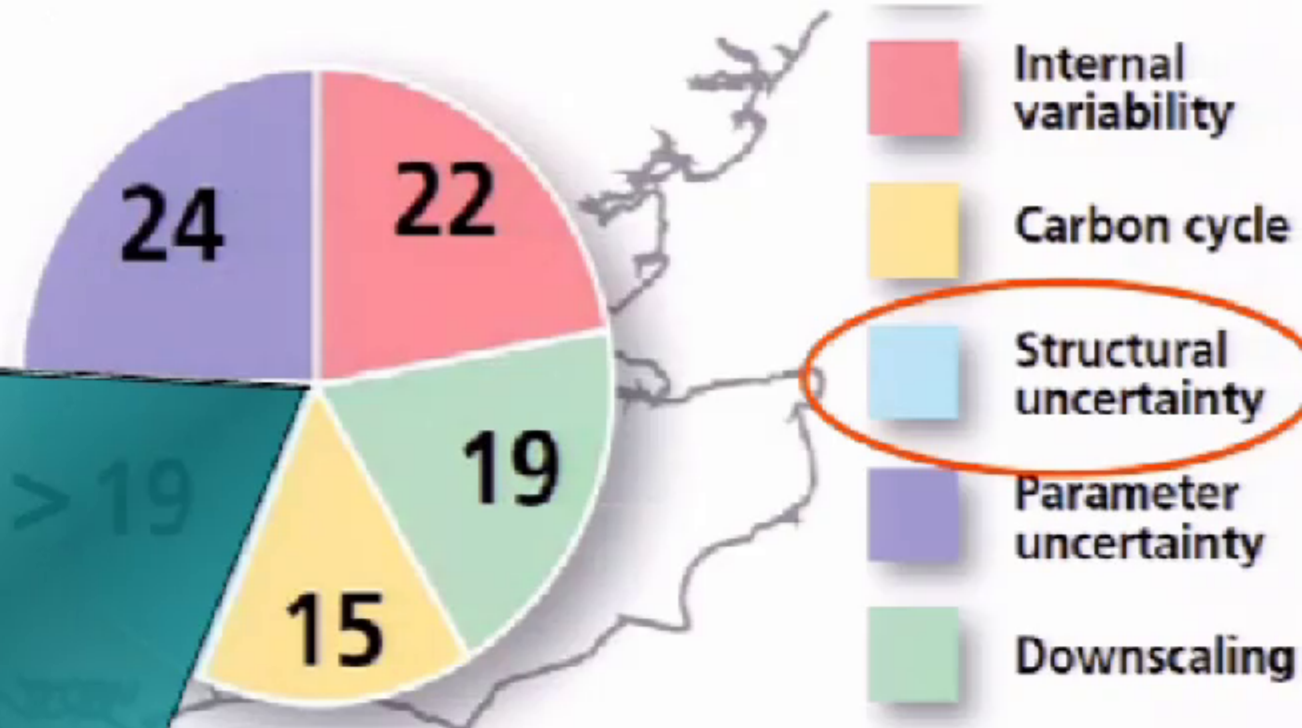
How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



Uncertainties in winter precipitation changes for the 2080s relative to 1961-90, at a 25km box in SE England

Source: Met Office

Can we, nonlinear dynamicists, help?

The uncertainties might be ***intrinsic***, rather than mere "tuning problems"

If so, maybe ***stochastic structural stability*** could help!

Might fit in nicely with recent taste for "stochastic parameterizations"

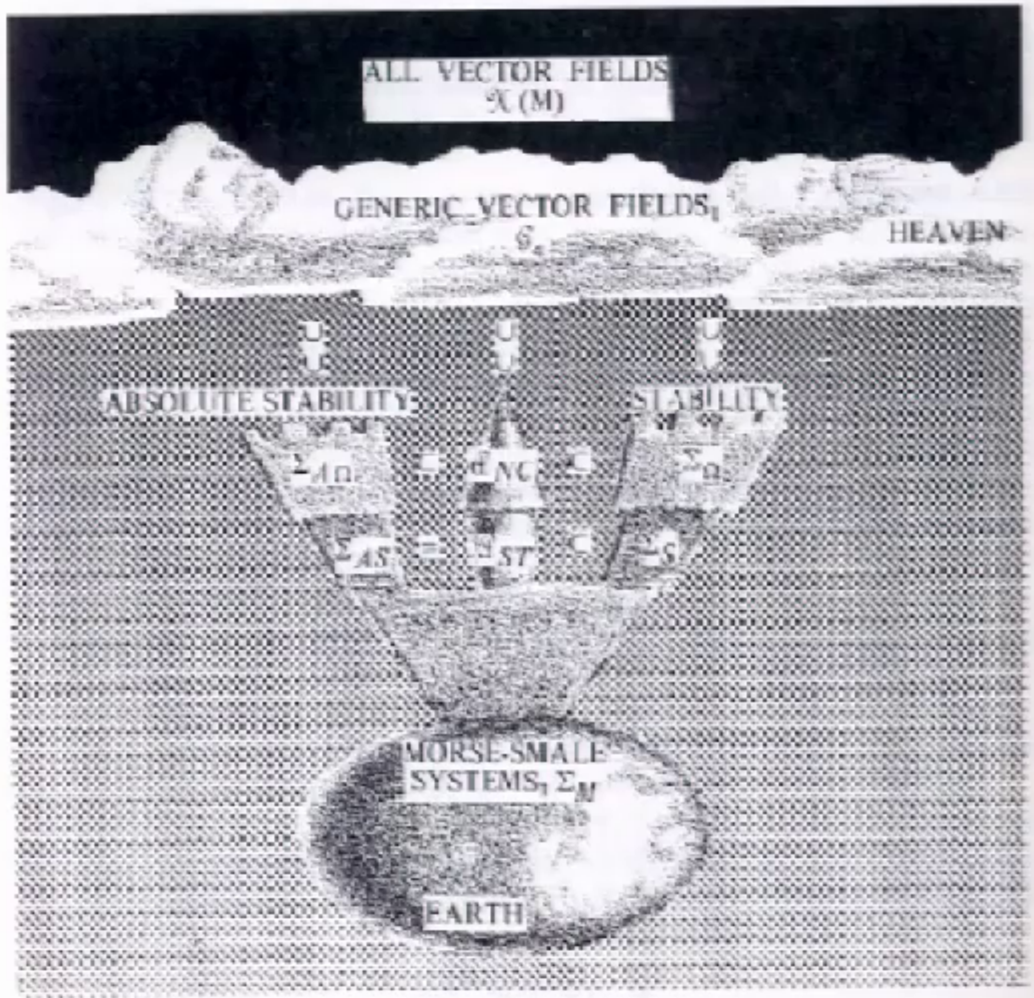


Figure 7.5-1. The three towers of differentiable dynamics.

The DDS dream of structural stability (from Abraham & Marsden, 1978)

Non-autonomous Dynamical Systems

A linear, dissipative, forced example: *forward vs. pullback attraction*

Consider the scalar, linear ordinary differential equation (ODE)

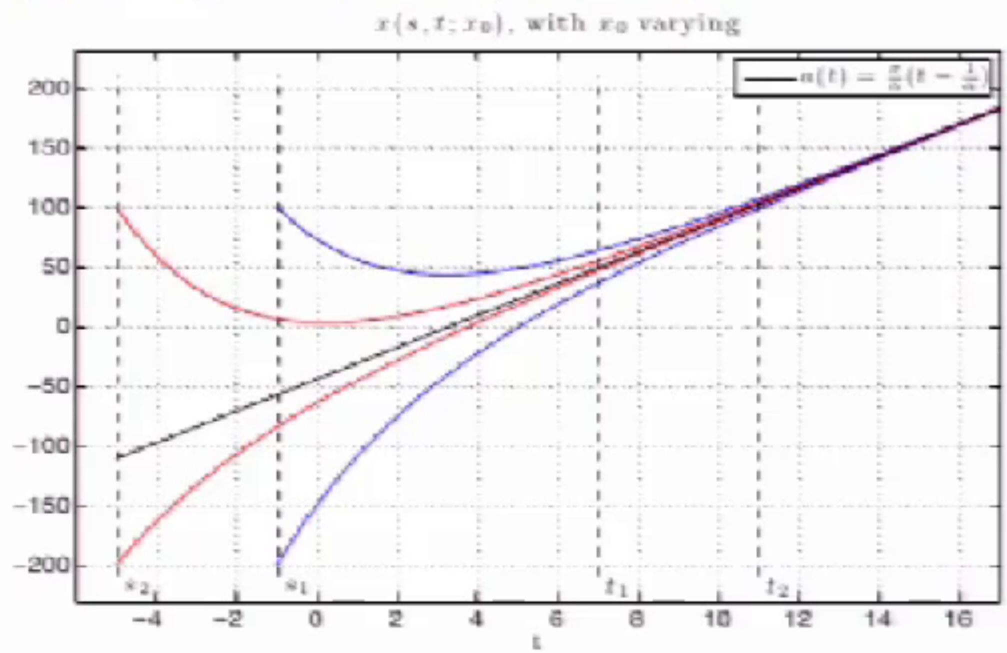
$$\dot{x} = -\alpha x + \sigma t\,, \quad \alpha > 0\,,\ \sigma > 0\,.$$

The autonomous part of this ODE, $\dot{x} = -\alpha x$, is dissipative
and all solutions $x(t; x_0) = x(t; x(0) = x_0)$ converge to 0 as $t \to +\infty$.

