Variational Approach to Image Segmentation and Inpainting

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joint research with
Michele Carriero & Antonio Leaci (Università del Salento, Italy).

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in **image segmentation** and **inpainting**,

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the decomposition of the image is performed by choosing a pattern of lines of steepest discontinuity for light intensity,
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In image restoration the term **inpainting** denotes
the process of filling in the missing information
over subdomains where a given image is damaged:
these domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or aging of canvas and colors in a painting.


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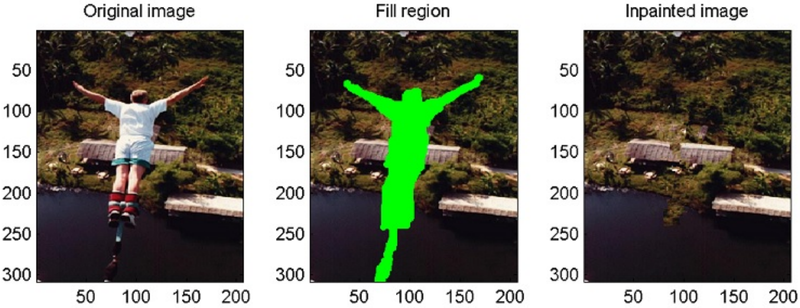
DEVELOPMENT THROUGH TECHNOLOGY


Original image
Fill region
Inpainted image
50
100
150
200
250
300
50
100
150
200

We focus on

the mathematical analysis of Blake & Zisserman functional,
[A.Blake, A.Zisserman, *Visual Reconstruction*, The MIT Press, Cambridge, 1987]

exploiting this variational approach for both segmentation and inpainting:

1. existence of strong solution
   with Dirichlet boundary condition is shown,
2. several extremality conditions on optimal segmentation are stated,
3. well-posedness of the problem is discussed,
4. non trivial local minimizers are analyzed
5. a variational approximation is introduced and implemented.

MS and TV approach to inpainting

A general introduction to Image Inpainting is contained in:

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- P. Markowich, *Applied Partial Differential Equations,* Springer, 2007.

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Total variation

- L.I. Rudin, S.Osher & E.Fatemi, *Nonlinear total variation based noise removal algorithms,* Physica D 60, (1992).
- ...

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- M.Fornasier & C.B.Schönlieb, *Subspace correction methods for total variation and* ℓ^1 *minimization,* SIAM J.Num.An., (2009).
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- A.Bertozzi-S.Esedoglu-A.Gillette *A Cahn-Hilliard model for binary image inpainting* (2007).

Exact reconstruction of damaged color images by a TV model

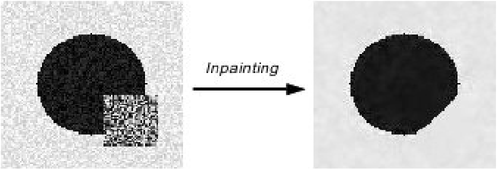
[I. Fonseca - G. Leoni - F. Maggi - M. Morini,
Ann. Inst. H. Poincaré Anal. Non Linéaire, 2010].

reconstruction of damaged piecewice constant color images studied using a RGB total-variation based model for colorization inpainting.

vector-valued functions

When used for Image Inpainting, the (Mumford-Shah based) first order variational approach fosters straight edges and produces artificial corners as shown in the following example taken from

- [S.Esedoglu, J.Shen, Europ.J.Applied Mathematics, 2002].



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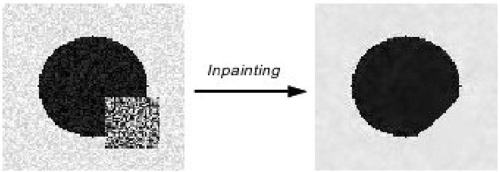

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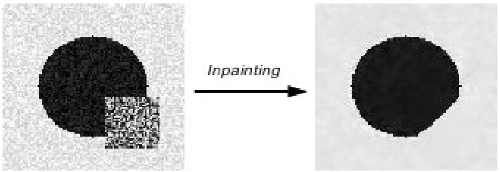

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Here we adopt another strategy based on a second order variational model with free discontinuity.

This by now classic variational model for image segmentation has been set by **Mumford & Shah**, who introduced the functional

$$\int_{\Omega \setminus K} \left(|Du(x)|^2 + \mu\, |u(\mathbf{x}) - g(\mathbf{x})|^2 \right) d\mathbf{x} \; + \; \gamma\, \mathcal{H}^{n-1}(K \cap \Omega) \qquad (1)$$

where

- $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is an open set,
- $K \subset \mathbb{R}^n$ is a closed set,
- u is a scalar function,
- Du denotes the distributional gradient of u,
- $g \in L^2(\Omega)$ is the datum (grey intensity levels of the given image),
- $\gamma > 0,\ \mu > 0$ are parameters related to the selected contrast threshold,
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According to this model
the segmentation of the given image is achieved by

minimizing (1) among admissible pairs (K,u),
say closed $K\subset\mathbb{R}^n$ and $u\in C^1(\Omega\setminus K)$.

This model led in a natural way to the study of
a new type of functional in Calculus of Variations:

free discontinuity problem.

Existence of minimizers of (1) was proven by

- De Giorgi, Carriero & Leaci (1989)

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Further regularity properties of optimal segmentation
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- [Ambrosio, Fusco & Pallara (2000)],
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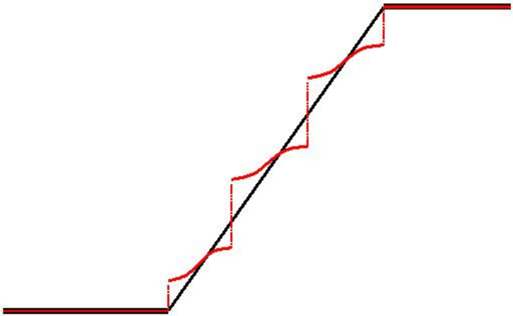
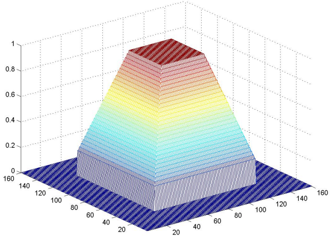
- [De Lellis, Focardi, (2013), dens.l.b. explicit constant,$n = 2$],
- [Bucur, Luckhaus, (2014), monotonicity formula].

