Coherent Structures in Run-and-Tumble Processes

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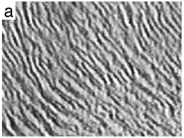
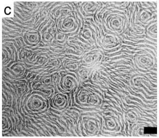
joint work with Angela Stevens, Universität Münster





Snowbird, May 2017

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Motivation — myxobacteria

Myxobacteria: interesting collective behavior!

Formation of fruiting bodies

high density, low nutrient

[Berleman, YouTube]

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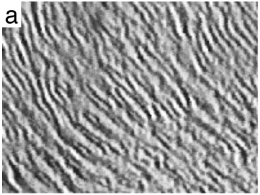
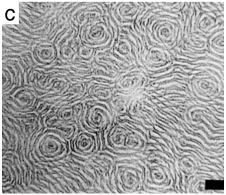
Rippling motion for "efficient depletion" of food source

[Berleman, YouTube]

Motivation — building blocks

Here: explain rippling!





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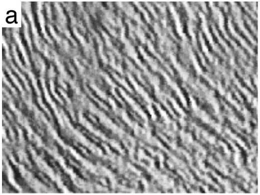

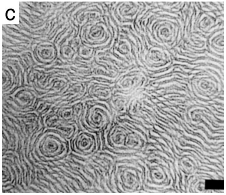

Phenomena: explain wavenumber selection, apparent “standing waves”!

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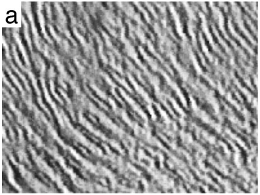

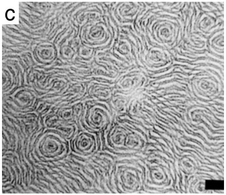

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Modeling: what are the "simplest" mechanisms that explain wavenumber selection here (compare Turing!)?

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Modeling: minimal ingredients "run" and "tumble"

Run: Populations of bacteria moving left and right, respectively:

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- self-propelled motion
- ripples $\perp$ motion
- "C-signal" transmitted upon contact $\Rightarrow$ reversal

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Models:

- internal clocks, delays
- struct' population models

[Bonilla PRE 93(2016), 012412]
[Börner Phys. Biol. 3(2006), 138]
[Igoshin PNAS 101(2004) 4256]
[Sliusarenko PNAS 103(2006), 1534]

What can we explain and what not?

Main results: our model explains

- ✓ rippling patterns
- × wavenumber selection from white noise
- ✓ wavenumber selection from shot noise or growth

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What's missing

- test hypotheses
- two-dimensional versions
- nonlinear analysis, stability,...

Analysis in three chapters

I) equilibria and stablity

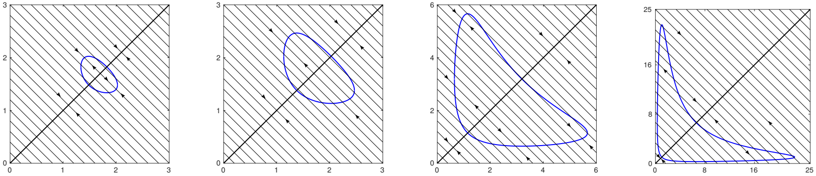
II) coherent structures

III) pointwise instability

Kinetics — nonlinear tumbling

Example: Turning rate increases with collisions in sigmoidal fashion

$$r(u,v) = u \cdot \left(1 + \frac{v^2}{1+\gamma v^2}\right)$$



Phase portraits for $\gamma = 0.122, 0.115, 0.07, 0.021$

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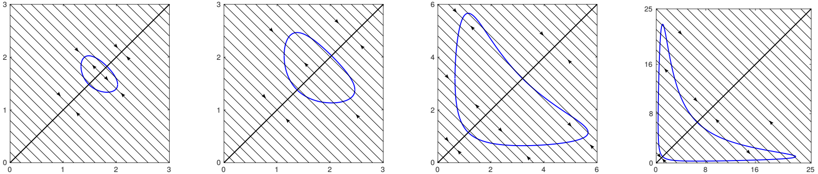

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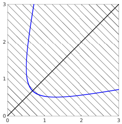
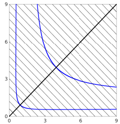
Other kinetics:

$u \cdot \left(1 + \frac{v^2}{1+\gamma(u+v)^2}\right)$

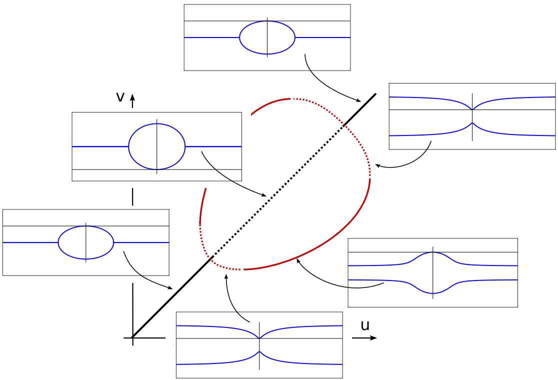


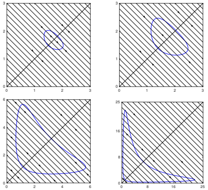
$u \cdot \left(1 + \frac{v^3}{1+\gamma v^2}\right)$