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we "pull back" far enough, replace $x(t; x_0)$ by $x(s, t; x_0) = x(s, t; x(s) = x_0)$,

and let $s \to -\infty$, we get th
pullback attractor $a = a(t)$
in the figure,

$$a(t) = \frac{\sigma}{\alpha}\left(t - \frac{1}{\alpha}\right).$$



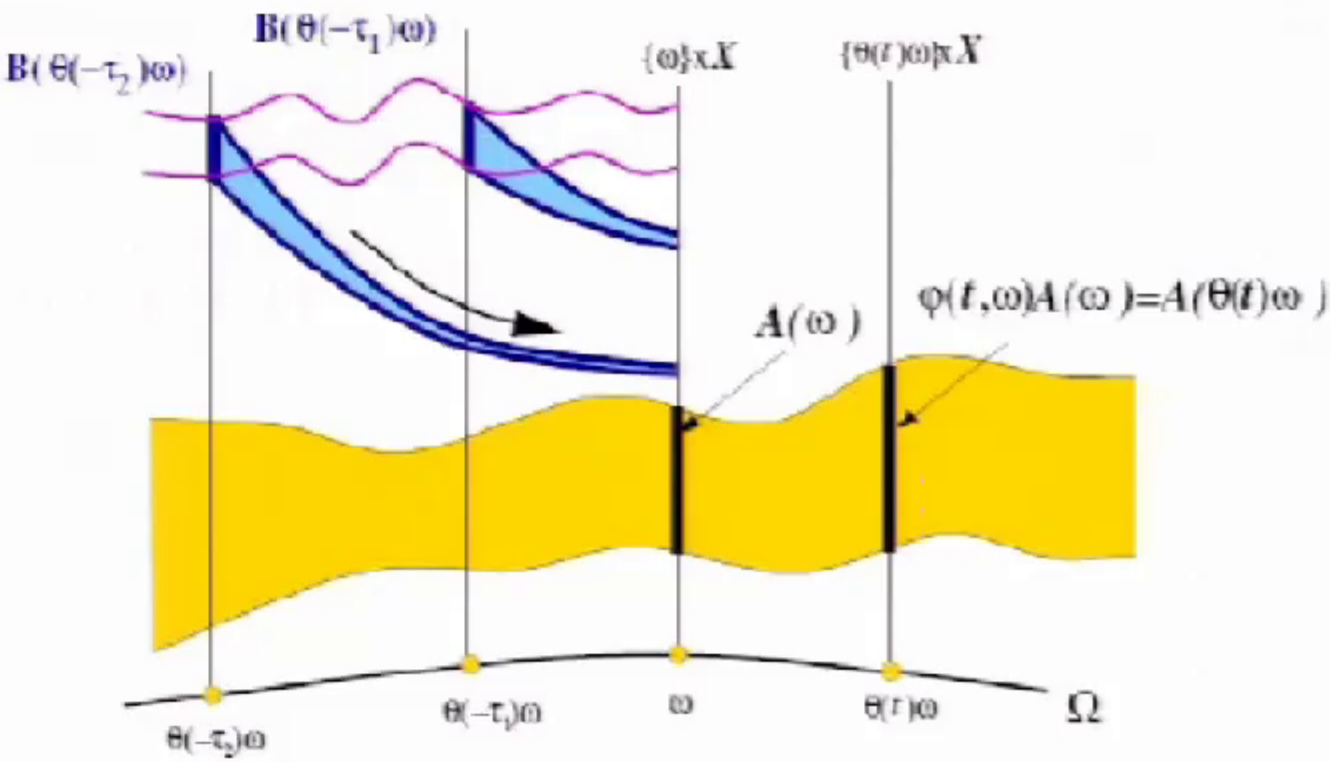
RDS, III- Random attractors (RAs)

A random attractor $\mathcal{A}(\omega)$ is both *invariant* and "pullback" *attracting*:

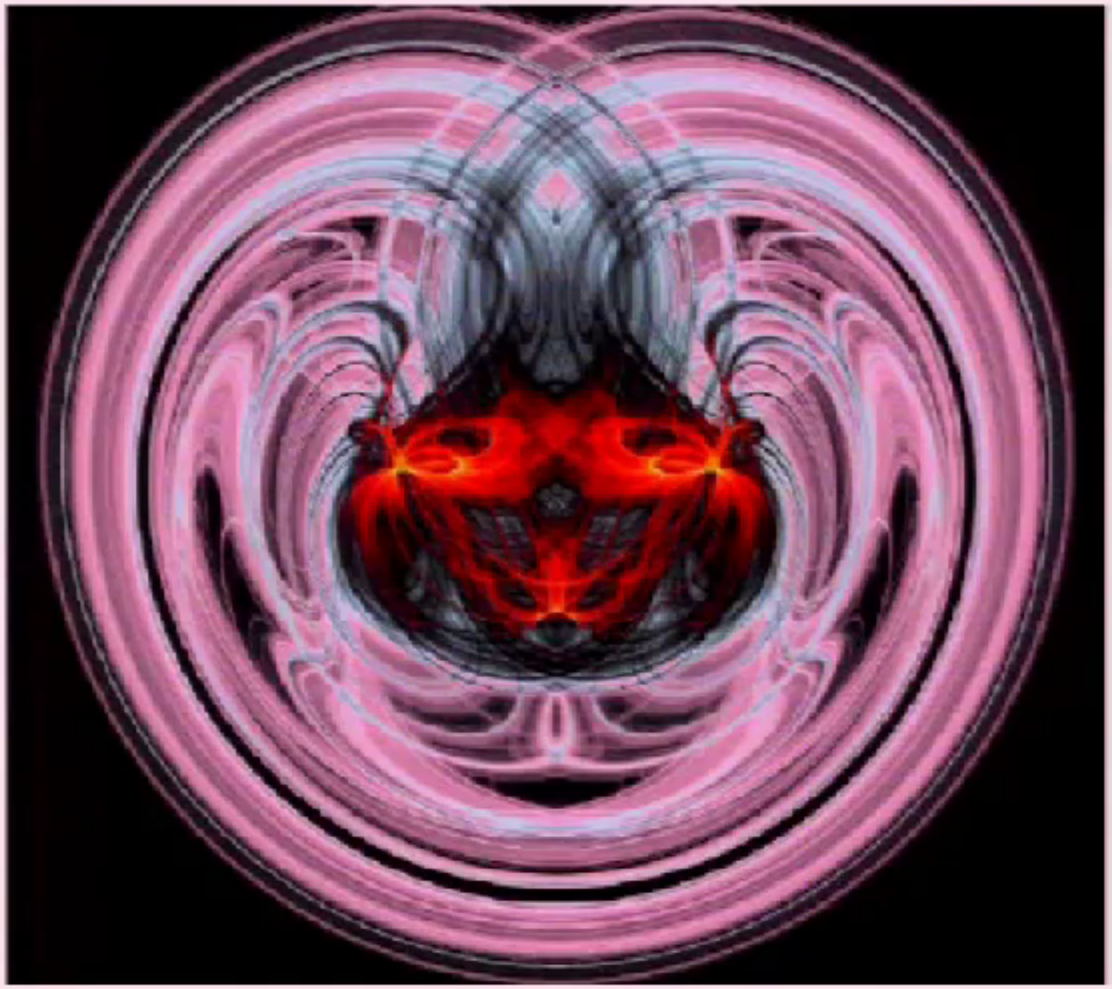
(a) **Invariant**: $\varphi(t,\omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$.