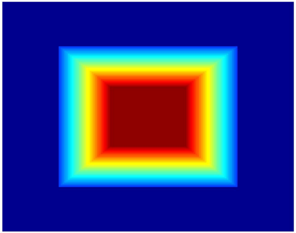
stair-casing effect



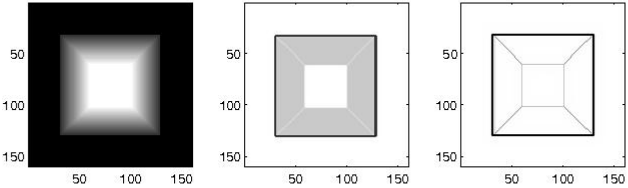
MS does not detect “crease discontinuities”

MS does not detect the crease discontinuities,
i.e. the points where u is continuous while ∇u is discontinuous:
indeed MS has no energy cost for creases.





BZ detects "crease discontinuities"



M-S B-Z

To overcome the problems and aiming to better description of stereoscopic images they proposed a different functional including second derivatives.

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Blake & Zisserman variational principle faces segmentation as a minimum problem:

input is given by intensity levels of a monochromatic image,

output is given by

- meaningful boundaries whose length is penalized (correspond to discontinuity set of the given intensity and of its first derivatives)
- a piece-wise smooth intensity function (smoothed on each region in which the domain is splitted by such boundaries).

another problem with free discontinuity: Blake & Zisserman functional

$$
\begin{aligned}
F(K_0, K_1, v) &= \\
&= \int_{\Omega \setminus (K_0 \cup K_1)} \left(\left| D^2 v(\mathbf{x}) \right|^2 + \mu \left| v(\mathbf{x}) - g(\mathbf{x}) \right|^2 \right) \, d\mathbf{x} + \\
&\qquad + \alpha \, \mathcal{H}^{n-1}(K_0) + \beta \, \mathcal{H}^{n-1}(K_1 \setminus K_0)
\end{aligned} \tag{2}
$$

another problem with free discontinuity: Blake & Zisserman functional

$$F(K_0, K_1, v) = \\ = \int_{\Omega\setminus(K_0\cup K_1)} \left(\left|D^2 v(\mathbf{x})\right|^2 + \mu \left|v(\mathbf{x}) - g(\mathbf{x})\right|^2 \right) d\mathbf{x} + \\ + \alpha\, \mathcal{H}^{n-1}(K_0) + \beta\, \mathcal{H}^{n-1}(K_1 \setminus K_0) \qquad (2)$$

to be **minimized among admissible triples (K_0, K_1, v)**:

- K_0 , K_1 Borel subsets of $\mathbb{R}^n$ s.t. $K_0 \cup K_1$ is closed
- $u \in C^2(\Omega \setminus (K_0 \cup K_1))$ and approximately continuous on $\Omega \setminus K_0$.

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with data:

- $\Omega \subset \mathbb{R}^n$ open set, $n \geq 1$,
- $g \in L^2(\Omega)$ grey level intensity of the given image,
- α, β, μ positive parameters
  (chosen accordingly to scale and contrast threshold).

Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure.

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- $n = 1$, [Coscia]
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via direct method in calculus of variations:
solution of a weak formulation of minimum problem
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Since we looked for a weak formulation
of a free discontinuity problem,
we wrote a suitable relaxed form relaxed version of BZ functional;
this form depends only on u (not on triples!):
optimal segmentation ($K_0 \cup K_1$) has to be recovered through
jumps (discontinuity set of u) and creases (discontinuity set of ∇u)
[C-L-T, in PNLDE, 25 (1996)]

We proved also several (interior) density estimates for minimizers energy and optimal segmentation:

- [C-L-T, Nonconvex Optim. Appl.55 (2001)],
- [C-L-T, C.R.Acad.Sci.(2002)],
- [C-L-T J. Physiol.(2003)].

Moreover, with Gamma-convergence techniques,

- [Ambrosio, Faina & March, SIAM J.Math.An. (2002)] obtained an efficient approximation of Blake & Zisserman functional with elliptic functionals,

and numerical implementation was performed also by

- [R.March]
- [M.Carriero, A.Farina, I.Sgura].

GNC and finite differences for BZ

- [F.Doveri, 1997] Γ–convergence and implementation of GNC algorithm proposed by Blake & Zisserman.
- [A.Braides, 2003] 1D

$$E_\varepsilon(u) \; = \sum_{\text{gridpoints}} \varepsilon\,\psi_\varepsilon\left(\frac{u(x+\varepsilon)+u(x-\varepsilon)-2u(x)}{\varepsilon^2}\right)$$

  where $\psi_\varepsilon(t) = \min(t^2, \mu/\varepsilon)$

- [A.Braides, A. DeFranceschi & E.Vitali, ESAIM Math.Model.Numer.Anal. 2012] 2D

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About uniqueness and well-posedness:

- [T.BOCCELLARI - F.T., Ist.Lombardo Rend.Sci. 2008,] ($n \geq 1$),
- [T.BOCCELLARI - F.T., Revista Mat. Complut., 2013] ($n = 1$).

notice that 1-dimensional case leads to a much simpler formulation, since (only in 1-d) strong and weak functional coincide.

1-d Blake & Zisserman 1-d functional

Given $g \in L^2(0,1)$, $\alpha, \beta\mu \in \mathbb{R}$ we set $F^g_{\alpha,\beta,\mu}$:

$$F^g_{\alpha,\beta,\mu}(u) = \int_0^1 |\ddot{u}(x)|^2 \, dx + \int_0^1 \mu \, |u(x) - g(x)|^2 \, dx + \alpha \, \sharp(S_u) + \beta \, \sharp(S_{\dot{u}} \setminus S_u) \quad (3)$$

to be minimized among $u \in L^2(0,1)$ s.t.
$\sharp\,(S_u \cup S_{\dot{u}}) < +\infty$ and $u', u'' \in L^2(I)$ $\forall$ interval $I \subseteq (0,1) \setminus (S_u \cup S_{\dot{u}})$.

