Low Dimensional Manifold Model for Image Processing

Wei Zhu

University of California Los Angeles

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Joint work with Zuoqiang Shi and Stanley Osher

General Image Processing Problems

Many image processing problems can be formulated as recovering an image $f \in \mathbb{R}^{m \times n}$ from its noisy and linear measurements:

$$b = \Phi f + \epsilon$$

Original Inpainting Denoising Deblurring

- Inpainting: $\Phi = \Phi_\Omega$ is the subsample operator, and $\epsilon = 0$.
- Denoising: $\Phi = Id$, and ϵ is the corresponding noise type.
- Deblurring: Φ is a convolution kernel.

Variational Model for Image Processing

Reconstructing f from b is an ill-posed problem, and some regularization is needed in a variational model:

$$\min_f R(f) \quad \text{subject to:} \quad b = \Phi f + \epsilon$$

- Total variation (TV):

$$R(f) = \|\nabla f\|_{L^1}$$

- Nonlocal total variation (NLTV):

$$R(f) = \|\nabla_w f\|_{L^1}$$

- Wavelet sparsity:

$$R(f) = \|Wf\|_{L^1}$$

- ......

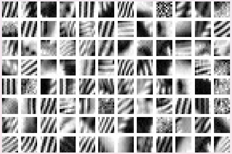
LDMM: dimension of the patch manifold.

Patch Set and Patch Manifold of an Image

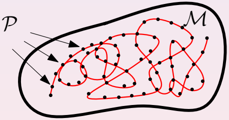
Image patches have been widely used in image processing.

Original Image f

Patch Set $\mathcal{P}(f)$



Patch Manifold $\mathcal{M}(f)$



- $\mathcal{P}(f) \subset \mathbb{R}^d$ is the collection of of all patches in the image f.
- $\mathcal{M}(f) \subset \mathbb{R}^d$ is the underlying patch manifold, discretely sampled by the point cloud $\mathcal{P}(f)$.

Low Dimensionality of the Patch Manifold $\mathcal{M}$

For most natural images, the dimension of the patch manifold $\mathcal{M}$ is usually much lower than that of the ambient space.

- If f is a smooth image, the patch at coordinate x, $p_x(f)$ can be approximated by a linear function

$$p_x(f)(y) \approx f(x) + (y - x) \cdot \nabla f(x).$$

  This implies that $\dim \mathcal{M} \approx 3$.

- If f is a piecewise constant function corresponding to a cartoon image, then each patch is characterized by the location and the orientation of the edge. This means $\dim \mathcal{M} \approx 2$.

- If f is an oscillatory function corresponding to a texture, then

$$f(x) \approx a(x) \cos \theta(x), \quad p_x f \approx a_L \cos \theta_L,$$

  where a_L and θ_L are linear approximation of a and θ. Hence $\dim M \approx 6$.

Low Dimensional Manifold Model

The idea of the low dimensional manifold model (LDMM) in image processing is to use the dimension of the patch manifold $\mathcal{M}$ as a regularization.

$$\min_{f,\mathcal{M}} \dim(\mathcal{M}), \quad \text{subject to:} \quad b = \Phi f + \epsilon, \mathcal{P}(f) \subset \mathcal{M}$$

Question: How to compute $\dim \mathcal{M}$?

Dimension of a Manifold

Proposition

Let $\mathcal{M}$ be a smooth submanifold embedded in $\mathbb{R}^d$. For any $x \in \mathcal{M}$,

$$\dim(\mathcal{M}) = \sum_{j=1}^{d} \|\nabla_{\mathcal{M}} \alpha_j(\mathbf{x})\|^2,$$

where $\alpha_i, i = 1, \ldots, d$ are coordinate functions,

$$\forall \mathbf{x} \in \mathcal{M}, \quad \alpha_i(\mathbf{x}) = x_i.$$

Dimension of a Manifold

Sanity check:
If $\mathcal{M} = S^1$, then $k = \dim(\mathcal{M}) = 1$, $d = \dim(\mathbb{R}^2) = 2$, and $\mathbf{x} = \psi(\theta) = (\cos\theta, \sin\theta)^t$ is the coordinate chart.
The metric tensor $g = g_{11} = \left\langle \frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\theta} \right\rangle = 1 = g^{11}$.
The gradient of α_i, $\nabla_{\mathcal{M}}\alpha_i = g^{11}\partial_1\alpha_i\partial_1 = \partial_1\alpha_i\partial_1$ can be viewed as a vector in the ambient space $\mathbb{R}^2$:

$$\nabla^j_{\mathcal{M}}\alpha_i = \partial_1\psi^j\partial_1\alpha_i$$

Therefore, we have

$$\begin{aligned}\nabla_{\mathcal{M}}\alpha_1 &= \left\langle \partial_1\psi^1\partial_1\alpha_1, \partial_1\psi^2\partial_1\alpha_1 \right\rangle = \left\langle \sin^2\theta, -\cos\theta\sin\theta \right\rangle, \\ \nabla_{\mathcal{M}}\alpha_2 &= \left\langle \partial_1\psi^1\partial_1\alpha_2, \partial_1\psi^2\partial_1\alpha_2 \right\rangle = \left\langle -\sin\theta\cos\theta, \cos^2\theta \right\rangle.\end{aligned}$$

Hence $\|\nabla_{\mathcal{M}}\alpha_1\|^2 + \|\nabla_{\mathcal{M}}\alpha_2\|^2 = \sin^2\theta + \cos^2\theta = 1$

Low Dimensional Manifold Model

The original optimization problem can be rewritten as:

$$\min_{\substack{f\in\mathbb{R}^{m\times n}\\ \mathcal{M}\subset\mathbb{R}^d}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M})} + \lambda\|y-\Phi f\|_2^2, \quad \text{subject to: } \mathcal{P}(f)\subset\mathcal{M},$$

where

$$\|\nabla_{\mathcal{M}}\alpha_i\|_{L^2(\mathcal{M})} = \left(\int_{\mathcal{M}} \|\nabla_{\mathcal{M}}\alpha_i(\mathbf{x})\|^2 d\mathbf{x}\right)^{1/2}$$

This optimization problem is nonconvex. It can be solved by alternating the direction of minimization with respect to f and $\mathcal{M}$. We also perturb the coordinate function $\boldsymbol{\alpha}$ at each step.

