Eidgenössische Technische Hochschule Zürich
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Objective Eulerian Coherent Structures

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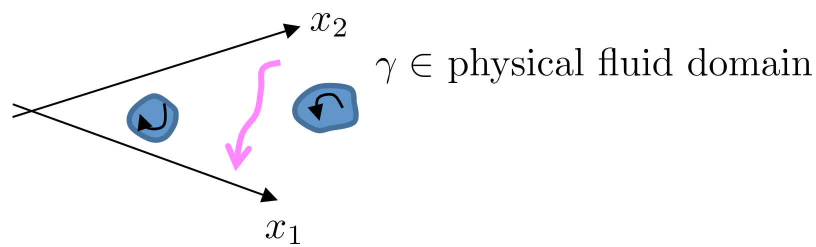
joint work with: George Haller (ETH)

Different approaches to coherent structures detection (next talk)

$$\dot{x} = v(x,t), \qquad x(t,t_0,x_0) := F_{t_0}^t(x_0)$$



Geometric methods

- Different types of coherence
- Codimension-1 material structures
- Effective over short times
- Requires data on a dense grid

Set-based methods

- Vortex-type structures
- Focus on the interior of coherent regions
- Work with sparse data

Diagnostic methods

Objective Eulerian Coherent Structures

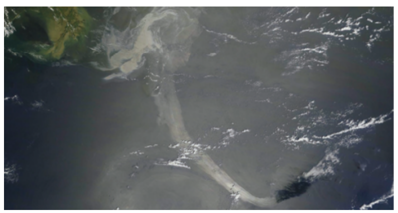
From the asymptotic to the instantaneous limit

stability

stable and unstable manifolds | Lagrangian Coherent Structures | ?

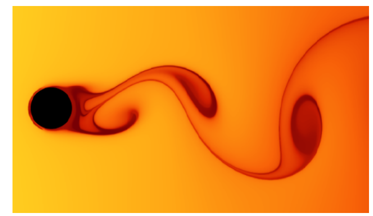
$t - t_0 \to \infty$ | $t - t_0 = T$ | $t - t_0 \to 0$

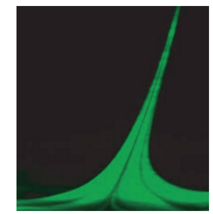
Motivation



Oil spill

Search & Rescue





Flow control

Challenges

- Multiple and **unknown time scales** → **Which** integration time **T=t-t$_0$?**
- **Finite-time** and **Finite-size** data

Need

- **Real-time** identification of the **Eulerian skeleton** of the flow
- **Objectivity** → self-consistent prediction for material transport

Objective deformation measures

Set-up and notation

$$\dot{x} = v(x,t),\ x \in U \subset \mathbb{R}^2,\ t \in [a,b]$$

Lagrangian (t-t_0=T)

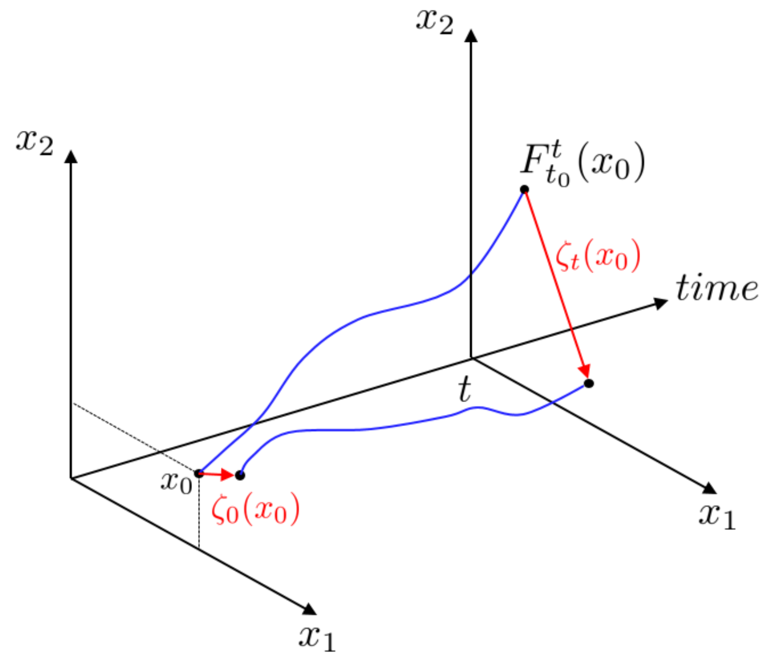
$$F_{t_0}^t(x_0) = x(t;x_0,t_0)$$

$$F_{t_0}^t(x_0+\zeta_0(x_0)) = F_{t_0}^t(x_0)\zeta_0(x_0) + \nabla F_{t_0}^t(x_0)\zeta_0(x_0) + \mathcal{O}(|\zeta_0(x_0)|^2)$$

$$\zeta_t(x_0) := \nabla F_{t_0}^t(x_0)\zeta_0(x_0) + \mathcal{O}(|\zeta_0(x_0)|^2)$$

$$|\zeta_t(x_0)| = \sqrt{\langle \zeta_0(x_0), \underbrace{[\nabla F_{t_0}^t(x_0)]^\top \nabla F_{t_0}^t}_{C_{t_0}^t(x_0)}(x_0)\zeta_0(x_0)\rangle}$$

Right Cauchy-Green tensor



Eulerian (t-t_0→0)