Linear stability

Stability depends on normal vector (n_1, n_2) of equilibrium curve:





Fastest-growing mode $k = 0, \infty$

$\Rightarrow$ No "Turing" finite wavenumber selection!

$$\lambda_{\pm}(k) = \frac{1}{2}\left(n_1 - n_2 \pm \sqrt{(n_1 - n_2)^2 + 4ik\,(n_1 + n_2 + ik)}\right)$$

Coherent structures

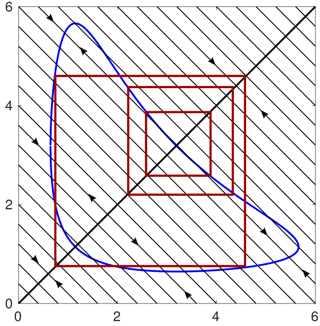
Many questions. . . Here: "standing" waves!

$$u_t = +u_x - r(u,v) + r(v,u)$$

$$v_t = -v_x + r(u,v) - r(v,u)$$

Look for sol's $u_0(x+t), v_0(x-t)$

$$\Longleftrightarrow r(u_0,v_0) = r(v_0,u_0)$$



E.g. $u \in \{u_-, u_+\}$, $v \in \{v_-, v_+\}$,

$r(u_\pm, v_\pm) = r(v_\pm, u_\pm)$

Plethora of waves with jumps! Stability. . .

Pointwise instabilities

- linearization at constant state can be solved "explicitly"

$$U_t = \mathcal{L}U \Rightarrow U(t,x) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}\int_y G_\lambda(x-y)U(0,y)\,dy\,d\lambda$$

[Holzer & S., Criteria for pointwise growth and their role in invasion processes, JNLS 2014]

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- pointwise singularities generically from pinched double roots:

  Ansatz $U(t,x) = U_0 e^{\lambda t + \nu x}$ gives dispersion relation $d(\lambda, \nu)$

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- pinched double roots (λ, ν) solve

$$d(\lambda,\nu) = 0, \qquad \partial_\nu d(\lambda,\nu) = 0, \qquad \nu_\pm(\lambda) \to \pm\infty$$

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Spreading speeds and wavenumber selection

Ptwise instabilities/pinched dble roots depend on coordinate frame!

$$U(t,x) \to U(t,x-ct) \quad \Rightarrow \quad d(\lambda,\nu) \to d_c(\lambda,\nu) = d(\lambda - c\nu, \nu)$$

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- $\lambda = i\omega$ gives frequency, c spreading speed

  $\Rightarrow$ wavenumber from nonlinear dispersion relation...

Finding pinched double roots

A calculation with $n_{1/2} = \partial_{u/v}(-r(u,v) + r(v,u))$, gives

$$d_c(\lambda,\nu) = (\lambda - c\nu)^2 - (\lambda - c\nu)(n_1 - n_2) - (n_1 + n_2)\nu - \nu^2,$$

$$\partial_\nu d_c(\lambda,\nu) = -2c\lambda - (n_1 + n_2) + c(n_1 - n_2) - 2\nu + 2c^2\nu,$$

and

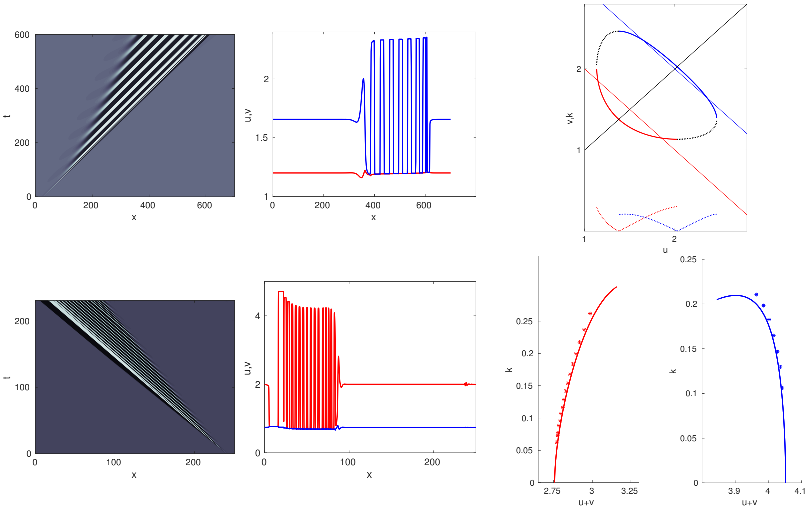
$$\lambda_* = \frac{1}{2}\left(n_1 - n_2 - c(n_1 + n_2) \pm 2\sqrt{-n_1 n_2(1 - c^2)}\right),$$

$$\nu_* = \frac{1}{2(1 - c^2)}\left((n_1 + n_2)(c^2 - 1) \mp 2c\sqrt{-n_1 n_2(1 - c^2)}\right).$$