Notation:
$\dot{u}$ denotes the absolutely continuous part of u',
$\ddot{u}$ the absolutely continuous part of $(\dot{u})'$,
$S_u \subseteq (0,1)$ the set of jump points of u,
$S_{\dot{u}} \subseteq (0,1)$ the set of jump points of $\dot{u}$,
$\sharp$ the counting measure.

$n = 1$

Summary of analytic results:

- Euler equations for local minimizers,
- compliance identity for local minimizers,
- a priori estimates on minimum value and minimizers,
- continuous dependence of minimum value $m^g(\alpha, \beta, \mu)$ with respect to g, α, β, μ.

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Theorem

$F^g_{\alpha,\beta}$ achieves its minimum provided the following conditions are fulfilled:

$$0 < \beta \leq \alpha \leq 2\beta < +\infty \tag{4}$$

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Uniqueness fails

Generic uniqueness

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Nevertheless we can exhibit a generic uniqueness result:

Theorem ([T.Boccellari - F.T., Revista Mat. Complut. 2013]) $n = 1$

For any α, β s.t.

$$0 < \beta \le \alpha \le 2\beta \,, \qquad \alpha/\beta \notin \mathbb{Q} \,,$$

there is a G_δ set (*countable intersection of dense open sets*) $E_{\alpha,\beta,\mu} \subset L^2(0,1)$ such that

$$\sharp \left(\operatorname{argmin} F^g_{\alpha,\beta,\mu}\right) = 1 \qquad \text{for all } g \in E_{\alpha,\beta,\mu} \,.$$

The whole picture is coherent with
the presence of sporadic instable patterns,
each of them corresponding to a bifurcation of optimal segmentation under variation of parameters α e β, related to:

- contrast threshold ($\sqrt{\alpha}$),
- “luminance sensitivity”,
- resistance to noise,
- crease detection ($\sqrt{\beta}$),
- double edge detection.

[CARRIERO, LEACI & T.],
A survey on the BZ functional, MILAN J. MATH. (2015)

Dirichlet problem for BZ functional, $n = 2$

Image InPainting refers to reconstruction of missing or partially occluded regions of an image.

Minimizing Blake & Zisserman functional is useful to achieve contour continuation in the whole image region $\widetilde{\Omega}$ when occlusion or local damage occur in $\widetilde{\Omega} \setminus \Omega$ e.g. blotches in a fresco or a movie film.

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Dirichlet problem ($U \subset\subset \Omega \subset\subset \widetilde{\Omega} \subset \mathbb{R}^2$) : minimize the energy F:

$$F(K_0, K_1, v) \;=\; E(K_0, K_1, v) + \mu \int_{\Omega \setminus U} |v(\mathbf{x}) - g(\mathbf{x})|^2 \; d\mathbf{x} =$$

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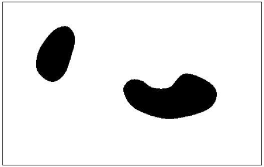
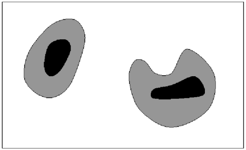
among **admissible triples** (K_0, K_1, v) which assume prescribed data w on $\widetilde{\Omega} \setminus \Omega$: **say** $v = w$ **a.e.** $\widetilde{\Omega} \setminus \Omega$



minimization of functional E: the image domain is the rectangle $\widetilde{\Omega}$. The blotches $\Omega \subset\subset \widetilde{\Omega}$ with complete loss of information are represented by the black region Ω.



minimization of functional F: the image domain is the rectangle $\widetilde{\Omega}$. The blotches $\Omega \subset\subset \widetilde{\Omega}$ correspond to some loss of information: complete loss in the black region $U \subset\subset \Omega \subset\subset \widetilde{\Omega}$, partially damaged image in the gray region $\Omega \setminus U$.

Weak formulation of Dirichlet pb for BZ functional

Minimize $\mathcal{F} : X \to [0, +\infty]$ defined by

$$\mathcal{F}(v) \;=\; \mathcal{E}(v) \,+\, \mu \int_{\Omega \setminus U} |v - g|^2 \, d\mathbf{x} \tag{7}$$

where $\Omega \subset\subset \widetilde{\Omega} \subset \mathbb{R}^2$ are open sets, $\mathbf{x} = (x, y) \in \Omega$,
$X = GSBV^2(\widetilde{\Omega}) \cap L^2(\widetilde{\Omega}) \cap \left\{ v = w \ a.e. \widetilde{\Omega} \setminus \Omega \right\}$
and the main part of $\mathcal{F}$ is denoted by $\mathcal{E}$:

$$\mathcal{E}(v) = \int_{\widetilde{\Omega}} \left(|\nabla^2 v|^2 + \delta \, |v - 1/2|^2 \right) d\mathbf{x} + \alpha \mathcal{H}^1(S_v) + \beta \mathcal{H}^1(S_{\nabla v} \setminus S_v) \tag{8}$$

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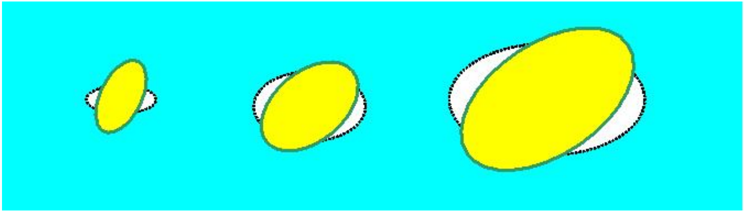
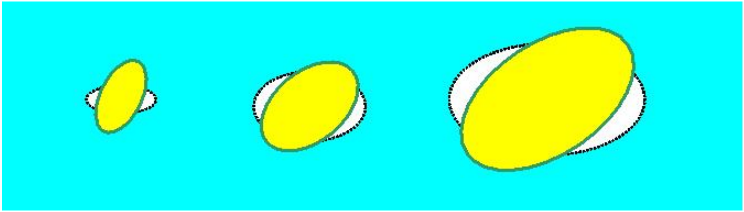
Theorem (C-L-T)