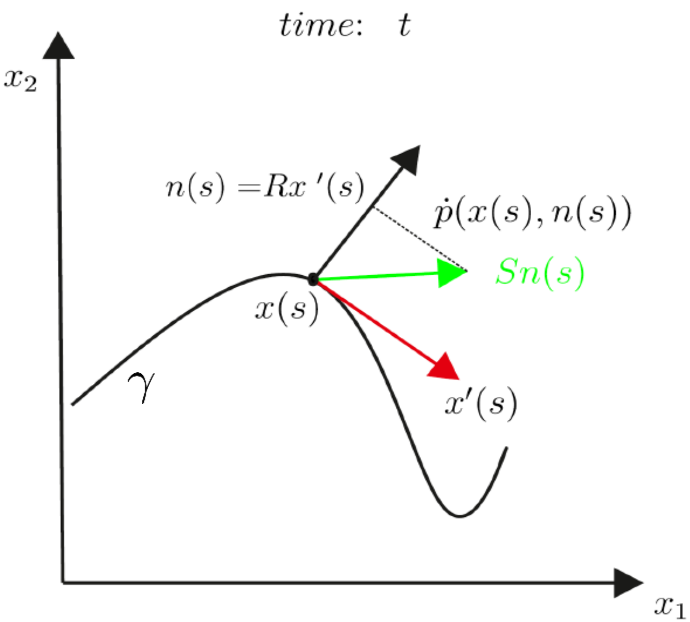
Rate-of-strain tensor $S(x,t)$

$$\frac{d}{dt}|\zeta_t(x_0)|_{t=t_0} = \frac{\langle \zeta_0(x_0), \overbrace{\tfrac{1}{2}([\nabla v(x_0,t_0)]^\top + \nabla v(x_0,t_0))}^{S(x,t)}\zeta_0(x_0)\rangle}{\langle \zeta_0(x_0),\zeta_0(x_0)\rangle}$$

$$\frac{d}{dt}C_{t_0}^t(x_0)|_{t=t_0} = I + 2S(x_0,t_0)(t-t_0) + \mathcal{O}(t-t_0)^2$$

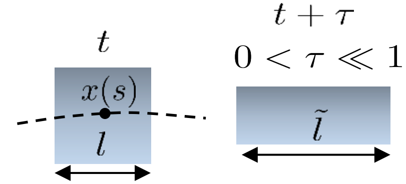
Objective deformation-rate measures

Types of deformation



Material stretch rate

$$\dot{q}(x, x', t) := \frac{\langle x', S(x,t)x' \rangle}{\langle x', x' \rangle}$$



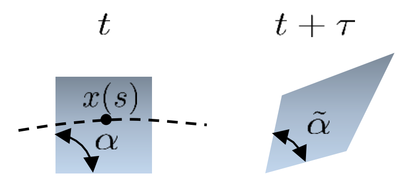
$$\dot{q}(x, x', t) = \frac{d}{dt} \frac{(\tilde{l} - l)}{l}$$

Averaged material stretch rate

$$\dot{Q}_t(\gamma) := \int_\gamma \dot{q}(x, x', t) ds$$

Material shear rate

$$\dot{p}(x, x', t) := \frac{\langle x', 2S(x,t)Rx' \rangle}{\langle x', x' \rangle}$$



$$\dot{p}(x, x', t) = \frac{d}{dt}\alpha$$

Averaged material shear rate

$$\dot{P}_t(\gamma) := \int_\gamma \dot{p}(x, x', t) ds$$

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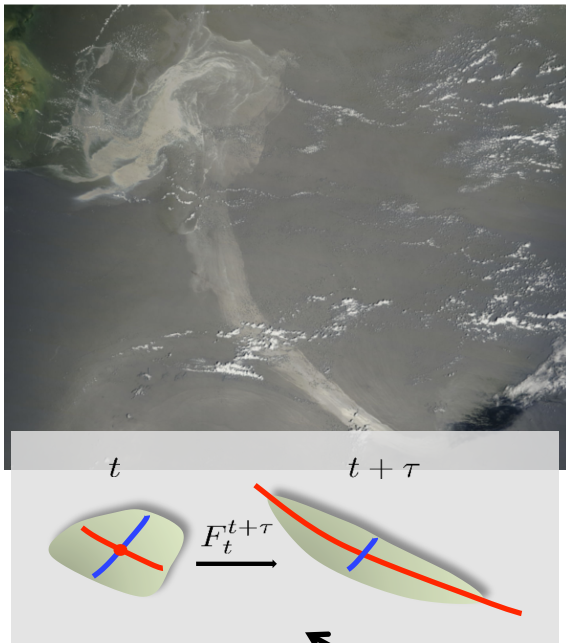
Objective Eulerian Coherent Structures

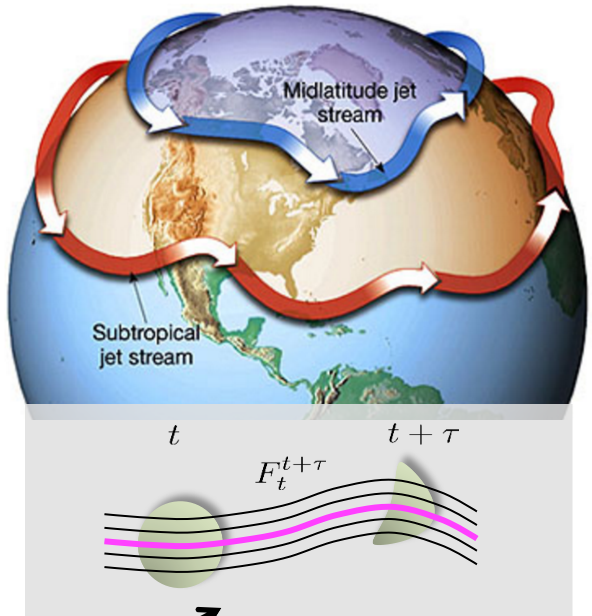
Different types of coherence

$$\lim_{T \to 0} LCSs$$

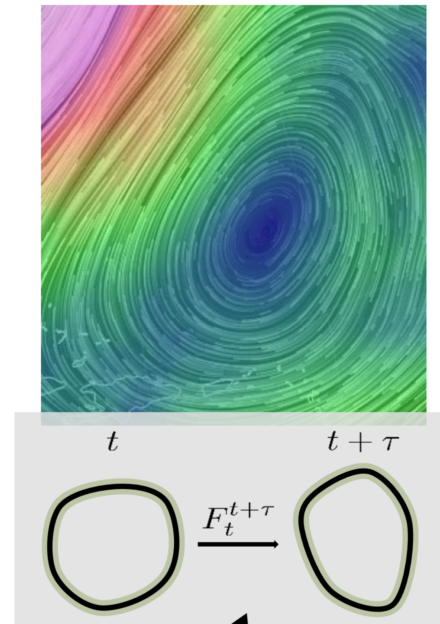
Hyperbolic OECS



Parabolic OECS



Elliptic OECS



No leading order **variation** in **material shear rate**

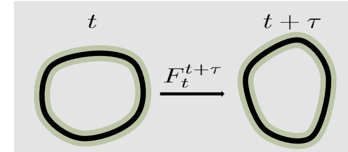
$$\delta \dot{P}_t(\gamma) = 0$$

No leading order **variation** in **material strain rate**

$$\delta \dot{Q}_t(\gamma) = 0$$

M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos,* (2016)