(b) **Attracting**: $\forall B \subset X$, $\lim_{t\to\infty} \mathrm{dist}(\varphi(t,\theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$ a.s.

Pullback attraction to A(ω)



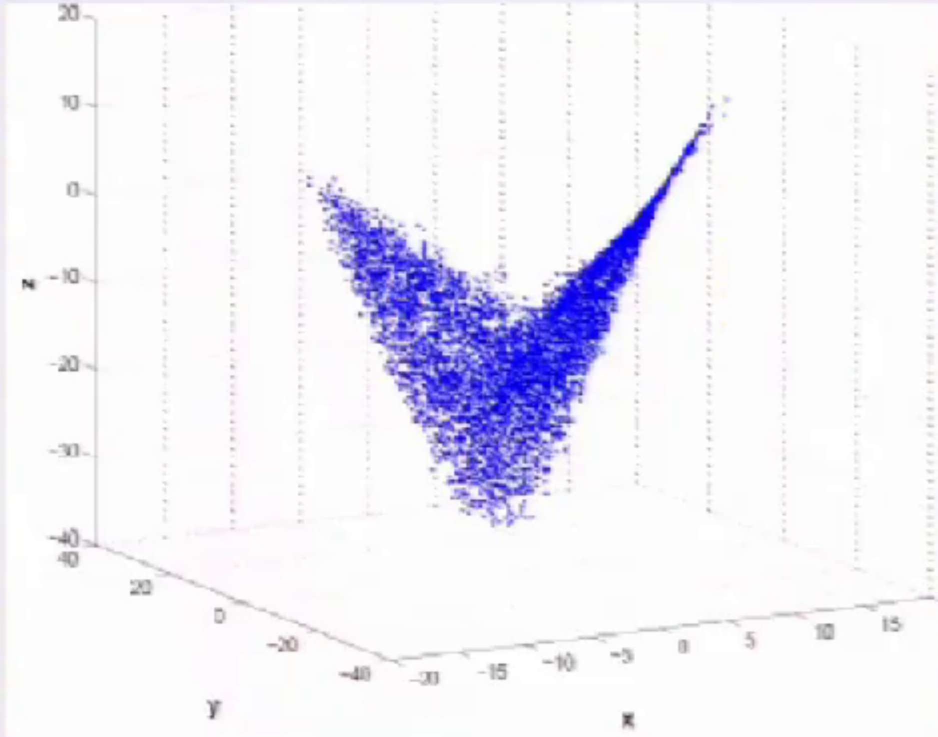
Sample measure supported by the R.A.



- Still 1 Billion I.D., and $\alpha = 0.5$. Another one?

Random attractor of the stochastic Lorenz system

Snapshot of the random attractor (RA)



- A snapshot of the RA, $\mathcal{A}(\omega)$, computed at a fixed time t and for the same realization ω; it is made up of points transported by the stochastic flow, from the remote past $t - T$, $T >> 1$.
- We use multiplicative noise in the deterministic Lorenz model, with the classical parameter values $b = 8/3$, $\sigma = 10$, and $r = 28$.
- Even computed pathwise, this object supports meaningful statistics.

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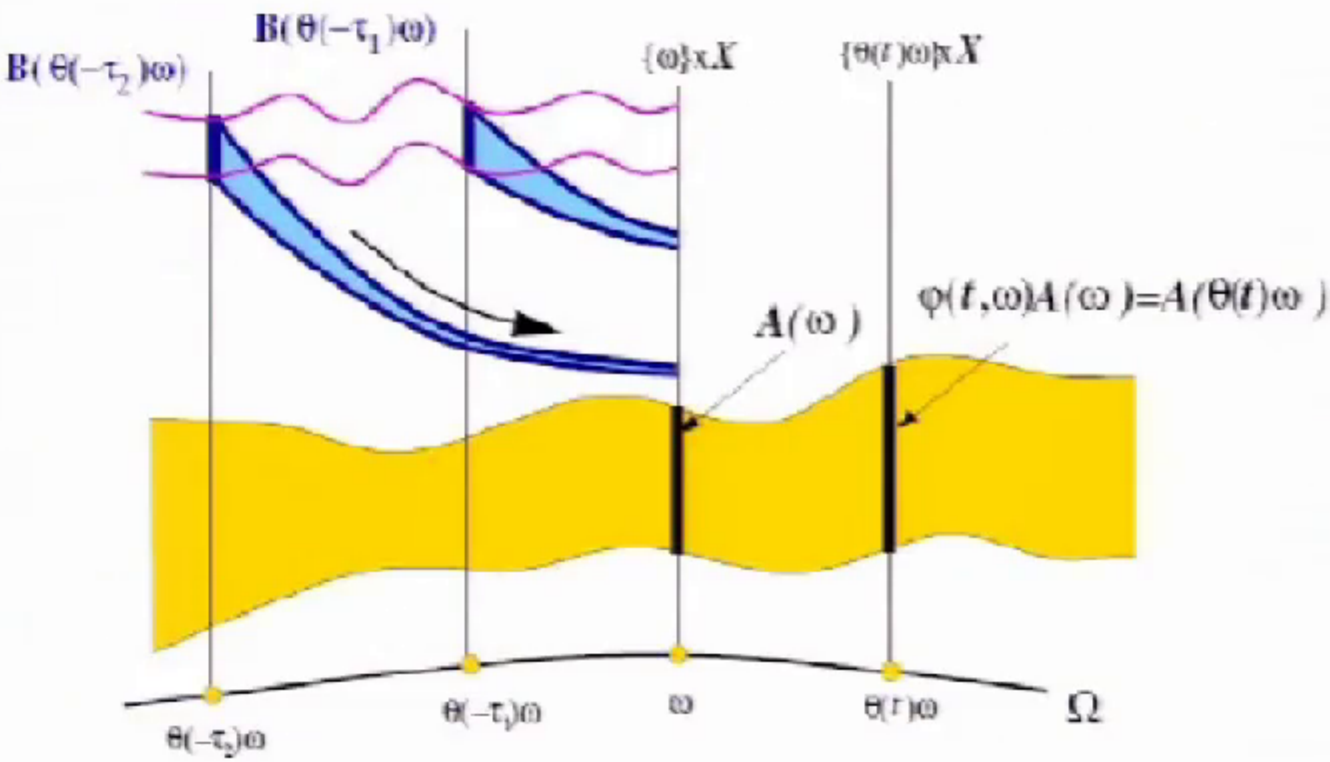

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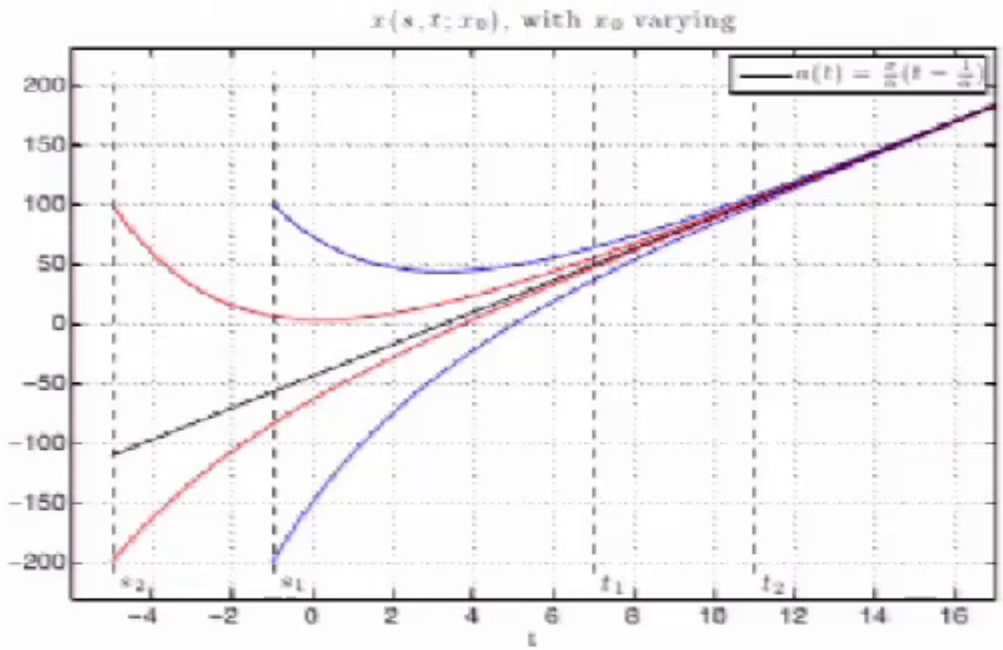