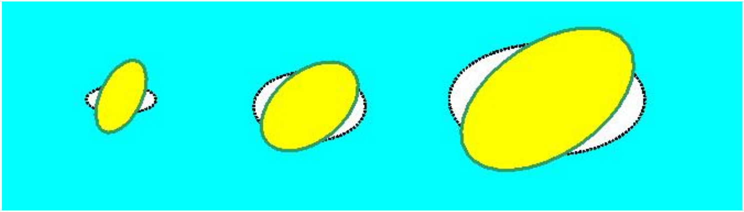
If $g \in L^2(\widetilde{\Omega}), \; w \in X$ and $\beta \le \alpha \le 2\beta$
then $\mathcal{F}$ has at least one minimizer in X.





consider a sequence with smaller and smaller disks
where datum is a steeper and steeper affine function:

1. sublevels of functional $\mathcal{E}$ are not compact on
$$\{\, v \in SBV(\Omega) \;:\; \nabla v \in SBV(\Omega) \times SBV(\Omega) \,\}$$



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2. by letting untilted some of the big disks
we find functions with unbounded gradient
with arbitrarily small energy $\mathcal{E}$

We recall the definitions of some function spaces where derivatives are special measures in the sense introduced by Ennio De Giorgi:

$SBV(\Omega)$ denotes the class of functions $v \in BV(\Omega)$ s.t.

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, dy + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1 .$$

$$SBV_{loc}(\Omega) \; = \; \{ v \in SBV(\Omega') : \; \forall \Omega' \subset\subset \Omega \} \, ,$$

$$GSBV(\Omega) = \big\{ v : \Omega \to \mathbb{R} \; Borel ; \; -k \vee v \wedge k \in SBV_{loc}(\Omega) \; \forall k \big\}$$

$$GSBV^2(\Omega) \; = \; \big\{ v \in GSBV(\Omega), \; \nabla v \in \big(GSBV(\Omega) \big)^2 \big\}$$

disadvantages (of dealing with GBV)

We emphasize that

$GSBV(\Omega)$, $GSBV^2(\Omega)$ are neither vector spaces
(e.g. $1/x \in GSBV$, $1/x + \sin(1/x) \in GSBV$, $n = 1$)
nor subsets of distributions in Ω ($\not\subset L^1$)

Nevertheless
smooth variations of a function in $GSBV^2(\Omega)$
still belong to the same class.

Notice that,

- if $v \in GSBV(\Omega)$, then S_v is countably $(\mathcal{H}^1, 1)$ rectifiable and ∇v exists a.e. in Ω.
- possibly $Dv \neq \nabla v$ in $GSBV^2(\Omega)$
- $S_{\nabla v} = \bigcup_{i=1}^{2} S_{\nabla_i v}$

advantages (of dealing with *GBV*)

Remark

1. $v \in BV \cap L^\infty\,, \quad P(E) < +\infty \quad \Rightarrow \quad v\,\chi_E \in BV$
2. $v \in BV\,, \quad P(E) < +\infty \quad \not\Rightarrow^{\,*} \quad v\,\chi_E \in BV$
3. $v \in BV\,, \quad P(E) < +\infty \quad \Rightarrow \quad v\,\chi_E \in GBV$

* the trace of v could be not integrable, e.g.:

$$n = 2 \quad \Omega = B_1 \quad v = \varrho^{-1/2} \in W^{1,1}(B_1)$$

$$E = \left\{ \mathbf{x} = \{x, y\} \;:\; \frac{1}{k^2+1} < \varrho < \frac{1}{k^2}\,,\; k \in \mathbf{N} \right\}$$

Theorem [C-L-T, Adv.Math.Sci.Appl., 2010] existence of strong minimizer

Assume

$$0 < \beta \le \alpha \le 2\beta,\ \mu > 0,\ \ g \in L^2(\widetilde{\Omega}) \cap L^4_{loc}(\widetilde{\Omega}),\ \ w \in L^2(\widetilde{\Omega}),$$

Ω is a bounded open set with C^2 boundary $\partial\Omega$,

$$w \in C^2\left(\widetilde{\Omega}\right), \qquad D^2 w \in L^\infty(\widetilde{\Omega}).$$

Then there is at least one triple minimizing the functional (6)

$$F(K_0, K_1, v)$$

with finite energy, among admissible triples (K_0, K_1, v) s.t.

$$\begin{cases} K_0\,,\ K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0 \cup K_1 \text{ closed}, \\ v \in C^2\left(\widetilde{\Omega} \setminus (K_0 \cup K_1)\right),\ v \text{ approximately continuous in } \left(\widetilde{\Omega} \setminus K_0\right), \\ v = w \text{ a.e. in } \widetilde{\Omega} \setminus \Omega\,. \end{cases}$$

Steps of the proof

Existence of minimizing triples is achieved by showing partial regularity of the weak solution with penalized Dirichlet datum. The novelty consists in the regularization at the boundary for a free gradient discontinuity problem;

regularity is proven at points with 2-dimensional energy density by:

1. **blow-up technique**
2. **suitable joining along lunulae filling half-disk**
3. **a decay estimate for weak minimizers**

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In the blow-up procedure, two refinements of relevant tools are

- **hessian decay of a function which is bi-harmonic in half-disk and vanishes together with normal derivative on the diameter**
- **a Poincaré-Wirtinger inequality for *GSBV* functions vanishing in a sector** [C-L-T, *Note di Matematica*, 2011] ($v \in GSBV^2(\Omega)$ does not even entail that either v or ∇v belongs to $L^1_{loc}(\Omega)$).

Theorem (Biharmonic extension and L^2 decay of Hessian) [CLT]

Set $B_R^+ = B_R(\mathbf{0}) \cap \{y > 0\} \subset \mathbb{R}^2 \,,\ R > 0\,.$

Assume $\ z \in H^2(B_R^+)\,,\ \ \Delta^2 z \equiv 0\ B_R^+\,,\ \ z = z_y \equiv 0$ on $\{y = 0\}$.)

Then there exists an (obviously unique) extension Z of z in whole B_R such that $\Delta^2 Z \equiv 0\ B_R$.

This extension may increase a lot the L^2 hessian norm of $D^2 Z$ nevertheless it implies nice decay on half-ball:

$$\| D^2 z \|^2_{L^2(B^+_{\eta R})} \ \le\ \eta^2 \| D^2 z \|^2_{L^2(B^+_R)}\,.$$

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nor preserves orthogonality in L^2 or H^2:
cancelation of big norms
may take place in one half-disk and not in the other: Fig.(21)

Duffin extension formula

Assume
$z \in H^2(B_1^+)$,
z is bi-harmonic in B_1^+
$z = \partial z/\partial y = 0$ on $B_1(\mathbf{0}) \cap \{y = 0\}$.