Objective Eulerian Coherent Structures



Variational definition of OECSs

$$\dot{Q}_t(\gamma) := \int_\gamma \dot{q}(x,x',t)ds \qquad \dot{q}(x,x',t) := \frac{\langle x', S(x,t)x'\rangle}{\langle x', x'\rangle} \qquad \delta\dot{Q}_t(\gamma) = 0$$

First variation w.r.t. free endpoints

$$\delta\dot{Q}_t(\gamma) = \frac{1}{\sigma}[\langle\partial_{x'}\dot{q},h\rangle]_0^\sigma + \frac{1}{\sigma}\int_0^\sigma\left[\partial_x\dot{q} - \frac{d}{ds}\partial_{x'}\dot{q}\right]h\,ds = 0$$

Euler-Lagrange equations

$$\partial_x\dot{q}(x,x',t) - \frac{d}{ds}[\partial_{x'}\dot{q}(x,x',t)] = 0$$

4D

Initial conditions ?

…

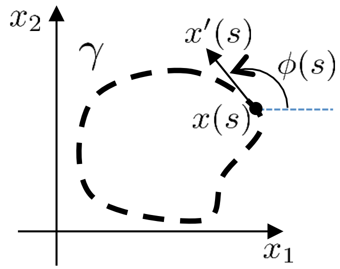
Equivalent Null-Geodesics formulation of OECSs

OECSs are Null Geodesics of the Lorentzian metrics:

Type of OECS	**Metric :** $g(u,u) = \langle u, Au\rangle$
Hyperbolic & Parabolic	$A(x,t) = 2S(x,t)R$
Elliptic	$A_\mu(x,t) = S(x,t) - \mu I, \quad \mu \in \mathbb{R}$

M. Serra & G. Haller, <u>**Objective Eulerian Coherent Structures**</u>, *Chaos,* (2016)

Closed null geodesics = Coherent Vortex Boundaries



Type of OECS	Metric : $g(u,u) = \langle u, Au \rangle$
Hyperbolic & Parabolic	$A(x,t) = 2S(x,t)R$
Elliptic	$A_\mu(x,t) = S(x,t) - \mu I, \quad \mu \in \mathbb{R}$

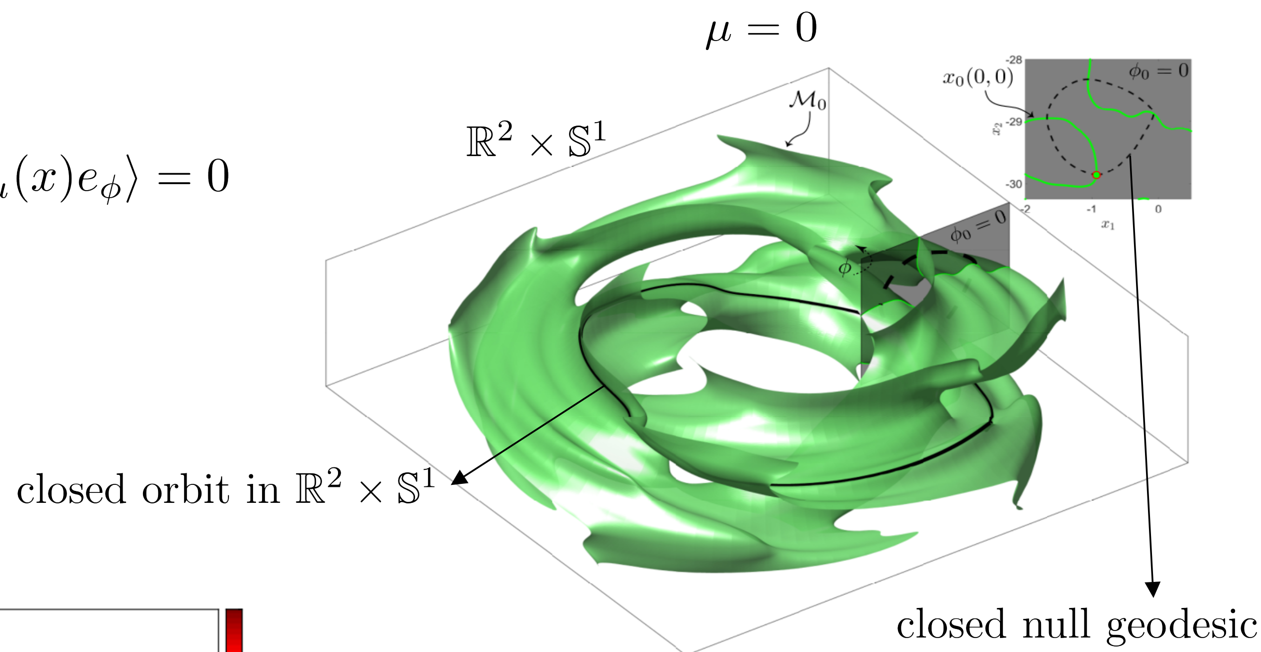
Theorem

1) Along null geodesics, $\langle e_\phi, A_\mu(x) e_\phi \rangle = 0$

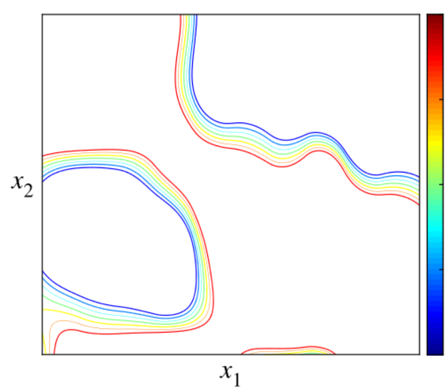
2) $\dot{x} = e_\phi := [\cos\phi, \sin\phi]^\top$
$\dot{\phi} = -\frac{\langle e_\phi, (D_x A(x) e_\phi) e_\phi \rangle}{2\langle e_\phi, R^\top A(x) e_\phi \rangle}$

3) I.C. pick any $\phi_0 \in \mathbb{S}^1$

4) Seek closed solutions



Dependence on parameters:



M. Serra & G. Haller, Efficient Computation of Null Geodesics with Applications to Coherent Vortex Detection, *Proc. Roy. Soc. A,* (2017)

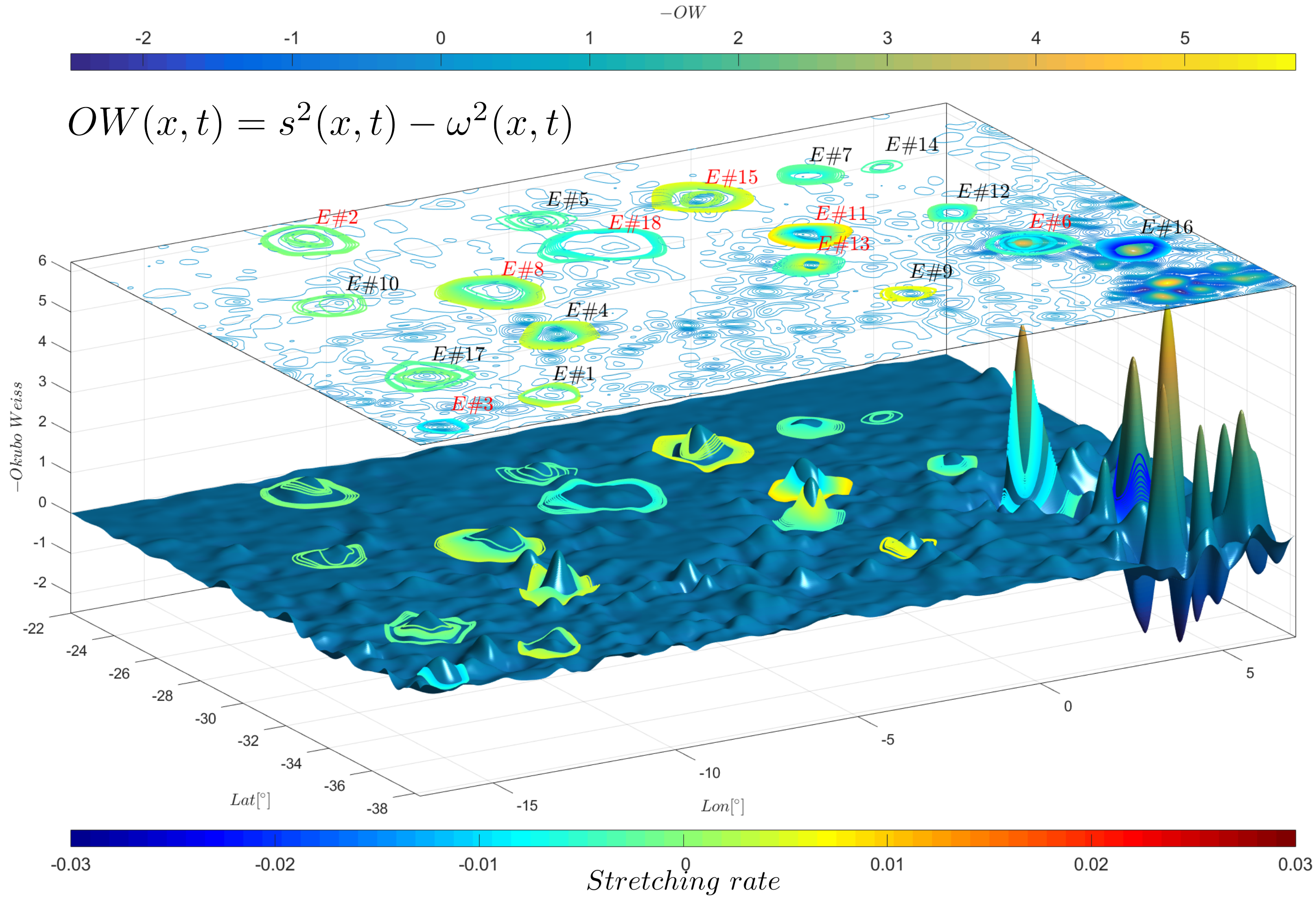
Examples

- OECSs from **satellite-inferred** ocean velocity data
- OECSs from **measured** ocean velocity data
- LCSs from **global atmospheric reanalysis** velocity data

Elliptic OECSs in the ocean (vortical structures)

$\dot{x} \approx \nabla^{\perp} h(x,t);\ h(x,t)$ =sea-surface height at x at time t.

$E\#i :=$ *elliptic OECSs*, $E\#i :=$ *elliptic OECSs in correspondence of Lagrangian coherent vortices*



Relevant for:

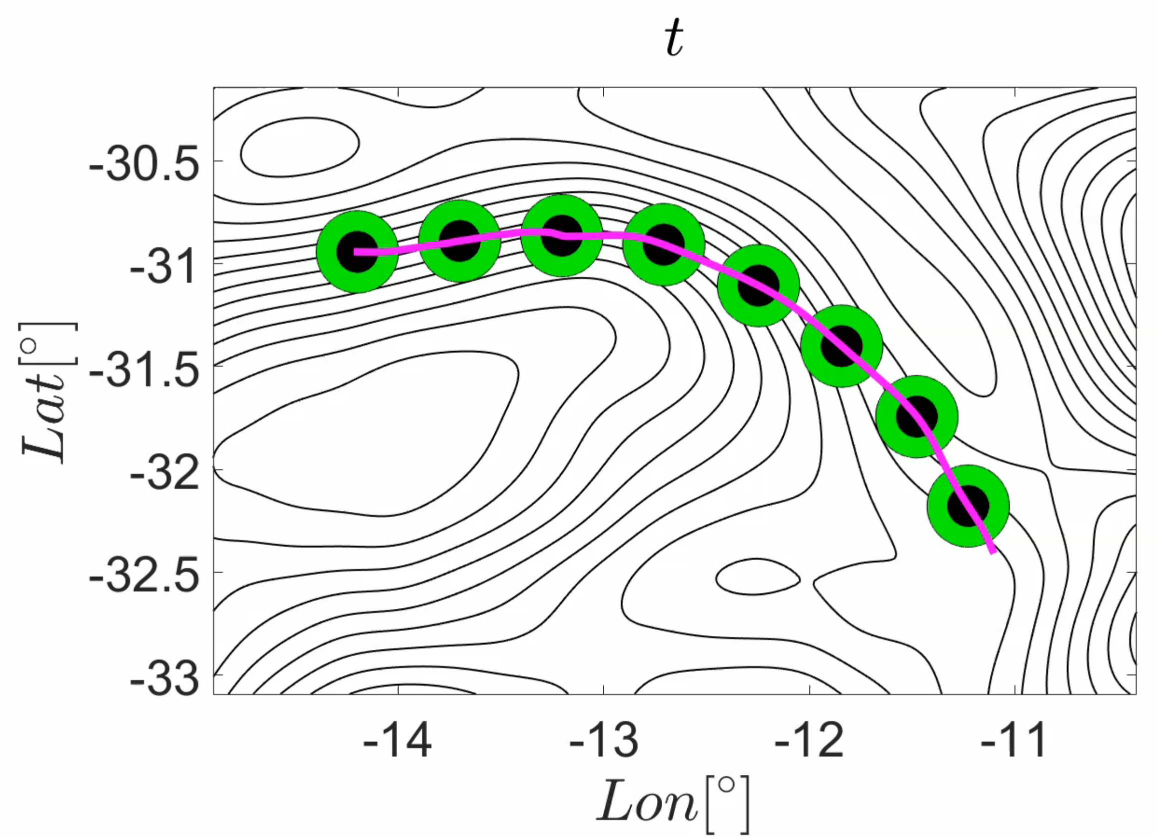
- Short-term forecasts
- Transport predictions