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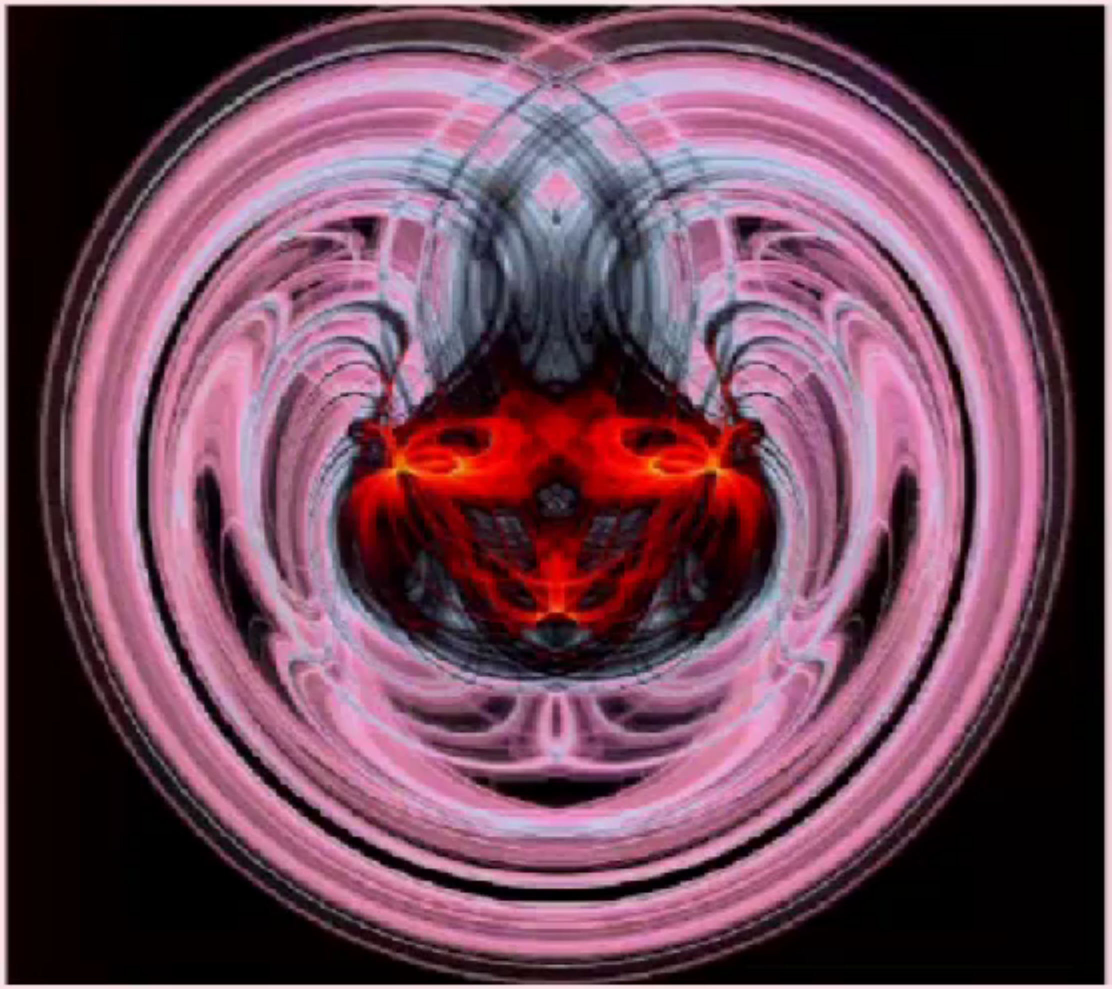

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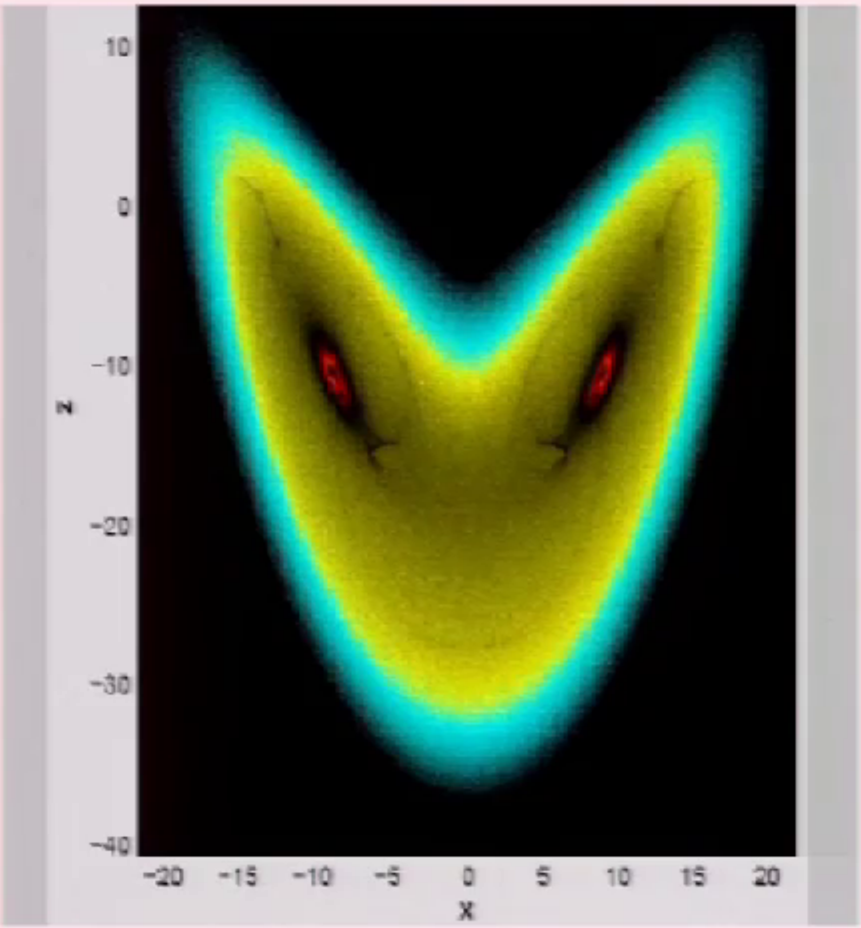
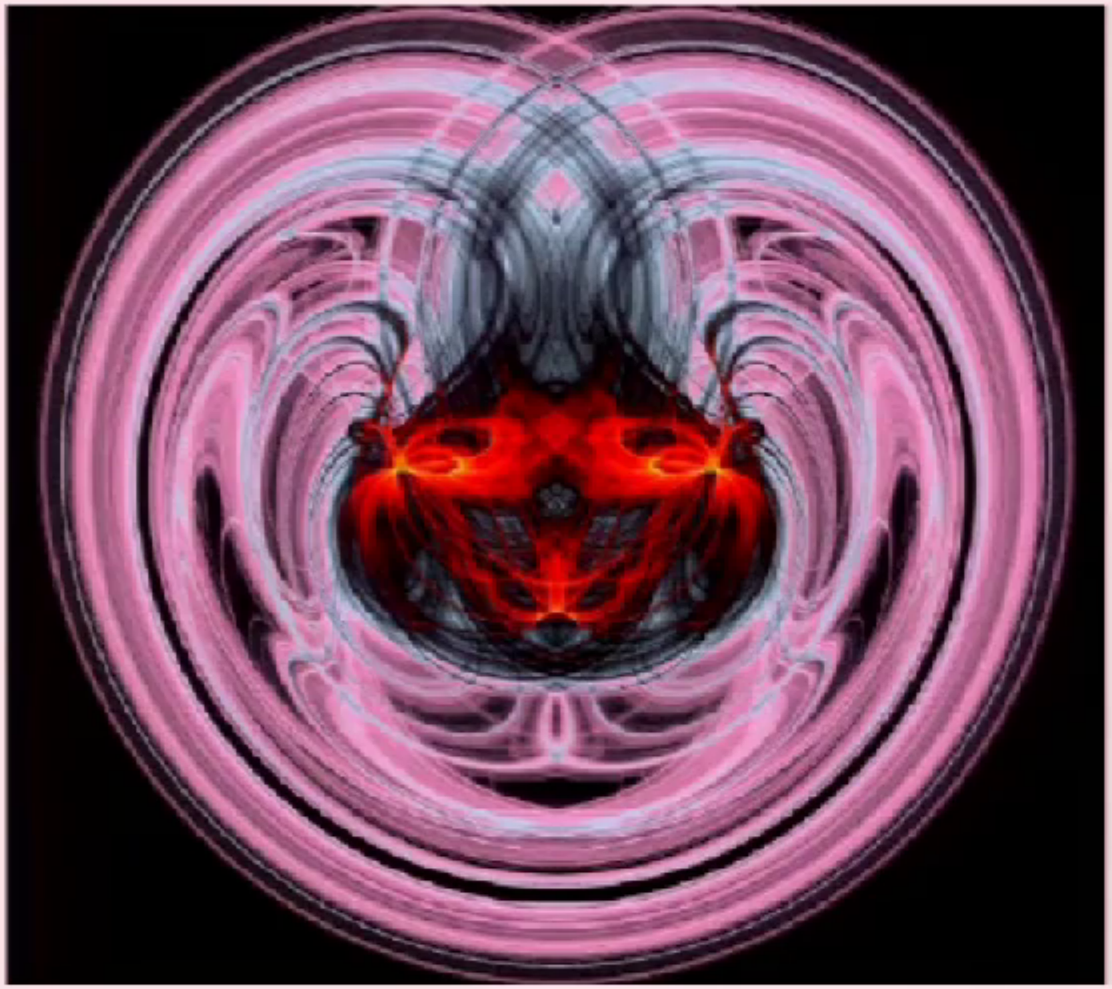
Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time t, and for a fixed realization ω. We show a "projection", $\int \mu_\omega(x, y, z)dy$, with multiplicative noise: dx_i=Lorenz$(x_1, x_2, x_3)dt + \alpha\, x_i dW_t; i \in \{1, 2, 3\}$.
- 10 million of initial points have been used for this picture!