Then
z has a bi-harmonic extension Z in B_1 defined by

$$\begin{cases} Z(x,y) = z(x,y) & \forall\,(x,y) \in B_1^+\,, \\ Z(x,-y) = -z(x,y) + 2y\,z_y(x,y) - y^2 \Delta z(x,y) & \forall\,(x,-y) \in B_1^-\,. \end{cases}$$

Almansi-type decomposition ([Ann.Mat.Pura Appl., 1899]) (revisited: [CLT, J.Math.Pures Appl., 96, 2011])

Let $u \in H^2(B_R \setminus \Gamma)$ and set $\Gamma =$ negative x axis.

Then

$$\Delta_{\mathbf{x}}{}^2 u = 0 \qquad B_R \setminus \Gamma \tag{9}$$

iff

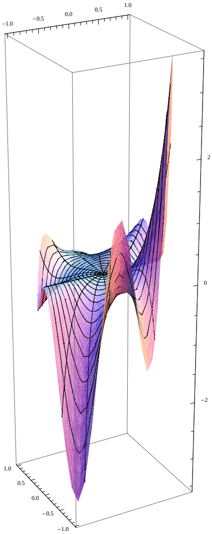
$$\exists\ \varphi, \psi :\ u(\mathbf{x}) = \psi(\mathbf{x}) + \|\mathbf{x}\|^2\,\varphi(\mathbf{x}), \quad \Delta_{\mathbf{x}}\,\varphi(\mathbf{x}) = \Delta_{\mathbf{x}}\,\psi(\mathbf{x}) \equiv 0,\ B_R \setminus \Gamma. \tag{10}$$

Moreover decomposition (10) is unique
up to possible linear terms in ψ:

say $A\varrho\cos\vartheta = Ax$ and $B\varrho\sin\vartheta = By$
that can switch independently to
respectively $A\varrho^{-1}\cos\vartheta$ and $B\varrho^{-1}\sin\vartheta$ in φ.

Back to hessian decay estimate (18)

$$v_k = \varrho^{k+1}\left(\sin((k-1)\vartheta) - \frac{k-1}{k+1}\sin((k+1)\vartheta)\right)$$
$$\omega_k = \varrho^{k+1}\left(\cos((k-1)\vartheta) - \cos((k+1)\vartheta)\right)$$



Several Euler equations in 2 dimensional case [C.L.T] Calc.Var.Part.Diff.Eq, 2008 [C.L.T] J.Math. Pures Appl., 2011

- $\Delta^2 u + \mu u = \mu g \qquad \Omega \setminus (K_0 \cup K_1)$
- Neumann boundary operators (plate-type bending moments) vanishing in $K_0 \cup K_1$
- $\left[\!\left[|D^2 u|^2 + \mu |u - g|^2 \right]\!\right] = \alpha\, \mathcal{K}(K_0)$
- $\left[\!\left[|D^2 u|^2 \right]\!\right] = \beta\, \mathcal{K}(K_1 \setminus K_0)$
- Integral and geometric conditions at the “boundary” of singular set: crack-tip and crease-tip.

Theorem - Uniform density estimates up to the bdry [C-L-T, Discr.Cont.Dyn.Sis.-A 2011]

(**Minkowski content of the segmentation**)
Let (K_0, K_1, u) be an essential locally minimizing triple for the functional F under: structural assumptions $g \in L^4(\widetilde{\Omega})$.

Then $K_0 \cup K_1$ is $(\mathcal{H}^1, 1)$ rectifiable and

$$\lim_{\varrho \downarrow 0} \frac{|\{\mathbf{x} \in \widetilde{\Omega} \, ; \; \mathrm{dist}(\mathbf{x}, (K_0 \cup K_1) \cap \overline{\Omega}) < \varrho \, \}|}{2\varrho} \; = \; \mathcal{H}^1 \, (K_0 \cup K_1) \; .$$

Variational approximations of BZ functional

- [G.Bellettini, A.Coscia] (n=1)
- [L.Ambrosio, L.Faina, R.March, 2001, SIAM J.Math.Anal.] (n=2) (segmentation)
- [M.Carriero, I.Farina, A.Sgura, 2004] implementation of finite difference on Euler-Lagrange equations (segmentation)
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- [M.Carriero, A.Leaci, & F.T., Adv.Calc.Var., 2014] (inpainting)

$$\mathcal{F}_k(v,s,\sigma) = \int_{\widetilde{\Omega}} ((\sigma^2+\kappa_k)|\nabla^2 v|^2) + \mu \int_{\Omega\setminus U} |v-g|^2 + \delta \int_{\Omega} |v-1/2|^2$$

$$+(\alpha-\beta)\,\mathcal{G}_k(s) + \beta\,\mathcal{G}_k(\sigma) + \xi_k \int_{\widetilde{\Omega}} (s^2+\xi_k)|\nabla u|^\gamma$$

with κ_k, ξ_k, ζ_k suitable infinitesimal weights and

$$\mathcal{G}_k(s) = \int_{\widetilde{\Omega}} \left(\frac{1}{k}\,|\nabla s|^2 + k\,\frac{(s-1)^2}{4} \right)$$

Theorem - $\mathcal{F}_k \xrightarrow{\Gamma} \mathcal{F}$ (weak BZ functional for inpainting). (4) E.L.

Euler-Lagrange system.

The system of Euler-Lagrange equations,
associated to the k-th (elliptic) approximating functional $\mathcal{F}_k$
when the term κ_k is neglected, is given by:

$$\begin{cases} \sigma^2 \Delta^2 u + 2\sigma\Big(2D\sigma \cdot D(\Delta u) + u_{xx}\sigma_{xx} + 2u_{xy}\sigma_{xy} + u_{yy}\sigma_{yy}\Big) \\ \qquad + 2(D^2 u D\sigma) \cdot D\sigma - \xi_k \Big(s^2 \Delta u + 2sDu \cdot Ds\Big) \\ \qquad\qquad + \delta\left(u - \frac{1}{2}\right)\chi_\Omega + k\,(u - w)\,\chi_{\widetilde{\Omega}\setminus\Omega} = 0, \\ 4(\alpha - \beta)\Delta s = 4\xi_k k\, s\,|Du|^2 + (\alpha - \beta)k^2\,(s - 1), \\ 4\beta\Delta\sigma = 4k\,\sigma\,|D^2 u|^2 + \beta k^2\,(\sigma - 1), \end{cases}$$

in the open set $\widetilde{\Omega}$.
For the sake of simplicity, we have set $\kappa_k = 0$, and to speed up computations we could assume $\alpha = \beta$, $\xi_k = 0$ and $s \equiv 1$.