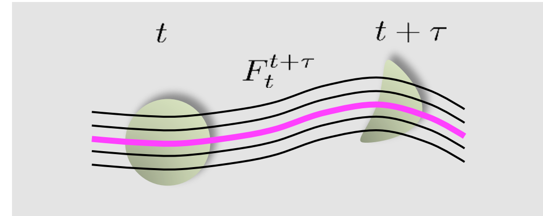
M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos* , (2016)

M. Serra & G. Haller, **Forecasting Long-Lived Lagrangian Vortices from their Objective Eulerian Footprints**, *J. Fluid Mech.*, (2017)

Parabolic OECSs in the ocean (jet-type structures)

$\dot{x} \approx \nabla^{\perp} h(x,t);\ h(x,t) =$sea-surface height at x at time t.





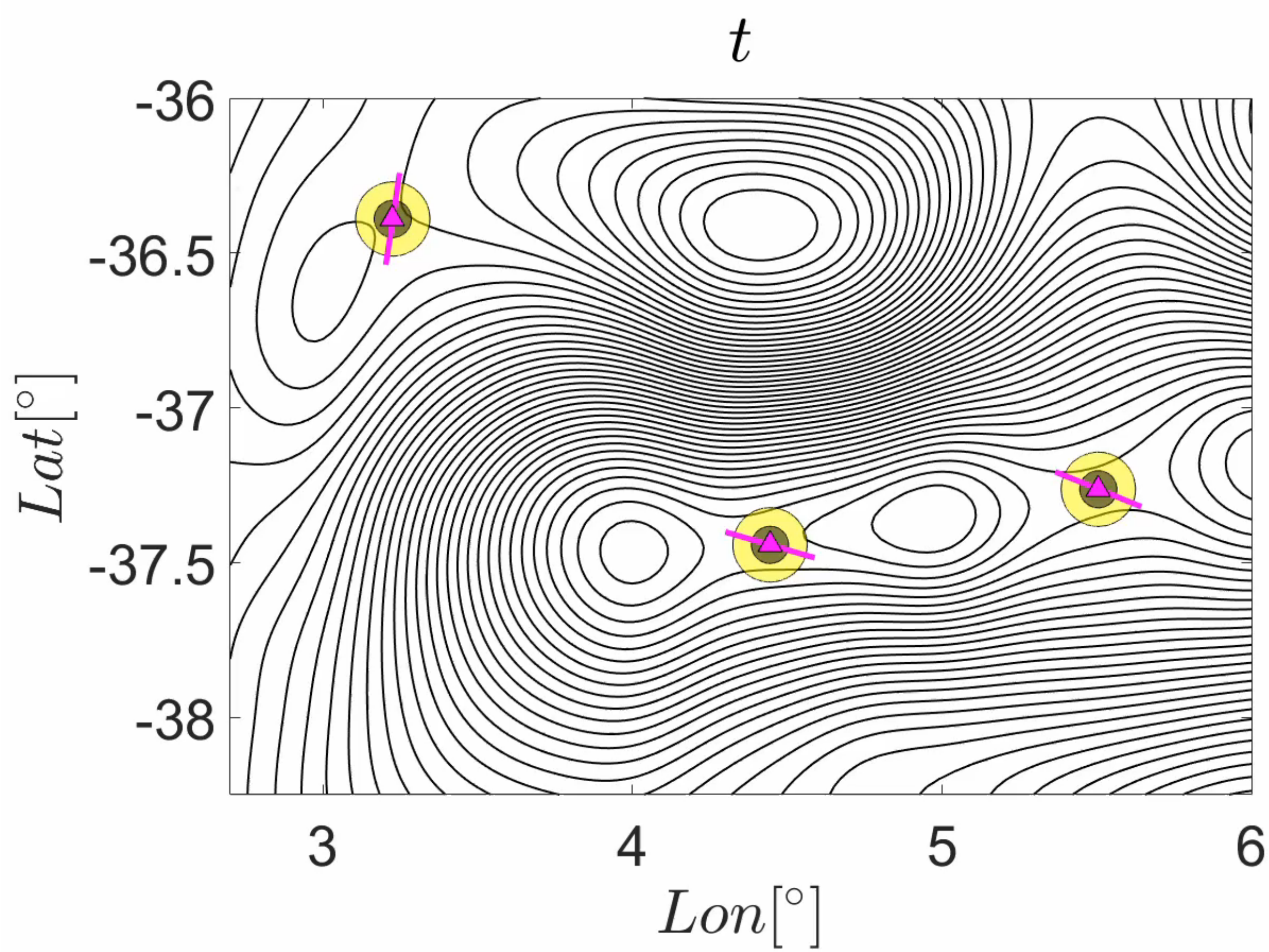
- **Short-term pathways** for material transport
- No clear signature in the streamline geometry

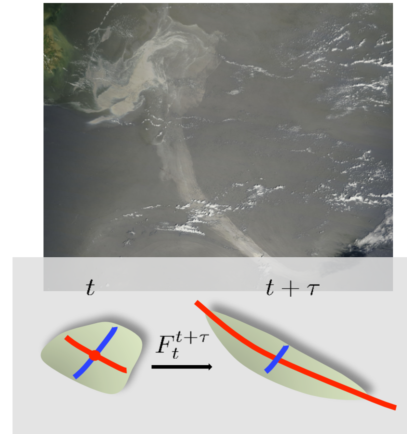
M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos,* (2016)

Where do particles go over short time ? (classic method)

$\dot{x} \approx \nabla^{\perp} h(x,t)$; $h(x,t)$ =sea-surface height at x at time t.