Sample measure supported by the R.A.



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Sample measure supported by the R.A.

Sample measures evolve with time.

- Recall that these sample measures are the frozen statistics at a time t for a realization ω.
- How do these frozen statistics evolve with time?
- Action!

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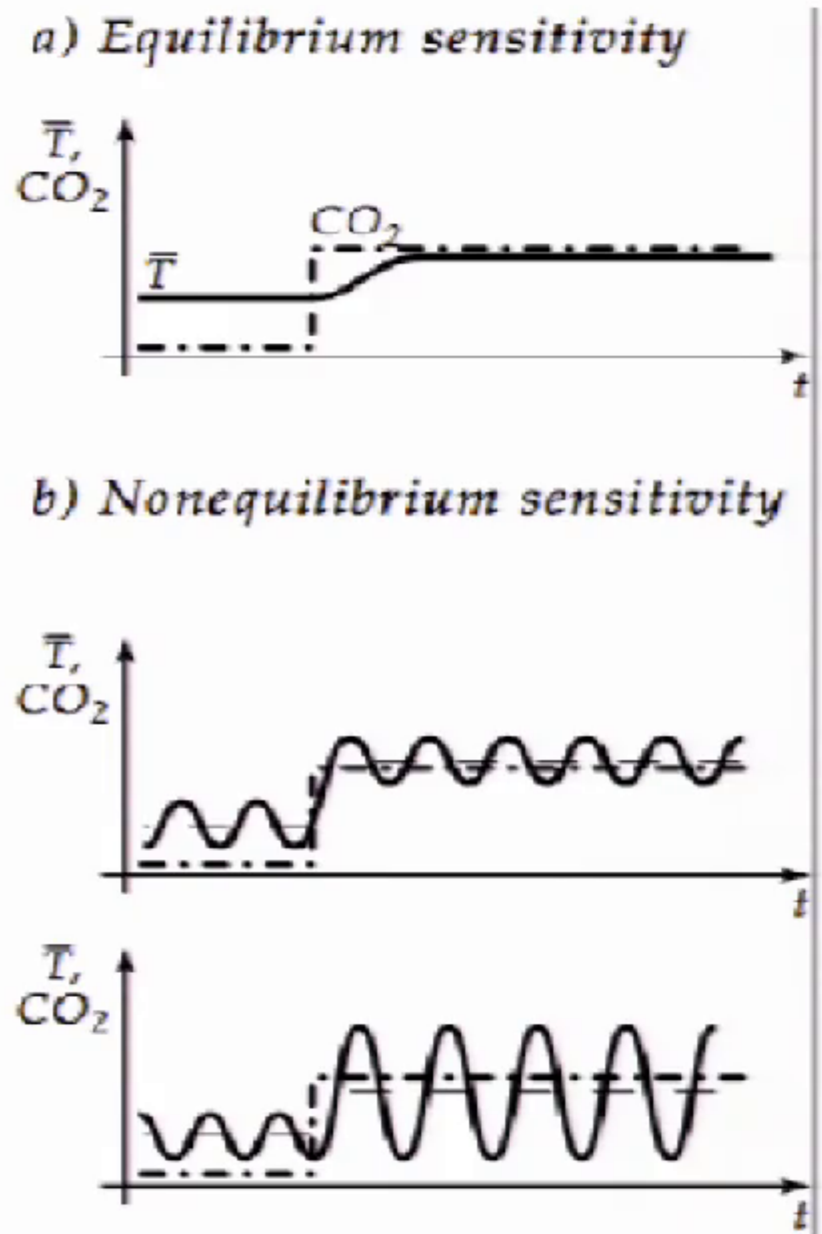
Ghil (Encycl. Global Environmental Change, 2002)

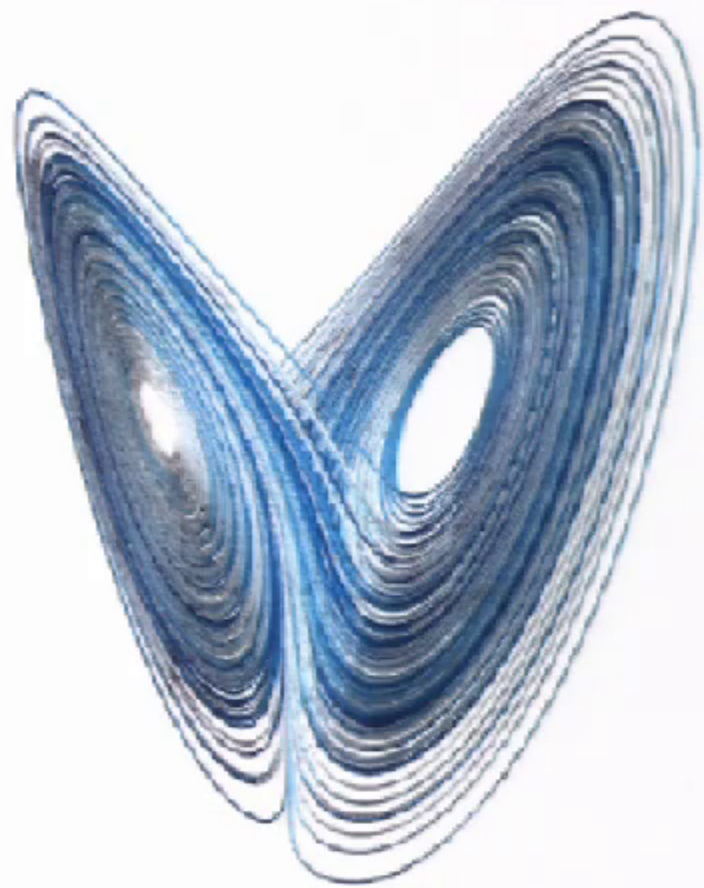


Classical Strange Attractor

Physically closed system, modeled mathematically as autonomous system: neither deterministic (anthropogenic) nor random (natural) forcing.

The attractor is strange, but still fixed in time ~ "irrational" number.

Climate sensitivity ~ change in the average value (first moment) of the coordinates (x, y, z) as a parameter λ changes.