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in the open set $\widetilde{\Omega}$.

For the sake of simplicity, we have set $\kappa_k = 0$, and to speed up computations we could assume $\alpha = \beta$, $\xi_k = 0$ and $s \equiv 1$.

The system seems closely related to the stationary/asymptotic version of the scheme for Bertozzi-Esedoglou-Gillette-Cahn-Hilliard model as implemented by [Cherfils-Fakhi-Miranville (2015), Inv.Pbs Imaging] for proving the existence of finite dimensional attractors.

... and numerical experiments

SEGMENTATION

Drone and battleship

Input image: BMP, 400x200 pixel, grey scale

Smooth output image

Segmentation (both jumps and creases)

Parameters: alpha=1e-5, beta=1e-5, mu=22

Drone and battleship...indeed battleships!

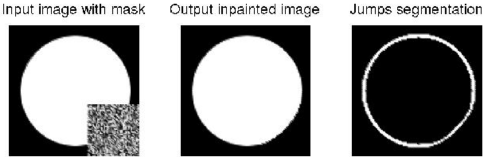
Input image: BMP, 400x200 pixel, grey scale — Smooth output image — Segmentation (both jumps and creases)

Parameters: alpha=1e-5, beta=1e-5, mu=22

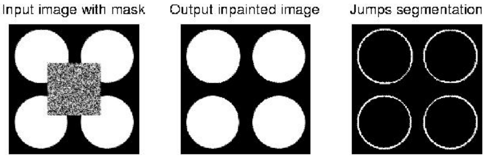
INPAINTING

We conclude by showing some numerical experiments of inpainting based on our variational approximation by elliptic functionals of BZ functional **without fidelity term** in the damaged portion of raw image ([C.L.& T.: St Petersburgh OTARIE 2011, J.Math.Sciences 2012]):

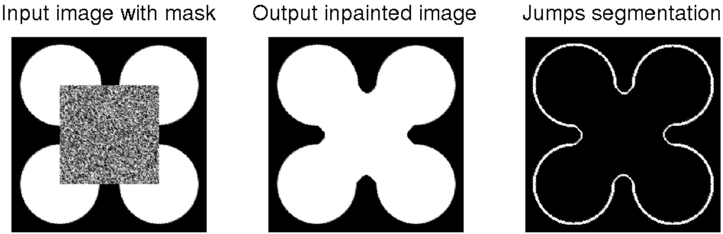
the inpainting algorithm removes masks or overlapping text.



Inpainting of a circle without introducing artificial corners.



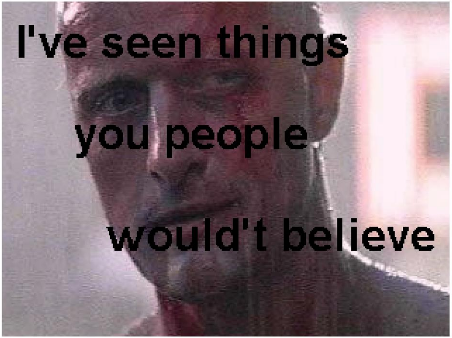
Inpainting of 4 circles (preserved by small mask).



Inpainting of four disks with a big mask is connected!.

INPAINTING (text removal) in color images

Input image



INPAINTING (text removal)

Output image

INPAINTING (text removal) in color images

input image

INPAINTING (text removal)

output image

Euler equations

From now on, for sake of simplicity,
we examine only the main part E of functional F:

$$E(K_0, K_1, v) = \\ = \int_{\Omega \setminus (K_0 \cup K_1)} \left|D^2 v(x)\right|^2 dx + \alpha\, \mathcal{H}^1(K_0) + \beta\, \mathcal{H}^1(K_1 \setminus K_0) \tag{11}$$

and the structural assumption $\beta \leq \alpha \leq 2\beta$ will be always understood.

Euler equations I : smooth variations

Theorem

Any essential locally minimizing triple (K_0, K_1, u) for functional F fulfils

$$\Delta^2 u + \mu(u-g) = 0 \qquad in\ \Omega \setminus (K_0 \cup K_1)\ .$$

Any essential locally minimizing triple (K_0, K_1, u) for the functional E fulfils

$$\Delta^2 u = 0 \qquad in\ \Omega \setminus (K_0 \cup K_1)\ .$$

(2)

Euler equations II : boundary-type conditions on singular set

Necessary conditions on jump discontinuity set K_0 for natural boundary operators

Assume (K_0, K_1, u) is an essential locally minimizing triple for the functional E, $B \subset\subset \Omega$ is an open disk such that $K_0 \cap B$ is a diameter of the disk and $(K_1 \setminus K_0) \cap B = \emptyset$. Then

$$\left(\frac{\partial^2 u}{\partial N^2}\right)^{\pm} = 0 \qquad \text{on } K_0 \cap B\,,$$

$$\left(\frac{\partial^3 u}{\partial N^3} + 2\,\frac{\partial}{\partial N}\left(\frac{\partial^2 u}{\partial \tau^2}\right)\right)^{\pm} = 0 \qquad \text{on } K_0 \cap B$$

where B^+, B^- are the connected components of $B \setminus K_0$, N is the unit normal to K_0 pointing toward B^+, v^+, v^- the traces of any v on K_0 respectively from B^+ and B^-,$\tau = (\tau_1, \tau_2) = (-N_2, N_1)$ the choice of the unit tangent vector to K_0 .