Saddle-type stagnation points (frame dependent)

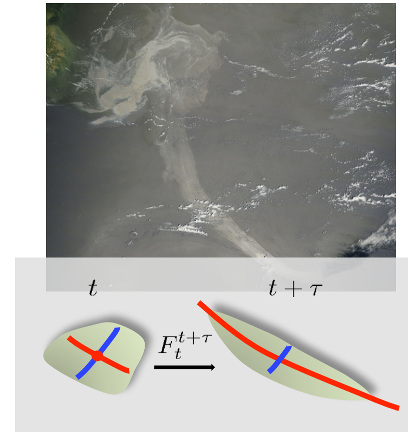




M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos* , (2016)

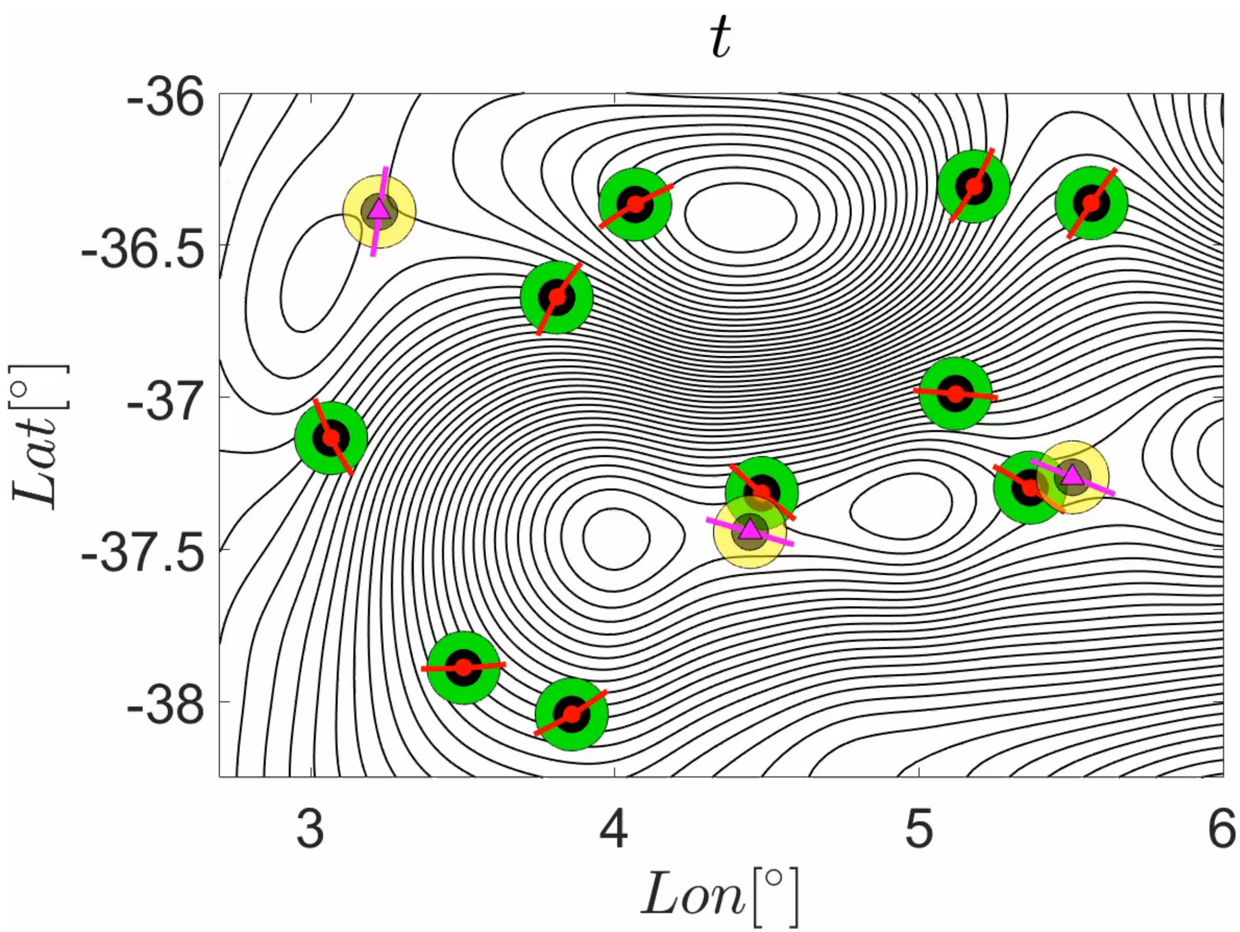
Where do particles go over short time ? (Hyperbolic OECSs)

$\dot{x} \approx \nabla^{\perp} h(x,t)$; $h(x,t)$ =sea-surface height at x at time t.



Saddle-type stagnation points (frame dependent)

Hyp. Attracting OECSs (frame independent)



Hyp. Attracting OECSs:

- The instantaneously most attracting material curves in the flow
- No signature in the streamline topology

M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos* , (2016)

Where do particles go over short time ?