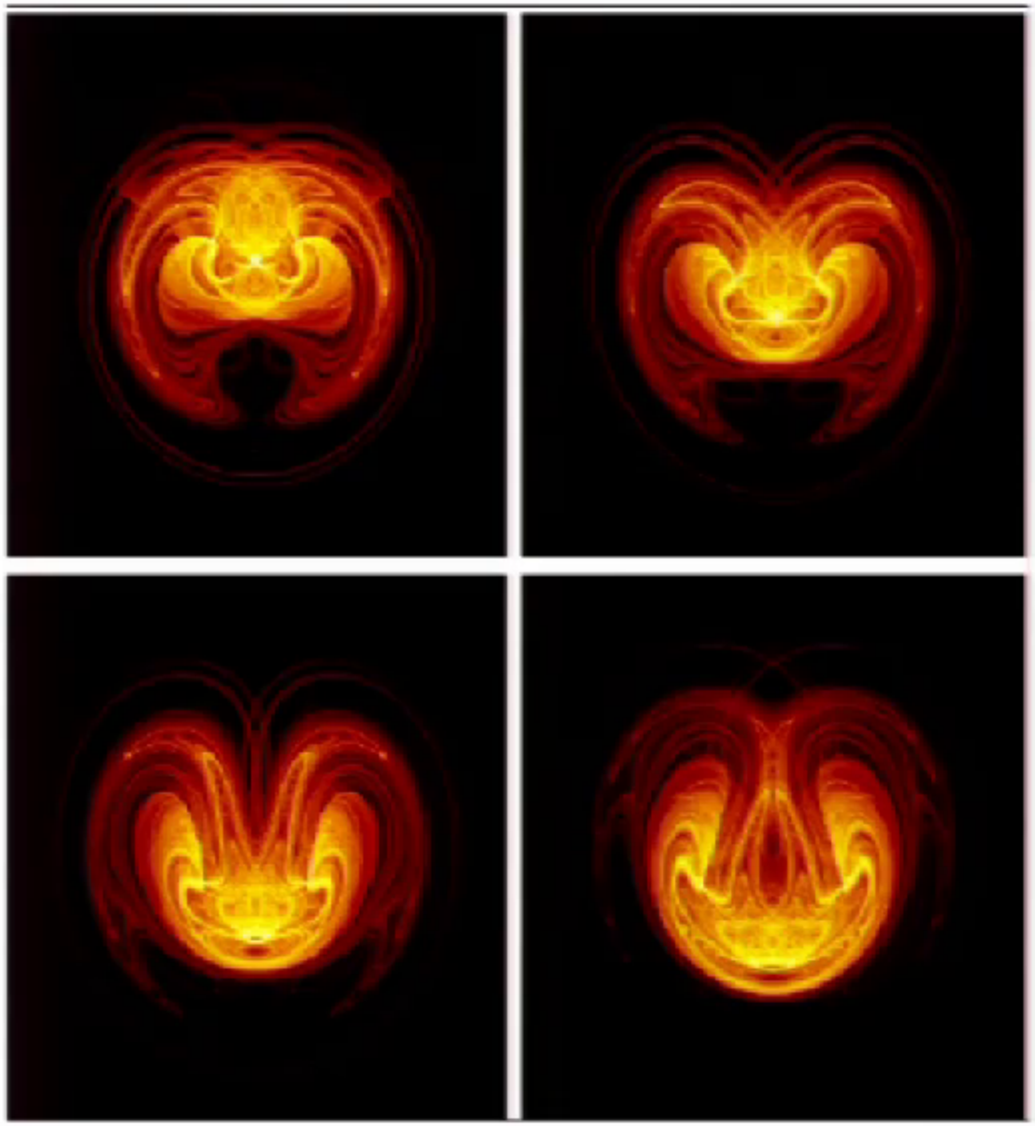
Random Attractor

Physically open system, modeled mathematically as non-autonomous system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The attractor is "pullback" and evolves in time ~ "imaginary" or "complex" number.

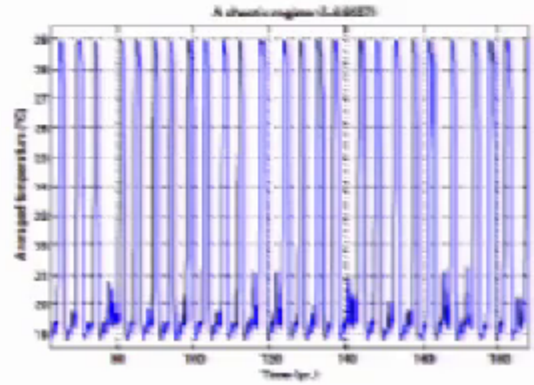
Climate sensitivity ~ change in the statistical properties (first and higher-order moments) of the attractor as one or more parameters (λ, μ, ...) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2nd ed., 2012)



Parameter dependence – I

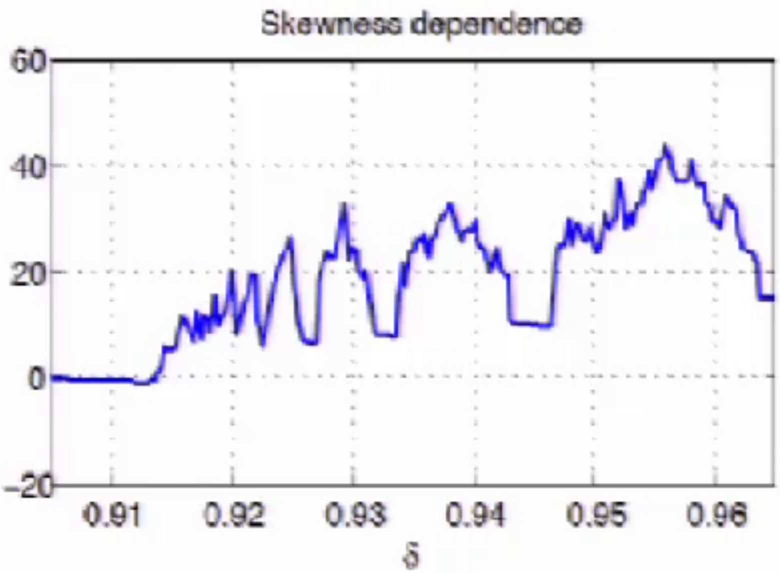
$\delta = 0.9557$

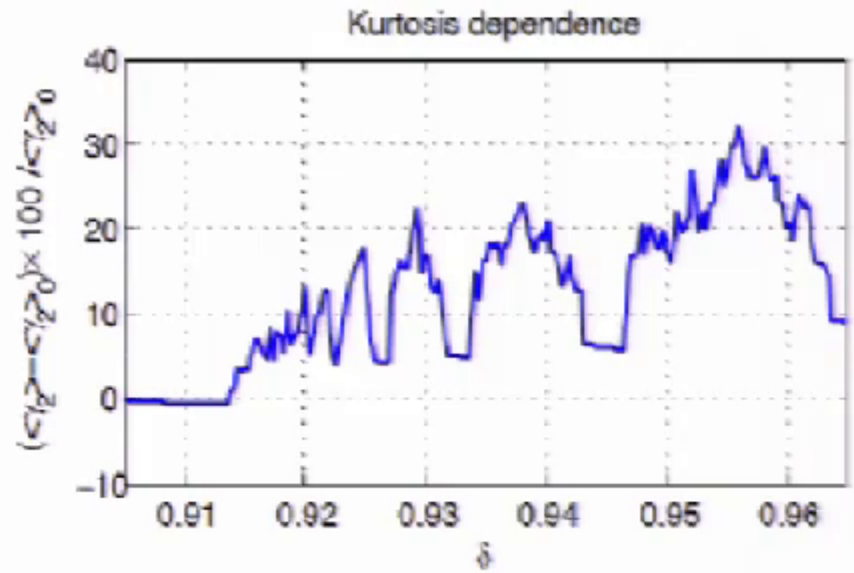


It can be smooth or it can be rough:
Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs:
time series of 4000 years,

$$\Delta\delta = 3 \cdot 10^{-4}$$





M. Chekroun & D. Kondrashov (work in progress)

Sample measures for an NDDE model of ENSO

The Galanti-Tziperman (GT) model (*JAS*, 1999)

$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

$$h(t) = M_1 e^{-\epsilon_m(\tau_1+\tau_2)} h(t-\tau_1-\tau_2) - M_2\tau_1 e^{-\epsilon_m(\frac{\tau_1}{2}+\tau_2)}\mu(t-\tau_2-\frac{\tau_1}{2})T(t-\tau_2-\frac{\tau_1}{2}) + M_3\tau_2 e^{-\epsilon_m\frac{\tau_2}{2}}\mu(t-\frac{\tau_2}{2})T(t-\frac{\tau_2}{2}).$$

Neutral delay-differential equation (NDDE), derived from Cane-Zebiak and Jin-Neelin models for ENSO: *T* is East-basin SST and *h* is thermocline depth.

Seasonal forcing given by $\mu(t) = 1 + \epsilon\cos(\omega t + \phi)$. The pullback attractor and its invariant measures were computed.