Euler equations III : singular set variations

Next we evaluate the first variation of the energy
around a local minimizer u,
under compactly supported smooth deformation of K_0 and K_1

Euler equations III : singular set variations

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around a local minimizer u,
under compactly supported smooth deformation of K_0 and K_1

Integral Euler equation

If (K_0, K_1, u) is a locally minimizing triple of E. Then $\forall \eta \in C_0^2(\Omega, \mathbb{R}^2)$

$$\int_{\Omega\setminus(K_0\cup K_1)} \Big(|D^2 u|^2 \operatorname{div} \eta - 2\big(D\eta\, D^2 u + (D\eta)^t\, D^2 u + Du\, D^2 \eta \big) : D^2 u \Big) d\mathbf{x}$$

$$+ \alpha \int_{K_0} \operatorname{div}^{\tau}_{K_0} \eta \, d\mathcal{H}^1 + \beta \int_{K_1\setminus K_0} \operatorname{div}^{\tau}_{K_1\setminus K_0} \eta \, d\mathcal{H}^1 \ = \ 0 \, ,$$

where $\operatorname{div}^{\tau}_S$ denotes the tangential (to set S) divergence and

$$\big(D\eta D^2 u + (D\eta)^t D^2 u + Du D^2 \eta\big)_{ij} =$$

$$= \sum_k \Big(D_k \eta_i D^2_{kj} u + D_i \eta_k D^2_{kj} u + D_k u D^2_{ij} \eta_k \Big)$$

Curvature of jump set K_0 and squared hessian jump

If (K_0, K_1, u) is an essential locally minimizing triple for functional E, $B \subset\subset U \subset \widetilde{\Omega}$ two open disks, s.t. $K_0 \cap U$ is the graph of a C^4 function, B^+ (resp. B^-) the open connected epigraph (resp. subgraph) of such function in B,.
$K_1 \cap U = \emptyset$, and u in $W^{4,r}(B^+) \cap W^{4,r}(B^-)$, $r > 1$.
Then

$$\left[\!\left[|D^2 u|^2 \right]\!\right] \; = \; \alpha\, \mathcal{K}(K_0) \qquad \text{on } K_0 \cap B\,.$$

where we denote
by $\mathcal{K}$ the curvature and by $[\![w]\!]$ the jump of a function w on K_0

Analogous results holds true for crease set $K_1 \setminus K_0$

Both results follows by plugging
(normal to singular set) vector fields in Integral Euler equation

Crack-tip

Now we perform a qualitative analysis of the “boundary” of the singular set, by assuming it is manifold as smooth as required by the computation of boundary operators.
The strategy is a new choice of the test functions in Euler equation: a vector field η tangential to K_0 (or K_1). (28)

Crack-tip

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Crack-tip Theorem

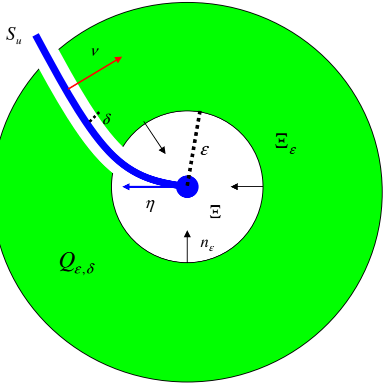
Assume (K_0, K_1, u) is an essential locally minimizing triple of E, $B = B(\mathbf{x}_0) \subset \Omega$ an open disk with center at $\mathbf{x}_0$ s.t. $(K_1 \setminus K_0) \cap B = \emptyset$, $K_0 \cap B = \overline{S_u} \cap B$ is a is a smooth curve from center to bdry of B and

$$\exists\, r > 1 \ : \quad u \in W^{4,r}\big(U \setminus (K_0 \cup B_k(\mathbf{x}_0))\big) \qquad \forall\, k > 0$$

Then u fulfils, for every $\eta \in C_0^3(B, \mathbb{R}^2)$ s.t. $\eta = \zeta\tau$
($\zeta \in C_0^\infty(B)$, $\tau \in C^3(B, S^1)$) and
η vector field tangent to K_0 pointing toward K_0 at $\mathbf{x}_0$)

$$\lim_{\varepsilon \to 0_+} \int_{\partial B_\varepsilon(\mathbf{x}_0) \setminus K_0} \mathcal{L}^\eta(u)\, d\mathcal{H}^1 \;=\; \alpha\, \zeta(\mathbf{x}_0)$$

Crack-tip


S_u
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n_ε
$Q_{\varepsilon,\delta}$

Mode 1 (JUMP) :

$$\varrho^{3/2}\,\omega(\theta) \;=\; \varrho^{3/2}\left(\sin\frac{\theta}{2} \;-\; \frac{5}{3}\,\sin\left(\frac{3}{2}\theta\right)\right) \qquad -\pi < \theta < \pi$$

Mode 2 (CREASE) :

$$\varrho^{3/2}\,\mathrm{w}(\theta) \;=\; \varrho^{3/2}\left(\cos\frac{\theta}{2} \;-\; \frac{7}{3}\,\cos\left(\frac{3}{2}\theta\right)\right) \qquad -\pi < \theta < \pi$$

CANDIDATE:

$$W \;=\; \pm\sqrt{\frac{\alpha}{193\pi}}\;\varrho^{3/2}\left(\sqrt{21}\,\omega(\theta) \pm \mathrm{w}(\theta)\right) \qquad -\pi < \theta < \pi$$

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W fulfils all Euler equations,
all constraints on jump and curvature of singular set and

Energy equipartition: $$\int_{B_\varrho(0)} |\nabla^2 u|^2\, d\mathbf{x} \;=\; \alpha\varrho$$

Candidate conjecture

Assume $0 < \beta \leq \alpha \leq 2\beta < +\infty$.

Then triple

$$(\, K_0 = \text{ negative real axis}\,,\ K_1 = \emptyset\,,\ \text{ function } W\,)\,)$$

is a locally minimizing triple for E in $\mathbb{R}^2$.

Moreover we conjecture that
there are no other nontrivial locally minimizing triples
with non empty jump set and different from triples

$$(K_0 = \text{ closed negative real axis},\ K_1 = \emptyset,\ \Phi)$$

$$\Phi = (A\,\omega(\vartheta) \,+\, B\,w(\vartheta))\ r^{3/2}, \quad 35\,A^2 + 37\,B^2 = \frac{4\,\alpha}{\pi}, \quad A \neq 0$$

possibly swayed by rigid motions of $\mathbb{R}^2$ co-ordinates
and/or addition of affine functions.