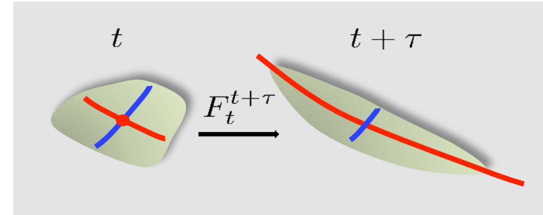
ODE for passive tracers:

$$\dot{x} = v(x,t)$$

ODE for non-passive tracers:

$$\dot{\tilde{x}} = v(\tilde{x},t) + \epsilon g(\tilde{x},t), \quad 0 < \epsilon \ll 1$$

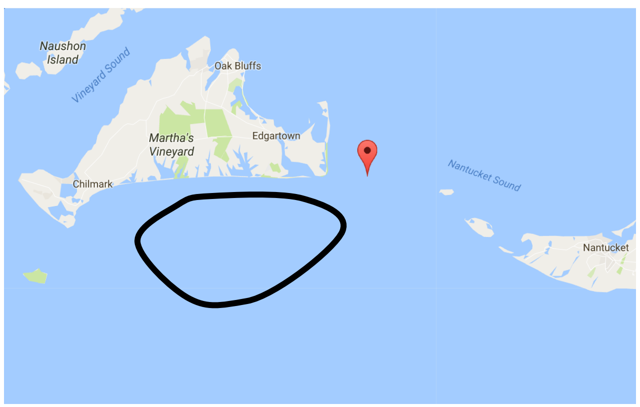
$$x, \tilde{x} \in U \subset \mathbb{R}^2$$
$$t \in [a,b], \quad v, g \in \mathcal{C}^k, \quad k > 1$$

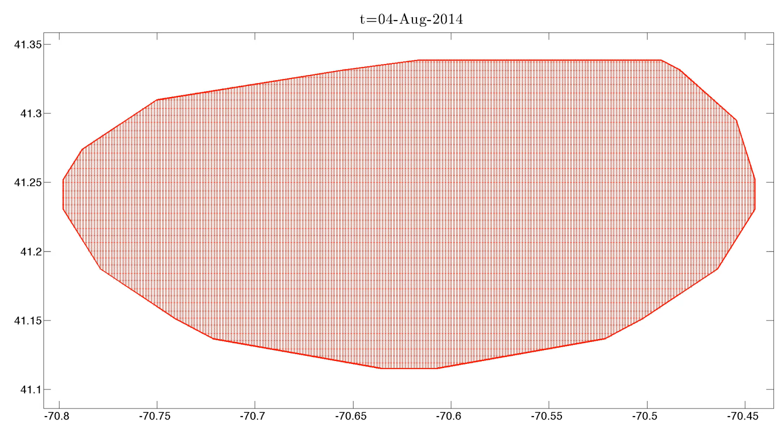


Goal:

Find features of $v(x,t)$ **which impact** $\tilde{x}(\tilde{x}_0, t, t_2)$

A real-life example:

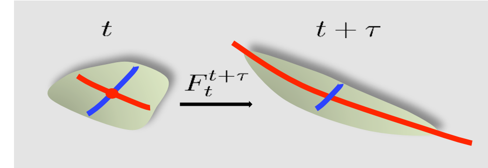




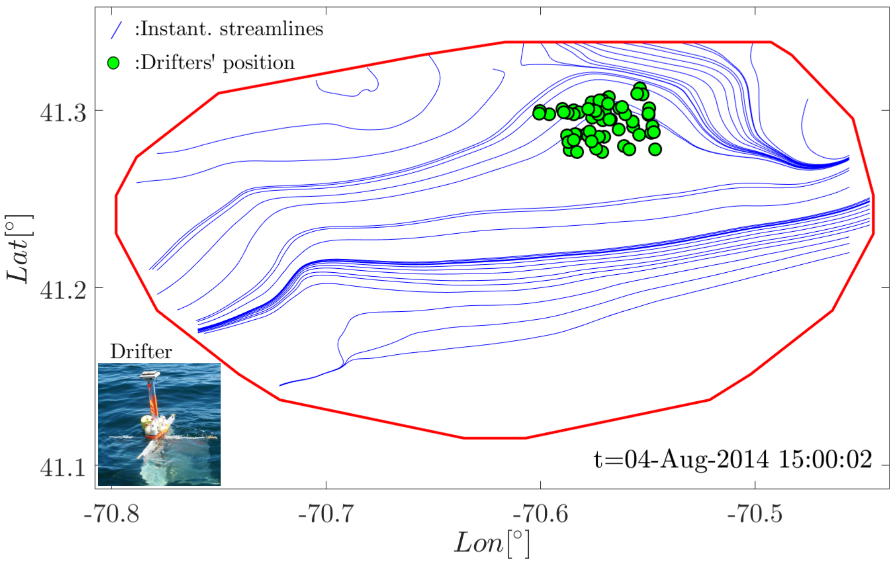
Joint Project with:
MIT
WHOI

- **Sub-mesoscale** velocity field **from High-Frequency Radar** (Anthony Kirincich, WHOI)
- Drifter positions from **experiment** (Irina Rypina, WHOI)

Where do particles go over short time ? (Hyperbolic OECSs)

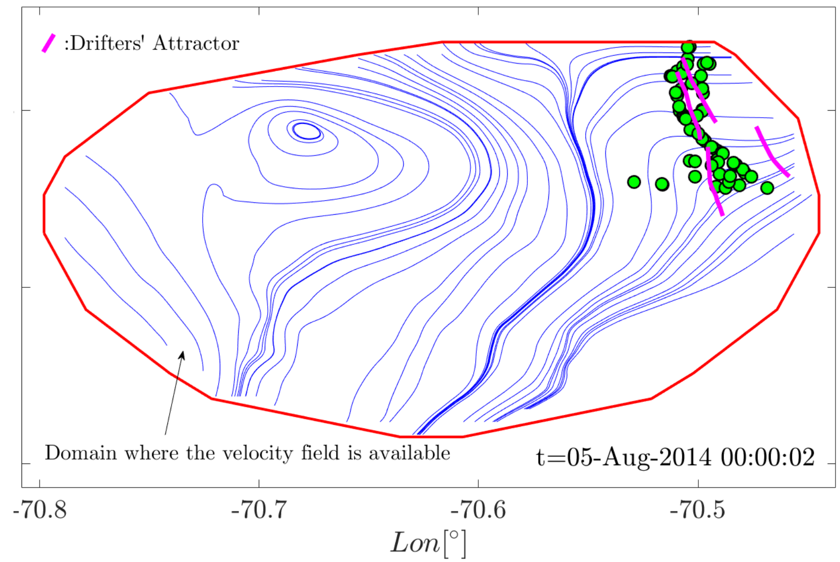


Real-life drifters emulate person fallen in water with uncertain initial position