Alternating Direction of Minimization

$$\min_{\substack{f\in\mathbb{R}^{m\times n}\\ \mathcal{M}\subset\mathbb{R}^d}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M})} + \lambda\|y-\Phi f\|_2^2, \quad \text{subject to: } \mathcal{P}(f)\subset\mathcal{M},$$

- With a guess $\mathcal{M}^n$ and f^n of the manifold and image, update the coordinate function $\alpha_i^{n+1}, i=1,\cdots,d$ and f^{n+1}:

$$(f^{n+1},\boldsymbol{\alpha}^{n+1}) = \arg \min_{\substack{f\in\mathbb{R}^{m\times n},\\ \alpha_1,\ldots,\alpha_d\in H^1(\mathcal{M}^n)}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}^n}\alpha_i\|^2_{L^2(\mathcal{M}^n)} + \lambda\|b-\Phi f\|_2^2,$$

$$\text{subject to:} \quad \boldsymbol{\alpha}(\mathcal{P}(f^n)) = \mathcal{P}(f)$$

- Update $\mathcal{M}$ by setting

$$\mathcal{M}^{n+1} = \boldsymbol{\alpha}(\mathcal{M}^n) = \left\{(\alpha_1^{n+1}(\mathbf{x}),\ldots,\alpha_d^{n+1}(\mathbf{x}))^T : \mathbf{x}\in\mathcal{M}^n\right\}.$$

Question: How to update f and $\boldsymbol{\alpha}$

Split Bregman Iteration

- Solve $\alpha_i^{n+1,k+1}, i = 1, \cdots, d$ with fixed $f^{n+1,k}$,

$$\min_{\alpha_1,\cdots,\alpha_d \in H^1(\mathcal{M}^n)} \sum_{i=1}^{d} \|\nabla \alpha_i\|_{L^2(\mathcal{M}^n)}^2 + \mu \|\boldsymbol{\alpha}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k}) + d^k\|_F^2.$$

- Update $f^{n+1,k+1}$ as

$$\min_{f \in \mathbb{R}^{m \times n}} \lambda \|b - \Phi f\|_2^2 + \mu \|\boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f) + d^k\|_F^2.$$

- Update d^{k+1}:

$$d^{k+1} = d^k + \boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k+1}).$$

Algorithm

Algorithm 1 LDMM Algorithm - Continuous version

1: **while** not converge **do**
2: **while** not converge **do**
3:
$$\alpha_i^{n+1,k+1} = \arg \min_{\alpha_i \in H^1(\mathcal{M}^n)} \|\nabla_{\mathcal{M}^n} \alpha_i\|_{L^2(\mathcal{M}^n)}^2 + \mu \|\alpha_i(\mathcal{P}(f^n)) - \mathcal{P}_i(f^{n+1,k}) + d_i^k\|^2$$
4:
$$f^{n+1,k+1} = \arg \min_{f \in \mathbb{R}^{m \times n}} \quad \lambda \|b - \Phi f\|_2^2 + \mu \|\boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f) + d^k\|_F^2$$
5:
$$d^{k+1} = d^k + \boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k+1}).$$
6: **end while**
7:
$$\mathcal{M}^{n+1} = \left\{ (\alpha_1^{n+1}(\mathbf{x}), \cdots, \alpha_d^{n+1}(\mathbf{x})) : \mathbf{x} \in \mathcal{M}^n \right\}.$$
8: **end while**

Graph Laplacian

The key step in the previous algorithm is to solve the following optimization:

$$\min_{u\in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|^2_{L^2(\mathcal{M})} + \mu \sum_{y\in\Omega} |u(\mathbf{y}) - v(\mathbf{y})|^2 \tag{1}$$

Normally, (1) is solved by discretizing $\nabla_{\mathcal{M}} u$ by the nonlocal gradient:

$$\nabla_w u(\mathbf{x},\mathbf{y}) = \sqrt{w(\mathbf{x},\mathbf{y})}\,(u(\mathbf{y}) - u(\mathbf{x}))\,.$$

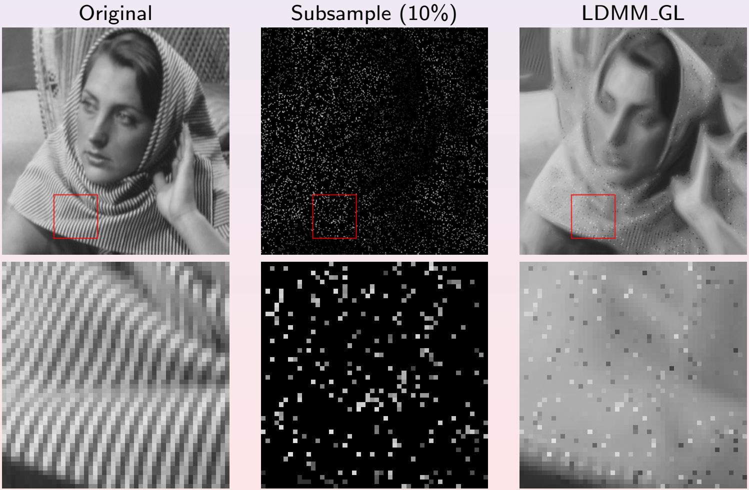
This leads to solving the following graph Laplacian (GL) problem:

$$\min_{u\in\mathbb{R}^{m\times n}} \sum_{x,y\in\Omega} w(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 + \mu \sum_{y\in\Omega} |u(\mathbf{y}) - v(\mathbf{y})|^2.$$