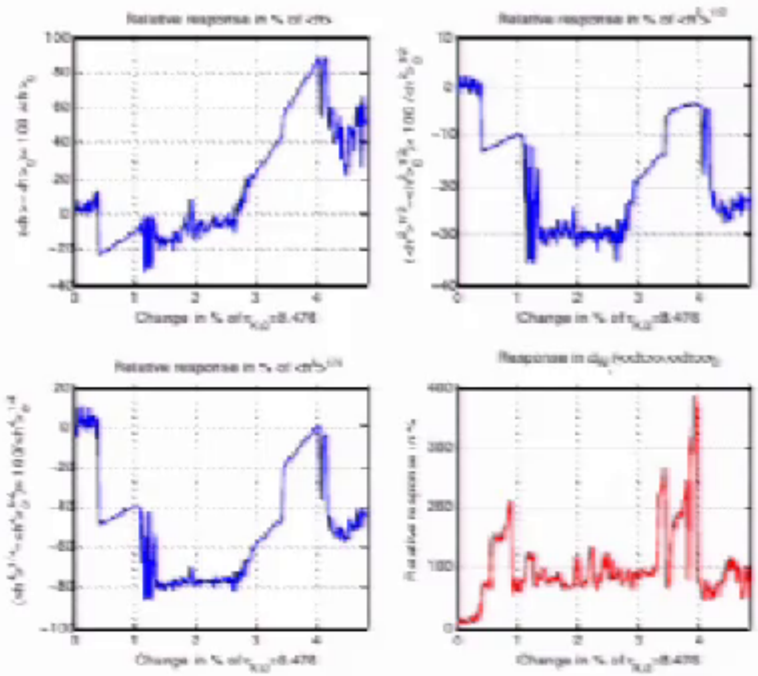


Figure shows the changes in the mean, 2nd & 4th moment of $h(t)$, along with the Wasserstein distance d_W, for changes of 0–5% in the delay parameter $\tau_{\kappa,0}$.

Note intervals of both smooth & rough dependence!

Parameter dependence – I

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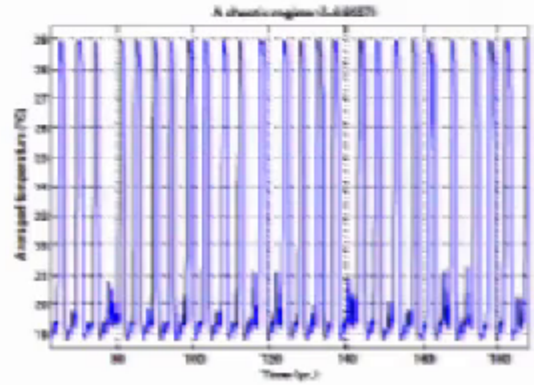

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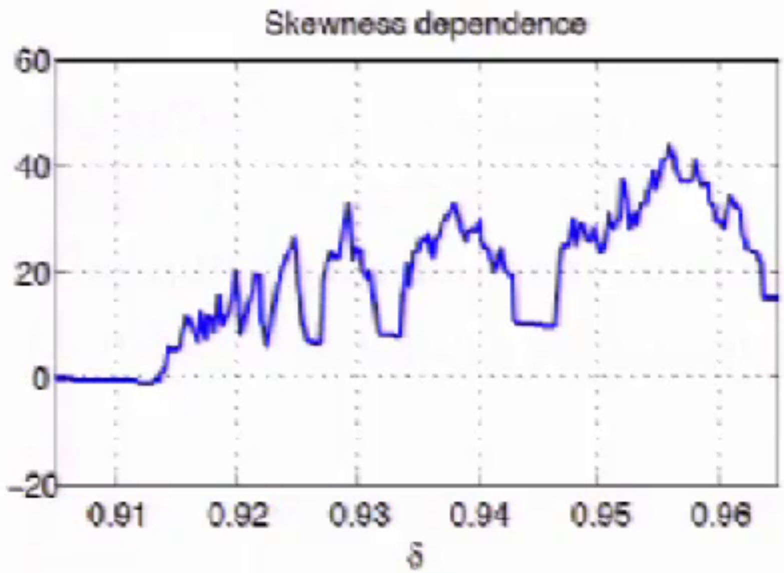

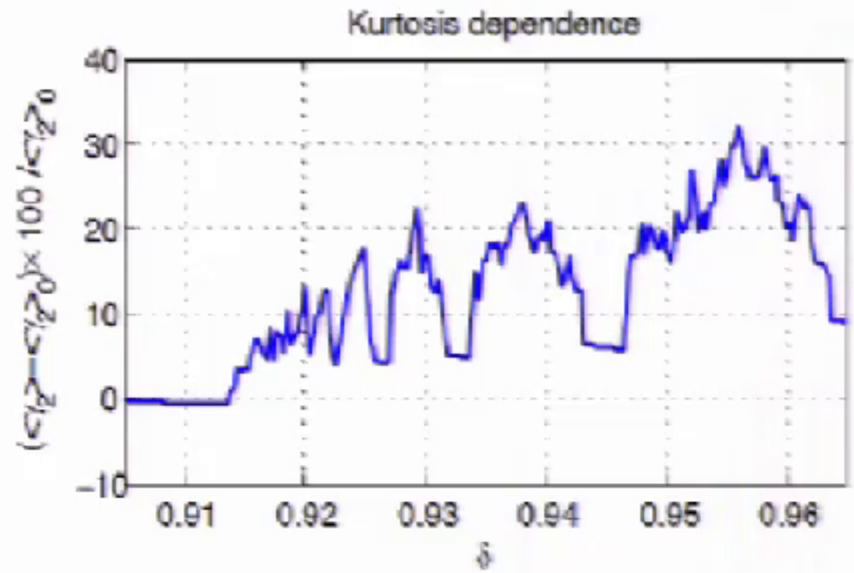

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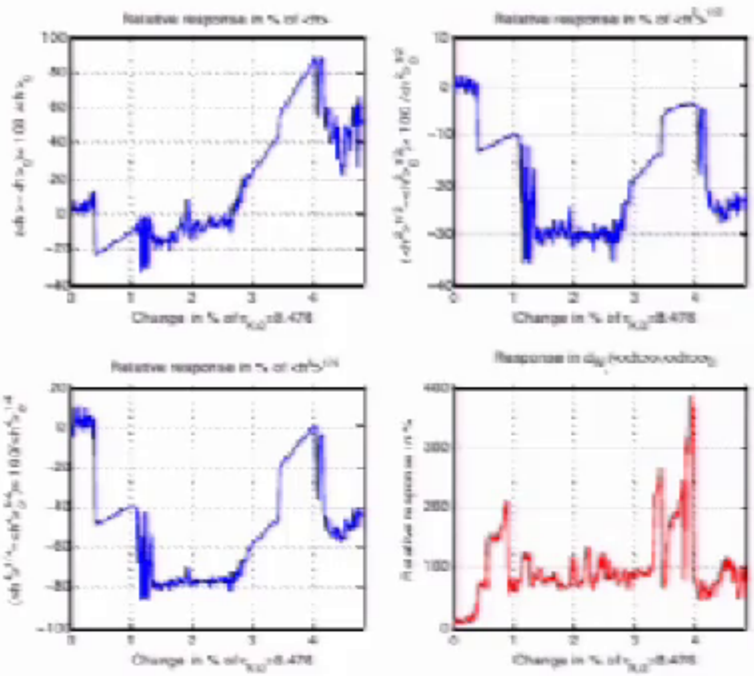

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Pullback attractor and invariant measure of the GT model