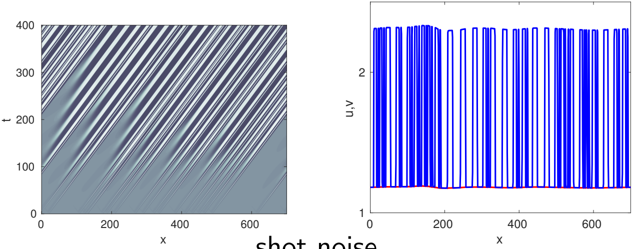
Pinching implies $|c| < 1$, $\omega_* = 2\frac{n_1 n_2}{n_1 + n_2}$,

$k_* = n_2$ when $n_1, n_2 > 0$, $\qquad k_* = -n_1$ when $n_1, n_2 < 0$.

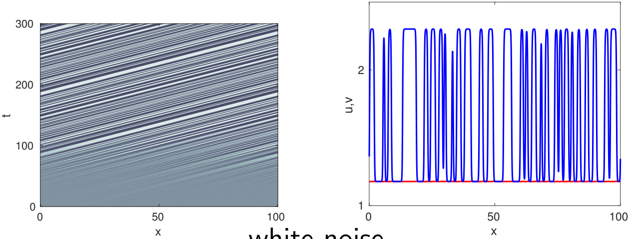
Perturbing asymmetric states locally



Perturbing asymmetric states — shot vs white noise

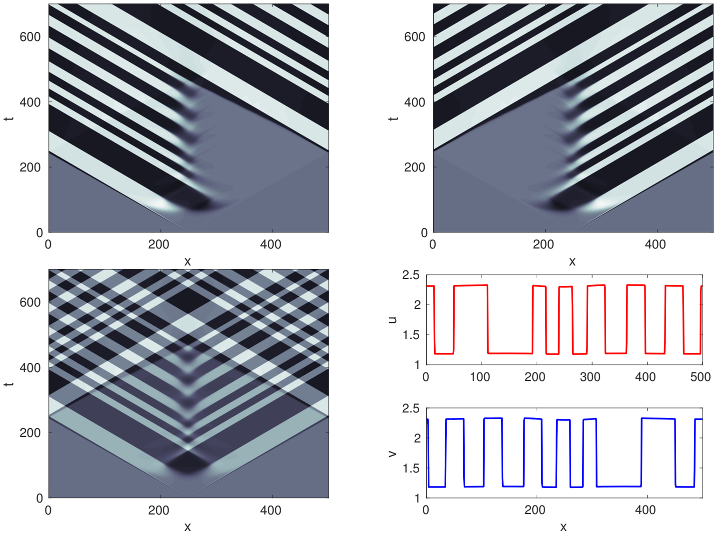


shot noise



white noise

Perturbing symmetric states



Spikes and blowup

Include small diffusive term

$$u_t = \varepsilon u_{xx} + u_x - r(u,v) + r(v,u)$$
$$v_t = \varepsilon v_{xx} - u_x + r(u,v) - r(v,u)$$

Bifurcation of family stationary spikes from $u = v = u_*$!

- unstable at small amplitude
- $\varepsilon \to 0 \Rightarrow$ Dirac-δ singularities
- stability for certain $r(u,v)$ for large amplitude $\Rightarrow$ blowup, cluster formation, fruiting?

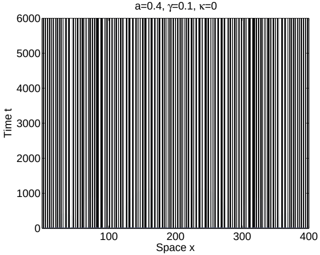
Diffusion and conversion

Very, very similar phenomena in reaction-diffusion

$$u_t = u_{xx} - f(u, v)$$

$$v_t = f(u, v)$$

white noise:



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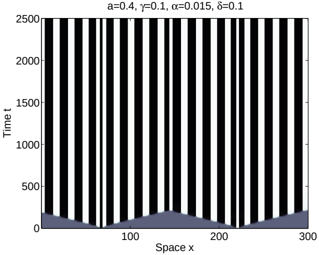

Kotzagiannidis et al., Stable pattern selection through invasion fronts in closed two-species reaction-diffusion systems, RIMS Kokyuroku Bessatsu B31 (2012)

Summary and open questions

- wavenumber selection — shot noise vs white noise

[Stevens,S., JMB, online first]

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Thank you!

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