Or equivalently,

$$\sum_{y\in\Omega} w(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \mu(u(\mathbf{x}) - v(\mathbf{y})) = 0, \quad \forall x \in \Omega.$$

Graph Laplacian



Laplace-Beltrami Equation

By a standard variational approach, we know that problem (1) is equivalent to the following PDE:

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) + \mu \sum_{\mathbf{y}\in\Omega} \delta(\mathbf{x}-\mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0, & \mathbf{x} \in \mathcal{M} \\ \dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}, \end{cases} \tag{2}$$

where $\partial\mathcal{M}$ is the boudary of $\mathcal{M}$ and $\mathbf{n}$ is the outer normal of $\partial\mathcal{M}$.

Point Integral Method

In the point integral method (PIM), the key observation is the following integral approximation:

$$\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\mathbf{y} \approx -\frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x})-u(\mathbf{y}))\, R\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\mathbf{y} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\tau_{\mathbf{y}}.$$

The function R is a positive function defined on $[0, +\infty)$ with compact support (or fast decay) and

$$\bar{R} = \int_r^{\infty} R(s) ds.$$

Local Truncation Error

Theorem

Let $\mathcal{M}$ be a smooth manifold and $u \in C^3(\mathcal{M})$, then

$$\left\| -\frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x}) - u(\mathbf{y}))\, R_t(\mathbf{x},\mathbf{y}) d\mathbf{y} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y}) d\tau_{\mathbf{y}} - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y}) d\mathbf{y} \right\|_{L^2(\mathcal{M})} = O(t^{1/4}),$$

where

$$R_t(\mathbf{x},\mathbf{y}) = \frac{1}{(4\pi t)^{k/2}} R\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right), \bar{R}_t(\mathbf{x},\mathbf{y}) = \frac{1}{(4\pi t)^{k/2}} \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right).$$

Proof of Theorem

Using integration by part, we have

$$\begin{aligned}\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} &= -\int_\Omega \nabla u(\mathbf{x})\cdot\nabla\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\\ &= \frac{1}{2t}\int_\Omega(\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial\mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\end{aligned}$$

We want to replace ∇u with function value u, which leads us to use the Taylor expansion

$$u(\mathbf{x})-u(\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y}) + O(\|\mathbf{x}-\mathbf{y}\|^3).$$

Proof of Theorem

$$u(\mathbf{x}) - u(\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y}) + O(\|\mathbf{x}-\mathbf{y}\|^3).$$

Integrating on both sides, we have

$$\begin{aligned}
&\frac{1}{2t}\int_\Omega (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) R_t(\mathbf{x},\mathbf{y})d\mathbf{x}\\
=&\frac{1}{2t}\int_\Omega \left(u(\mathbf{x}) - u(\mathbf{y})\right) R_t(\mathbf{x},\mathbf{y})d\mathbf{x}\\
&+\frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2}),
\end{aligned}$$

where $O(t^{1/2})$ is uniform with respect to $\mathbf{y}$. Next we need to estimate the $\mathbf{H}_u$ term.

Proof of Theorem

$$
\begin{aligned}
&\frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T \mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} \\
=&\frac{1}{4t}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)(\mathbf{x}_j-\mathbf{y}_j)\partial_{ij}u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} \\
=&-\frac{1}{2}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)\partial_{ij}u(\mathbf{x})\partial_j(\bar{R}_t(\mathbf{x},\mathbf{y}))d\mathbf{x} \\
=&\frac{1}{2}\int_\Omega \partial_j(\mathbf{x}_i-\mathbf{y}_i)\partial_{ij}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}+\frac{1}{2}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)\partial_{ijj}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} \\
&-\frac{1}{2}\int_{\partial\Omega} (\mathbf{x}_i-\mathbf{y}_i)\mathbf{n}_j\partial_{ij}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} \\
=&\frac{1}{2}\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}-\frac{1}{2}\int_{\partial\Omega} ((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}+O(t^{1/2}).
\end{aligned}
$$

Proof of Theorem

$$\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} = \frac{1}{2t}\int_\Omega (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}$$

$$= \frac{1}{2t}\int_\Omega \left(u(\mathbf{x})-u(\mathbf{y})\right)R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2})$$

$$+ \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}$$

$$= \frac{1}{2t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \frac{1}{2}\int_\Omega \Delta u\cdot\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}$$

$$- \frac{1}{2}\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2})$$

This implies that:

$$\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} = \frac{1}{t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + 2\int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}$$

$$- \int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2})$$

Proof of Theorem

$$\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} = \frac{1}{t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + 2\int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} - \int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2})$$

Although $\left\|\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\right\|_{L^\infty(\Omega)} = O(1)$, it can be easily estimated in $L^2(\Omega)$:

$$\left\|\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\right\|_{L^2(\Omega)} = O(t^{1/4}).$$

Therefore

$$\left\|-\frac{1}{t}\int_{\mathcal{M}}(u(\mathbf{x})-u(\mathbf{y}))\,R_t(\mathbf{x},\mathbf{y})d\mathbf{y} + 2\int_{\partial\mathcal{M}}\frac{\partial u}{\partial\mathbf{n}}(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\tau_{\mathbf{y}} - \int_{\mathcal{M}}\Delta_{\mathcal{M}}u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{y}\right\|_{L^2(\mathcal{M})} = O(t^{1/4}),$$

Integral Equation

The Laplace-Beltrami equation is:

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) + \mu \sum_{\mathbf{y}\in\Omega} \delta(\mathbf{x}-\mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0, & \mathbf{x} \in \mathcal{M} \\ \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}, \end{cases}$$

The integral approximation is:

$$\int_{\mathcal{M}} \Delta u(\mathbf{x}) \bar{R}_t(\mathbf{x},\mathbf{y}) d\mathbf{x} \approx \frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x}) - u(\mathbf{y})) R_t(\mathbf{x},\mathbf{y}) d\mathbf{x} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}} \bar{R}_t(\mathbf{x},\mathbf{y}) d\mathbf{x}$$

The integral equation is:

$$\int_{\mathcal{M}} (u(\mathbf{x}) - u(\mathbf{y})) R_t(\mathbf{x},\mathbf{y}) d\mathbf{x} + \mu t \sum_{\mathbf{y}\in\Omega} \bar{R}_t(\mathbf{x},\mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0.$$

Discretization

$$\frac{|\mathcal{M}|}{N}\sum_{j=1}^{N} R_t(\mathbf{x}_i,\mathbf{x}_j)(u_i-u_j)+\mu t\sum_{j=1}^{N}\bar{R}_t(\mathbf{x}_i,\mathbf{x}_j)(u_j-v_j)=0.$$

The matrix form is:

$$(Ł+\bar{\mu}\bar{\mathbf{W}})\mathbf{U}=\bar{\mu}\bar{\mathbf{W}}\mathbf{V},$$

where $\bar{\mu}=\mu t N/|\mathcal{M}|$,

$$Ł=\mathbf{D}-\mathbf{W},\quad \mathbf{W}=(w_{ij}),\quad \bar{\mathbf{W}}=(\bar{w}_{ij}),$$

and

$$w_{ij}=R_t(\mathbf{x}_i,\mathbf{x}_j),\quad \bar{w}_{ij}=\bar{R}_t(\mathbf{x}_i,\mathbf{x}_j),\quad \mathbf{x}_i,\mathbf{x}_j\in\mathcal{P}(f^n),\quad i,j=1,\cdots,N.$$

Algorithm (LDMM_PIM)

Algorithm 2 LDMM_PIM

while not converge **do**
    Compute the matrices $\boldsymbol{W} = (w_{ij})_{1 \le i,j \le N}$ from $\mathcal{P}(f^n)$
    for $k = 1 : K$ **do**

$$(\boldsymbol{L} + \bar{\mu}\bar{\boldsymbol{W}})\boldsymbol{U}_k = \bar{\mu}\bar{\boldsymbol{W}}\boldsymbol{V}_{k-1}.$$

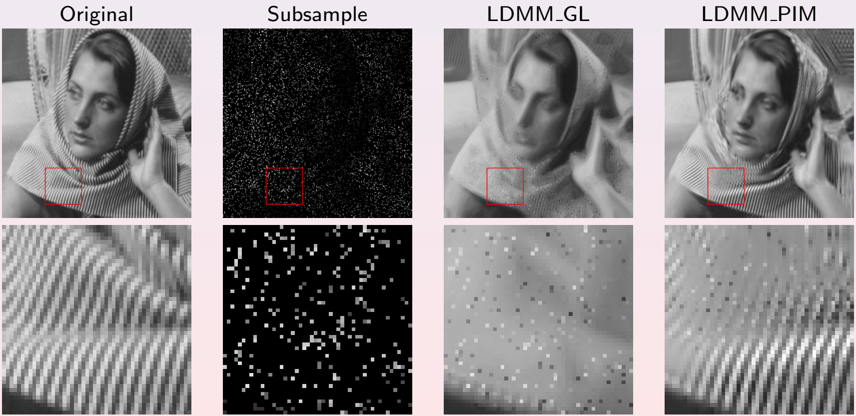
        where $\boldsymbol{V}_k = \left(\mathcal{P}(f^n) - d^k\right)^T$.
        Update f by solving a least square problem

$$f^{n+1,k} = \arg\min_{f \in \mathbb{R}^{m \times n}} \lambda \|b - \Phi f\|_2^2 + \bar{\mu}\|\boldsymbol{U}_k^T - \mathcal{P}(f) + d^{k-1}\|_F^2$$

$$d^k = d^{k-1} + \boldsymbol{U}_k^T - \mathcal{P}(f^{n+1,k})$$

    end for
    $f^{n+1} = f^{n+1,K}$
end while

LDMM_PIM in Image Inpainting



Another Reason Why Graph Laplacian Fails

Consider an unknown function u defined on a discrete set $\bar{\Omega} \subset \mathcal{M}$. Assume that we know the function value of u on a subset $\Omega \subset \bar{\Omega}$, $u(x) = b(x), \forall x \in \Omega$. Assume also that $|\Omega| \ll |\bar{\Omega}|$. The harmonic extension of u onto $\bar{\Omega}$ is modeled as

$$\min_{u \in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|^2, \quad \text{subject to:} \quad u(x) = b(x), \forall x \in \Omega$$

If we discretize the objective function above using graph Laplacian, we have

$$\begin{aligned}\|\nabla_{\mathcal{M}} u\|^2 &= \sum_{x \in \bar{\Omega}} \sum_{y \in \bar{\Omega}} w(x,y)\,(u(x) - u(y))^2 \\ &= \sum_{x \in \Omega} \sum_{y \in \bar{\Omega}} w(x,y)\,(u(x) - u(y))^2 + \sum_{x \in \bar{\Omega} \setminus \Omega} \sum_{y \in \bar{\Omega}} w(x,y)\,(u(x) - u(y))^2\end{aligned}$$

The first term on the right is of order $|\Omega|$, which is much smaller than that of the second term $|\bar{\Omega} \setminus \Omega|$. This causes the first term to be neglected in the minimization, and the algorithm sacrifices the continuity of u on Ω for small variation in $\bar{\Omega} \setminus \Omega$.