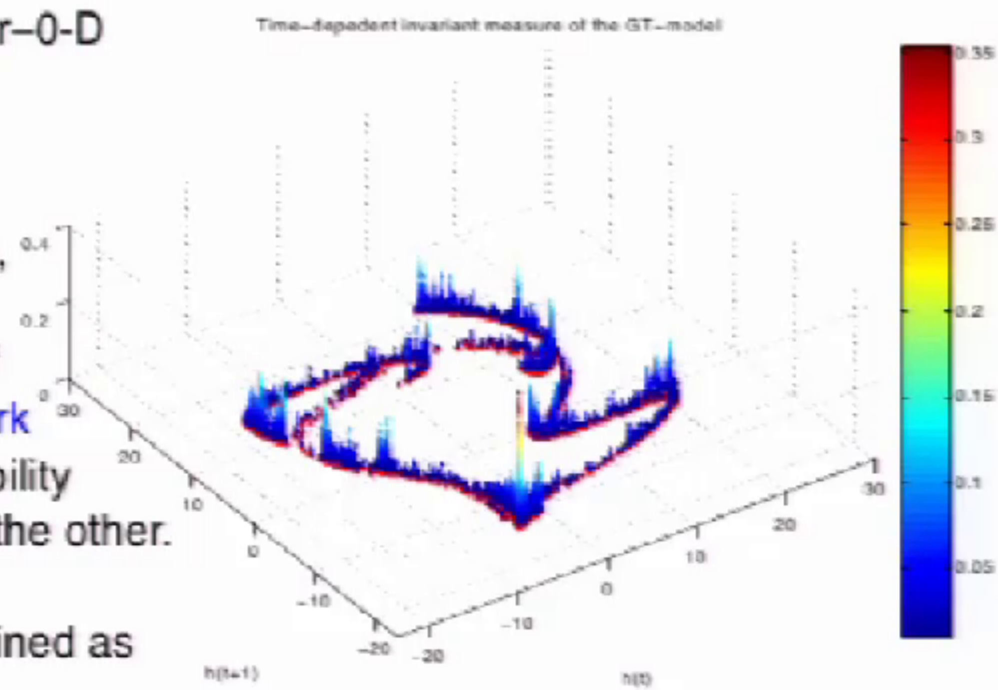
The time-dependent pullback attractor of the GT model supports an invariant measure $\nu = \nu(t)$, whose density is plotted in 3-D perspective.

The plot is in delay coordinates *h*(*t*+1) *vs.* *h*(*t*) and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near–0-D peaks on these filaments.

The Wasserstein distance d_W between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

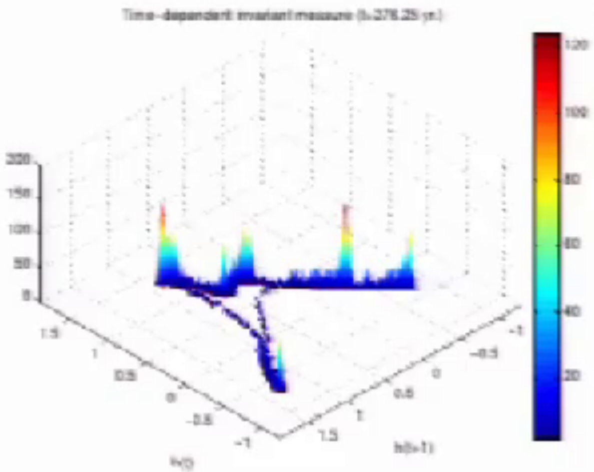
Climate sensitivity γ can be defined as

$$\gamma = \partial d_W / \partial \tau$$

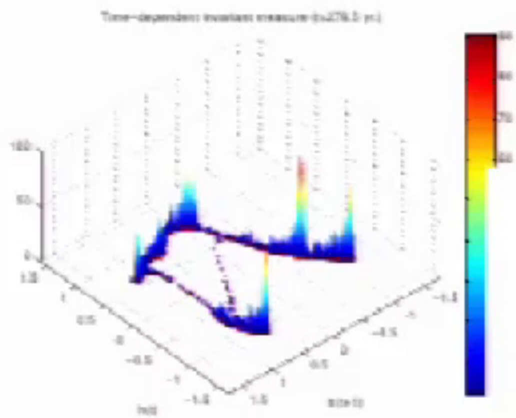


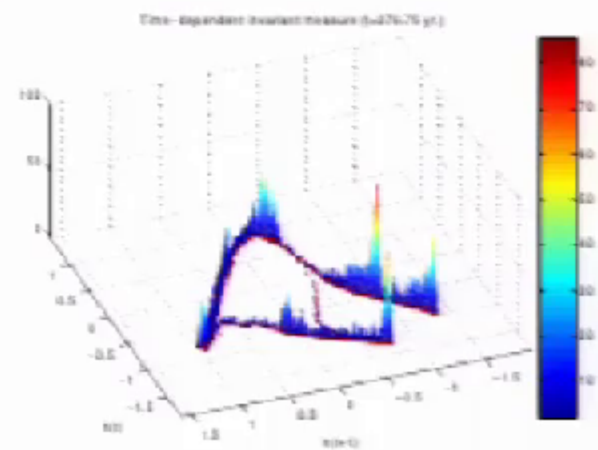
How to define climate sensitivity or, What happens when there's natural variability?

This definition allows us to watch how "the earth moves," as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:



$$\gamma = \partial d_{\mathrm{W}} / \partial \tau$$





Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Nonequilibrium climate sensitivity
- Pull vs. snap: a tale of two (kinds of) attractors
- Conclusions and references
  - natural variability and anthropogenic forcing: the "grand unification"
  - selected bibliography

Conjectures

- Snapshot attractors [*] approximate the mathematically rigorous pullback attractor [**].
- The convergence time of orbits started from a set of initial states to the pullback attractor is characterized by the system's least-negative Lyapunov exponent.
- Moreover, when measuring the convergence of the invariant measures by Wasserstein distance D_W, one has the following estimate on the lagged autocorrelations:

$$C(\tau) \leq \text{const} \times D_{\mathrm{W}}(\rho, \tilde{\rho}_\tau)$$

Here ρ is the sample measure on the pullback attractor, and $\tilde{\rho}_\tau$ is the sample measure on the τ-pullback attractor.

[*] Romeiras, Grebogi & Ott (*Phys. Rev. A*, 1990), Tél & colleagues — snapshot;
[**] Sell (*Trans. AMS*, 1967), L.-S. Young (*JSP*, 2002, etc.) — pullback & random.

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Concluding remarks, I – RDS and RAs

Summary

- A change of paradigm from **closed, autonomous systems** to **open, non-autonomous ones.**
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

Work in progress

- Study the effect of specific stochastic parametrizations on model robustness.
- Applications to intermediate models and GCMs.
- Implications for climate sensitivity.
- Implications for predictability?

Yet another (grand?) unification

Lorenz (*JAS*, 1963)

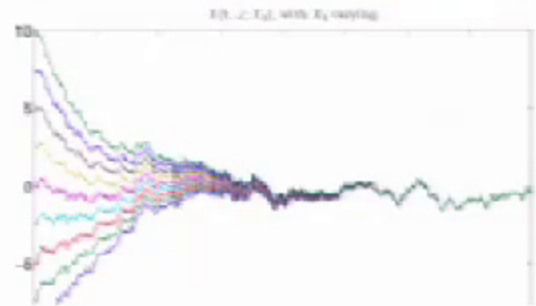
Climate is deterministic and autonomous,
but highly nonlinear.

Trajectories diverge exponentially,
forward asymptotic PDF is multimodal.

Hasselmann (*Tellus*, 1976)

Climate is stochastic and noise-driven,
but quite linear.

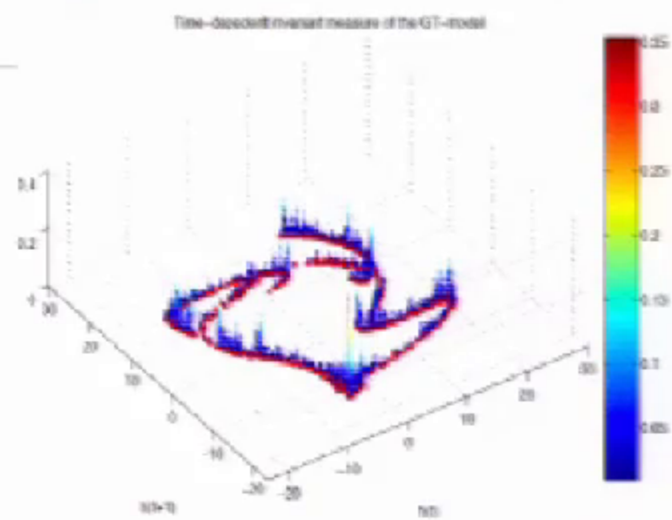
Trajectories decay back to the mean,
forward asymptotic PDF is unimodal.



Grand unification (?)

**Climate is deterministic + stochastic,
as well as highly nonlinear.**

**Internal variability and forcing interact
strongly, change and sensitivity
refer to both mean and higher moments.**



Concluding remarks, II –
Climate change & climate sensitivity

What do we know?

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

What do we know less well?

- By how much?
  - Is it getting warmer …
  - Do we contribute to it …
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

What to do?

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability.
  - linear response = response function + susceptibility function!
  - beyond linear response → use Wasserstein distance!!