later

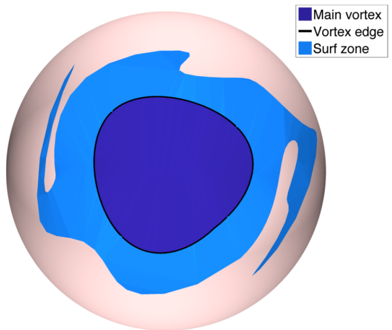
Hyp. Attracting OECSs



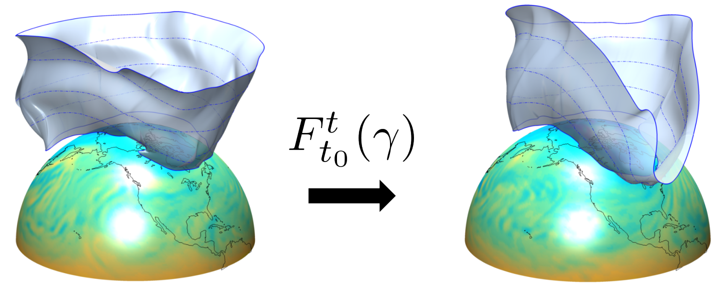
- **Sub-mesoscale velocity field from High-Frequency Radar** (Anthony Kirincich, WHOI)
- **Drifter positions from experiment** (Irina Rypina, WHOI)
- **OECSs attracts non-passive objects**
- **No signature in the instantaneous streamlines**

M. Serra, P. Sathe, I. Rypina, A. Kirincich, T. Peacock, P. Lermaseux & G. Haller , (in preparation)

Elliptic LCSs in the atmosphere (Lagrangian vortices)



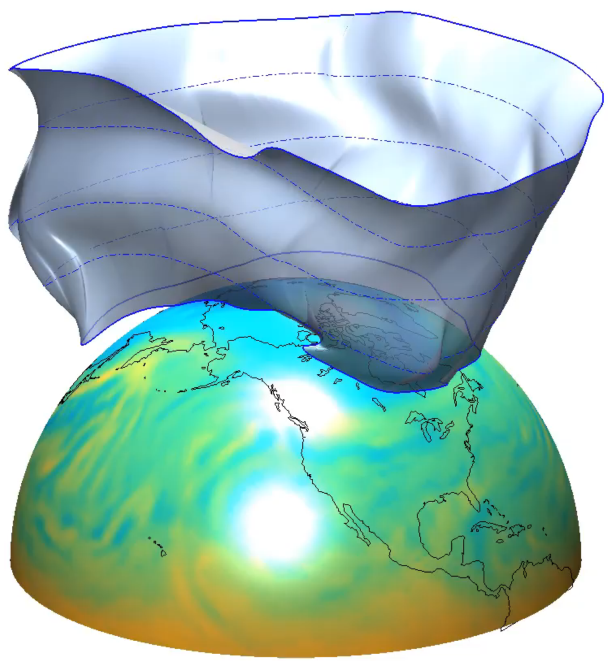
Dec. 2013 - Jan. 2014

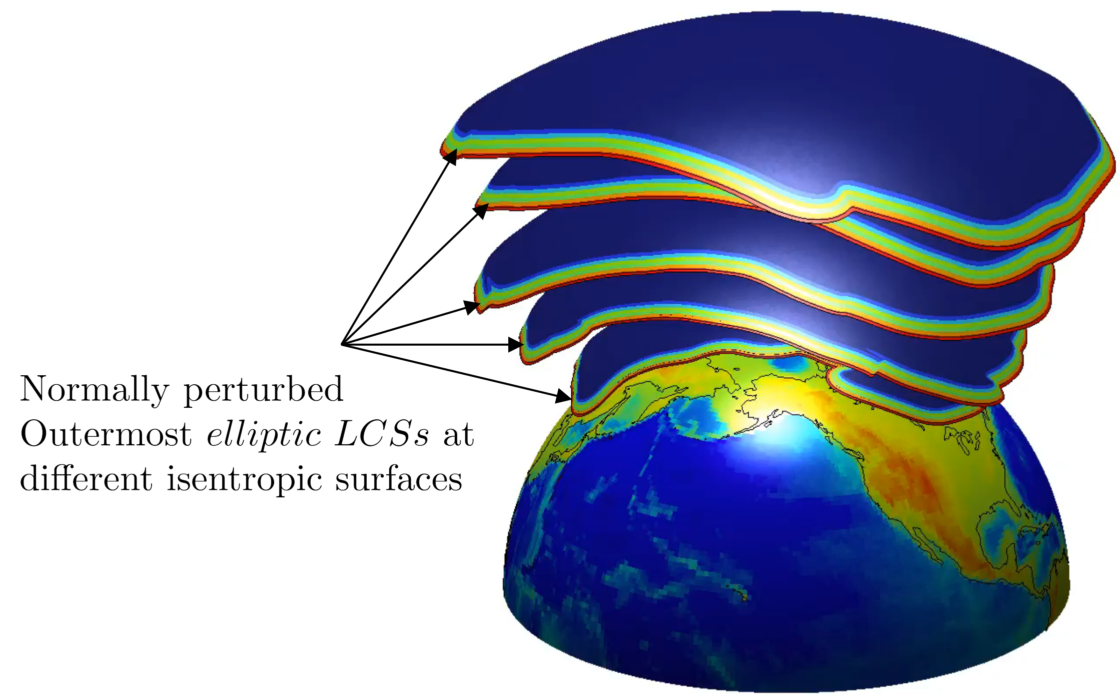


t_0= 28th Dec. 2013 **t= 7th Jan. 2014**

Material evolution of the polar vortex edge

Is it the polar vortex edge ?





M. Serra, P. Sathe, F. Beron-Vera & G. Haller, <u>Uncovering the edge of the Polar Vortex</u>, *submitted,* (2017)

Conclusions

- Short-term skeleton of the flow
- Time scales free
- Computable at any time instant

References:

M. Serra & G. Haller, **Objective Eulerian Coherent Structures**, *Chaos,* (2016)

M. Serra & G. Haller, **Forecasting Long-Lived Lagrangian Vortices from their Objective Eulerian Footprints**, *J. Fluid Mech.,* (2017)

M. Serra & G. Haller, **Efficient Computation of Null Geodesics with Applications to Coherent Vortex Detection**, *Proc. Roy. Soc. A,* (2017)

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