Weighted Graph Laplacian (WGL)

An easy fix is to put an extra weight μ in front of the first term.

$$\|\nabla_{\mathcal{M}} u\|^2 = \mu \sum_{x\in\Omega}\sum_{y\in\bar{\Omega}} w(x,y)\left(u(x)-u(y)\right)^2 + \sum_{x\in\bar{\Omega}\setminus\Omega}\sum_{y\in\bar{\Omega}} w(x,y)\left(u(x)-u(y)\right)^2$$

To balance the orders of the two terms, μ is chosen to be $\frac{|\bar{\Omega}|}{|\Omega|}$. The corresponding Euler-Lagrange equation is:

$$\begin{cases} \sum_{y\in P} 2w(x,y)(u(x)-u(y)) + (\mu-1)\sum_{y\in S} w(y,x)(u(x)-g(y)) = 0, & x\in P\setminus S, \\ u(x) = g(x), & x\in S. \end{cases} \tag{3}$$

On the other hand, if we use the point integral method to solve the interpolation problem, the resulting linear system would be:

$$\sum_{y\in P} R_t(x,y)(u(x)-u(y)) + \frac{2}{\lambda}\sum_{y\in S} R_t(x,y)(u(y)-g(y)) = 0, \tag{4}$$

where $0<\lambda\ll 1$. By comparing (3) and (4), it is clear that WGL can also be derived by replacing $u(y)$ in the second term of (4) by $u(x)$.

LDMM_WGL for Image Inpainting

Notice that a key step in LDMM for image inpainting is to solve the following optimization problem:

$$\min_{f\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M}^k)},$$

$$\text{subject to:} \quad \alpha_i\left(\mathcal{P}(f^k)(x)\right) = \mathcal{P}_i f(x), \qquad \forall x\in\bar{\Omega}, i=1,\cdots,d,$$

$$f(x) = b(x), \qquad \forall x \in \Omega \subset \bar{\Omega},$$

where $\mathcal{P}_i f(x)$ is the i-th element of the patch $\mathcal{P}f(x)$. We use the notation $x_{\widehat{i-1}}$ to denote the $(i-1)$-th element after x in a patch, i.e. $\mathcal{P}_i f(x) = f(x_{\widehat{i-1}})$.

If we use periodic padding near the boundary, the ajoint operator $\mathcal{P}_i^* = \mathcal{P}_i^{-1}$

LDMM_WGL for Image Inpainting

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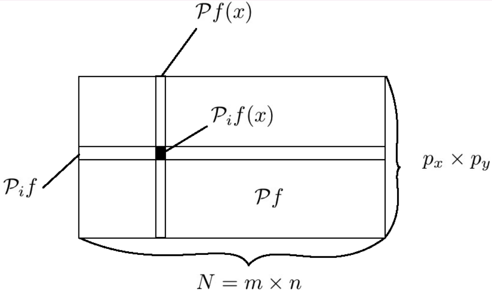
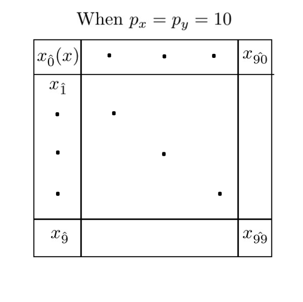




LDMM_WGL for Image Inpainting

$$\min_{f\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M}^k)},$$

$$\text{subject to:} \quad \alpha_i\left(\mathcal{P}(f^k)(x)\right) = \mathcal{P}_i f(x), \qquad \forall x\in\bar{\Omega}, i=1,\cdots,d,$$

$$f(x) = b(x), \qquad \forall x\in\Omega\subset\bar{\Omega},$$

Applying WGL, we have the following discretized optimization problem:

$$\min_{f\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} \left(\sum_{x\in\bar{\Omega}\backslash\Omega_i} \sum_{y\in\bar{\Omega}} \bar{w}(x,y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 + \frac{mn}{|\Omega|} \sum_{x\in\Omega_i} \sum_{y\in\bar{\Omega}} \bar{w}(x,y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 \right) \quad \text{subject to: } f(x) = b(x), \quad x\in\Omega\subset\bar{\Omega}$$

where $\Omega_i = \left\{x\in\bar{\Omega} : \quad \mathcal{P}_i f(x) \text{ is sampled}\right\}$, and $\bar{w}(x,y) = w(\mathcal{P}f(x), \mathcal{P}f(y))$

LDMM_WGL

Using a standard variational approach, the equivalent Euler-Lagrange equation is

$$\begin{cases} \left[\sum_{i=1}^{d} \mathcal{P}_i^*(h_i) + \mu \sum_{i=1}^{d} \mathcal{P}_i^*(g_i)\right](x) = 0, & x \in \bar{\Omega} \setminus \Omega \\ f(x) = b(x), & x \in \Omega \end{cases}$$

where

$$h_i(x) = \sum_{y \in \bar{\Omega}} 2\bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

$$g_i(x) = \sum_{y \in \Omega_i} \bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

LDMM_WGL

$$h_i(x) = \sum_{y \in \bar{\Omega}} 2\bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

$$\begin{aligned}
\mathcal{P}_i^* h_i(x) = h_i(x_{\widehat{1-i}}) &= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y) \left(\mathcal{P}_i f(x_{\widehat{1-i}}) - \mathcal{P}_i f(y)\right) \\
&= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y) \left(f(x) - f(y_{\widehat{i-1}})\right) \\
&= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}}) \left(f(x) - f(y)\right)
\end{aligned}$$

Therefore

$$\sum_{i=1}^{d} \mathcal{P}_i^*(h_i)(x) = \sum_{i=1}^{d} \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}}) \left(f(x) - f(y)\right)$$

Similarly,

$$\sum_{i=1}^{d} \mathcal{P}_i^*(g_i)(x) = \sum_{i=1}^{d} \sum_{y \in \Omega} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}}) \left(f(x) - f(y)\right)$$

LDMM_WGL

The Euler-Lagrange equation becomes:

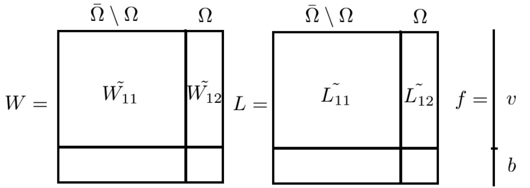
$$\begin{cases} \sum_{y\in\bar{\Omega}} \left(\sum_{i=1}^{d} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\right)(f(x)-f(y)) \\ + \mu \sum_{y\in\Omega} \left(\sum_{i=1}^{d} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\right)(f(x)-f(y)) = 0, & x\in\bar{\Omega}\setminus\Omega \\ f(x) = b(x), & x\in\Omega \end{cases}$$

Let $\tilde{w}(x,y) = \sum_{i=1}^{d} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})$, then

$$2\sum_{y\in\bar{\Omega}} \tilde{w}(x,y)(f(x)-f(y)) + \mu \sum_{y\in\Omega} \tilde{w}(x,y)(f(x)-f(y)) = 0, \quad x\in\bar{\Omega}\setminus\Omega$$

LDMM_WGL

$$2\sum_{y\in\bar{\Omega}} \tilde{w}(x,y)\left(f(x)-f(y)\right)+\mu\sum_{y\in\Omega} \tilde{w}(x,y)\left(f(x)-f(y)\right)=0, \quad x\in\bar{\Omega}\setminus\Omega$$



Let $\Delta = \text{diag}(\text{sum}(\tilde{W}_{12},2))$, then

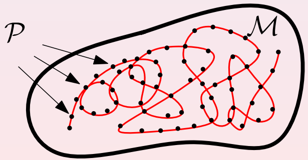
$$\begin{aligned} 2\tilde{L}_{11}v+2\tilde{L}_{12}b+\mu(\Delta v-\tilde{W}_{12}b)&=0\\ (2\tilde{L}_{11}+\mu\Delta)v&=\mu\tilde{W}_{12}b-2\tilde{L}_{12}b \end{aligned}$$

Semi-local Patches

The semi-local patches are obtained by adding local coordinates to the nonlocal patches with a weight λ, i.e.

$$\bar{\mathcal{P}}f(x) = [\mathcal{P}f(x), \lambda x] .$$

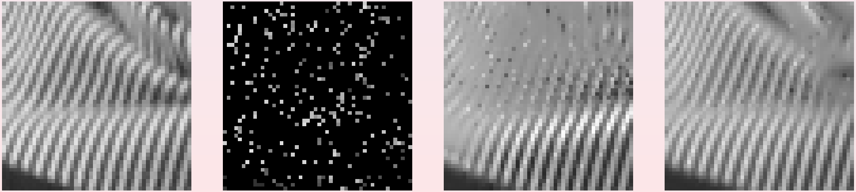
When $\lambda = 0$, semi-local patches are just nonlocal patches. When $\lambda \to \infty$, the patches are completely determined by local coordinates. We choose a proper λ to help LDMM update the "true" metric on the manifold $\mathcal{M}$ faster and more reliably.



LDMM_WGL with Semi-local Patches

Original | Subsample (10%) | LDMM_PIM | LDMM_WGL

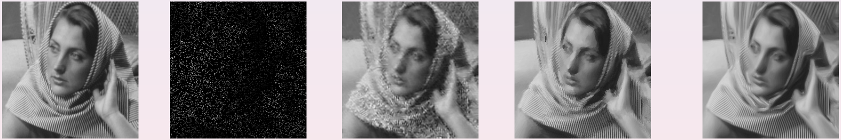
PSNR = 24.74 | PSNR = 26.16



Numerical Results

2D Image Inpainting with 10% Subsample

Original | Subsample | PLE (22.48dB) | EBI (23.91dB) | LDMM (**26.16dB**)



Original | Subsample | PLE (24.19dB) | EBI (23.74dB) | LDMM (**25.94dB**)

Image Denoising

original

noisy (8.13dB)

LDMM (**23.91dB**)

BM3D (23.71dB)

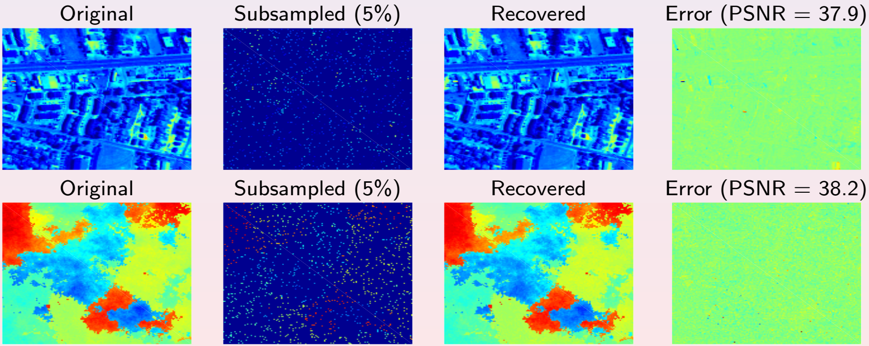
original

noisy (8.13dB)

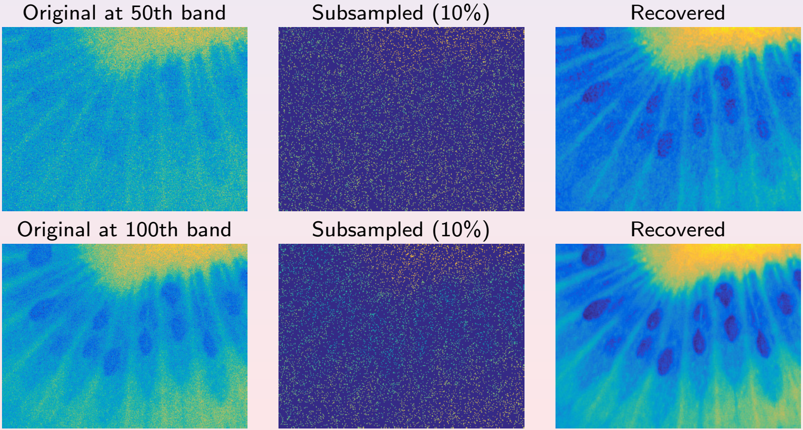
LDMM (24.41dB)

BM3D (**24.61dB**)

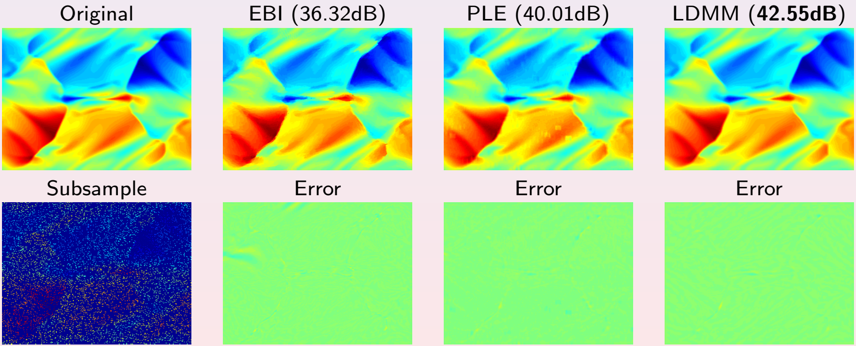
Hyperspectral Image Inpainting



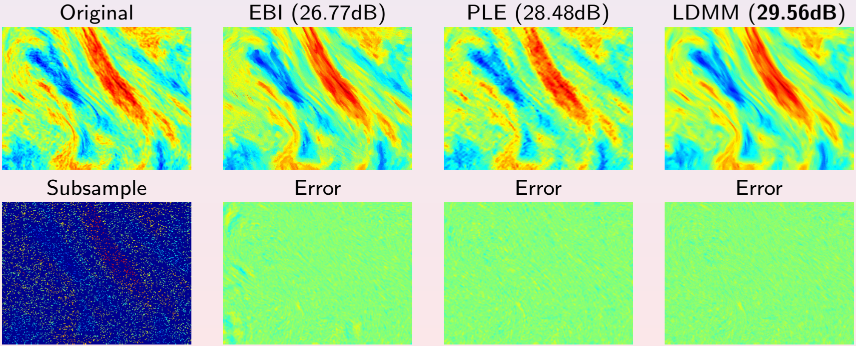
Noisy and Incomplete Hyperspectral Images



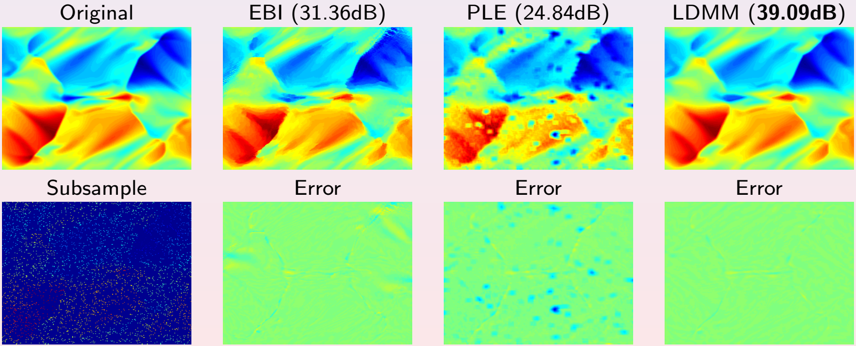
2D Scientific Data Interpolation with 10% Random Subsample



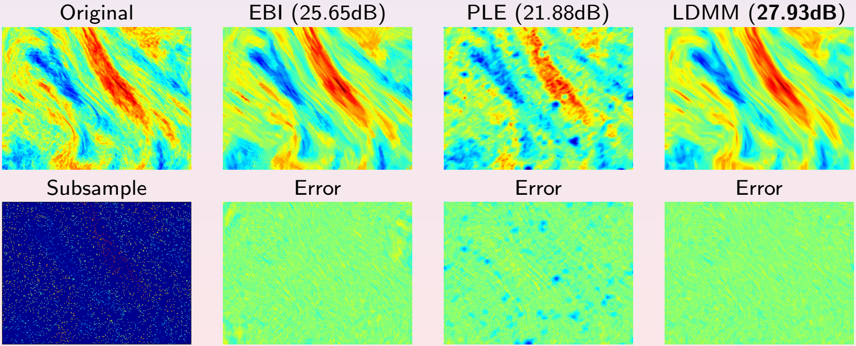
2D Scientific Data Interpolation with Random 10% Subsample



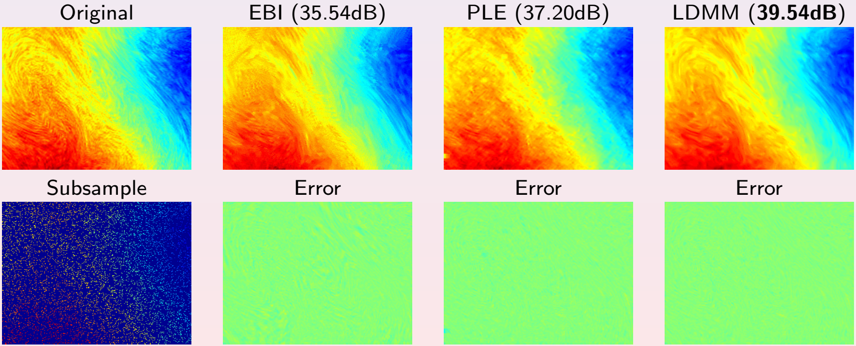
2D Scientific Data Interpolation with 5% Random Subsample



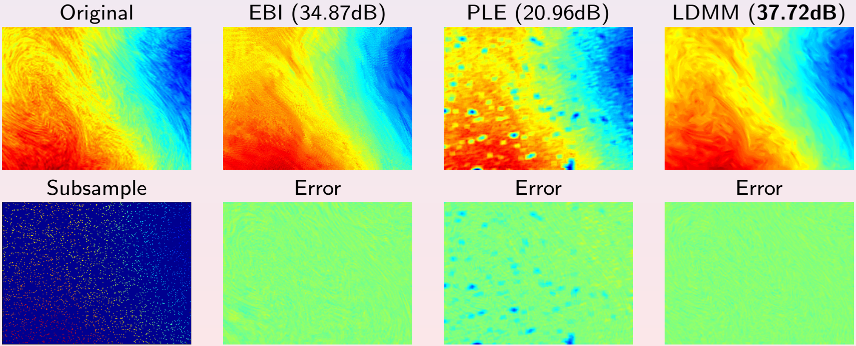
2D Scientific Data Interpolation with 5% Random Subsample



3D Scientific Data Interpolation with 10% Random Subsample

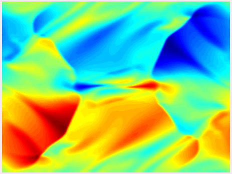


3D Scientific Data Interpolation with 5% Random Subsample

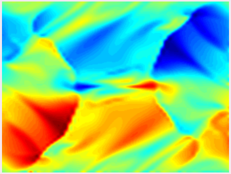


Interpolation of 2D Scientific Data from Regular Sampling with spacing 4×4

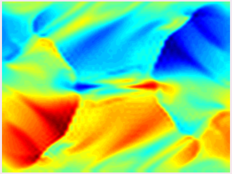
Original



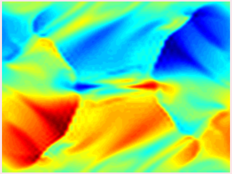
Cubic Spline (42.98dB)



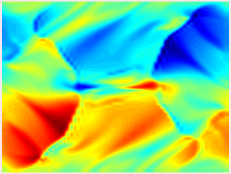
DCT (42.88dB)



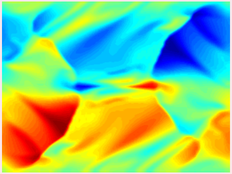
DFT (43.19dB)



Wavelet (40.48dB)

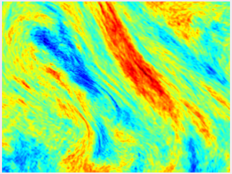


LDMM (**44.40dB**)

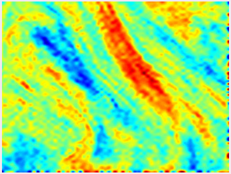


Interpolation of 2D Scientific Data from Regular Sampling with spacing 4×4

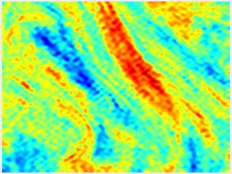
Original



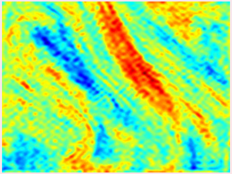
Cubic Spline (26.81dB)



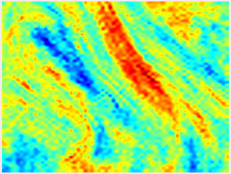
DCT (27.68dB)



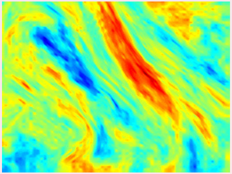
DFT (27.43dB)



Wavelet (27.34dB)

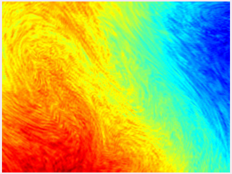


LDMM (**29.66dB**)

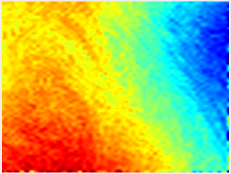


Interpolation of 3D Scientific Data from Regular Sampling with spacing $4 \times 4 \times 1$

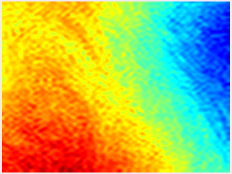
Original



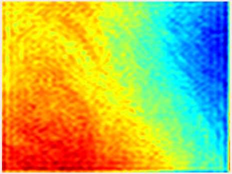
Cubic Spline (36.47dB)



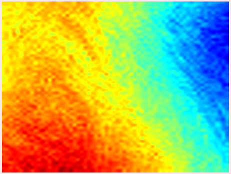
DCT (37.35dB)



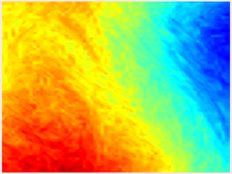
DFT (32.45dB)



Wavelet (37.02dB)



LDMM (**39.18dB**)



Conclusion

- LDMM uses the dimension of the patch manifold to regularize the variational problem.
- The Laplace-Beltrami equation can be solved via either the point integral method or the weighted graph Laplacian.
- Weighted graph Laplacian is much more efficient for image inpainting, because the equation is solved on the image domain instead of the patch domain.

Thank you for listening.