Proving the minimality of a given candidate for a free discontinuity problem is a difficult task in general.

As far as we know,
neither the calibration techniques [ALBERTI, BOUCHITTE, DALMASO], nor the method used by [BONNET, DAVID] (both successfully applied to Mumford & Shah functional to test non trivial minimizers) seem to apply to the present context of second order functionals.

Even the *excess identity* approach of [PERCIVALE & T.],
which succeeds with second order functionals related to elasto-plastic plates,
does not apply to the present context since Blake & Zisserman functional do not control $\int_{S_{Dv}} |[Dv]| d\mathcal{H}^1$.

Mumford-Shah functional

Theorem [M.Carriero, A.Leaci, D.Pallara, E.Pascali]

If $(\mathbb{R}^-, u)$ is a local minimizer in $\mathbb{R}^2$ of

$$\int_{B_1} |\nabla v|^2 + \alpha \mathcal{H}^1(S_v)$$

then

$$u(\rho,\theta) = a_0 \pm u^S(\rho,\theta) + u^R(\rho,\theta))$$

where

$$u^S(\rho,\theta) = \sqrt{\frac{2\alpha}{\pi}}\rho^{1/2}\sin\frac{\theta}{2}\,, \qquad u^R(\rho,\theta) = o(\rho^{1-\varepsilon})$$

CONJECTURE (E.De Giorgi)

$$\psi(\rho,\theta) \;=\; \sqrt{\frac{2\,\alpha}{\pi}}\,\rho^{1/2}\,\sin\frac{\theta}{2}$$

is a local minimizer of Mumford-Shah functional in $\mathbb{R}^2$.
ψ **is the only non trivial local minimizer in $\mathbb{R}^2$**
(up to the sign and/or a rigid motion and constant addition)

where local minimizer of M–S functional refers to
compactly-supported variation
(without topological restrictions)

With a slightly different definition
competitor for (u, K)**:** any pair (w, H) s.t. .... and
if $x, y \in \mathbb{R}^2 \setminus (K \cup B_R)$ are separated by K,
then also H separates them,

A.Bonnet & G.David proved the conjecture in a weak form.
(the difference does not play any role for candidate ψ.)

Theorem - Uniform density estimates up to the bdry [C-L-T, Pure Math.Appl., 2009]

(**Density upper bound for the functional** F)

Let (K_0, K_1, u) be an essential locally minimizing triple for the functional F under structural assumptions, $g \in L^4_{loc}(\Omega)$, and

$$\exists \bar{\varrho} > 0 \ : \ \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \partial\Omega\,,\ \forall \varrho \le \bar{\varrho}\,.$$

Then for every $0 < \varrho \le (\bar{\varrho} \wedge 1)$ and for every $x \in \overline{\Omega}$ such that $\overline{B}_\varrho(x) \subset \widetilde{\Omega}$ we have

$$F_{\overline{B}_\varrho(x) \cap \overline{\Omega}}(K_0, K_1, u) \le c_0\varrho$$

where $c_0 = C^2\pi + 2\pi^{\frac{1}{2}}\mu\big(\|w\|^2_{L^4(B_\varrho(\mathbf{x}))} + \|g\|^2_{L^4(B_\varrho(\mathbf{x}))}\big) + (2\pi + C)\alpha$.

Theorem - Uniform density estimates up to the bdry [C-L-T, Discr.Cont.Dyn.Sis.-A 2011]

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(**Density lower bound for the functional** F)
Then there exist $\varepsilon_0 > 0$, $\varrho_0 > 0$ such that, for every $0 < \varrho \le (\bar\varrho \wedge 1)$ and for every $x \in \overline{\Omega}$ such that $\overline{B}_\varrho(x) \subset \widetilde{\Omega}$ we have

$$F_{B_\varrho(\mathbf{x})}(K_0, K_1, u) \ \ge \ \varepsilon_0\varrho \qquad \forall x \in (K_0 \cup K_1) \cap \overline{\Omega}, \quad \forall \varrho \le \varrho_0$$

(**Density lower bound for the segmentation length**)
and there exist $\varepsilon_1 > 0$, $\varrho_1 > 0$ such that

$$\mathcal{H}^1\left((K_0 \cup K_1) \cap B_\varrho(\mathbf{x})\right) \ \ge \ \varepsilon_1\varrho \qquad \forall x \in (K_0 \cup K_1) \cap \overline{\Omega}, \ \forall \varrho \le \varrho_1.$$

Theorem - Asymptotic expansion of loc.min. triples with crack-tip

Assume $(\Gamma, \emptyset, u)$ is a locally minimizing triple of E in $\mathbb{R}^2$,
where $\Gamma =$ denotes the closed negative real axis.
Then there are constants A, B with $(A, B) \neq (0, 0)$ and A_h, B_h s.t.

$$
\begin{aligned}
u(r,\theta) = \\
= r^{3/2}\Big(A\Big(\sin\left(\tfrac{\theta}{2}\right) - \tfrac{5}{3}\sin\left(\tfrac{3}{2}\theta\right)\Big) + B\Big(\cos\left(\tfrac{\theta}{2}\right) - \tfrac{7}{3}\cos\left(\tfrac{3}{2}\theta\right)\Big)\Big) + \\
+ \sum_{h=1}^{+\infty} r^{h+\frac{3}{2}}\Big(A_h \cos\Big(\big(h+\tfrac{3}{2}\big)\theta\Big) + B_h \sin\Big(\big(h+\tfrac{3}{2}\big)\theta\Big) + \\
- \tfrac{2h+3}{2h+7} A_h \cos\Big(\big(h-\tfrac{1}{2}\big)\theta\Big) - \tfrac{2h+3}{2h-5} B_h \sin\Big(\big(h-\tfrac{1}{2}\big)\theta\Big)\Big)
\end{aligned}
$$

where u is expressed by polar coordinates in $\mathbb{R}^2$
with $\theta \in (-\pi, \pi)$ and $r \in (0, +\infty)$.

This expansion is strongly convergent in $H^2(B_\varrho \setminus \Gamma)$, moreover ...

... the lower order term ($h = 0$) in the expansion must have the following form

$$\begin{cases} W_0 \;=\; (A\,\omega(\vartheta) + B\,w(\vartheta))\; r^{3/2} \quad \text{in } B_\varrho \setminus \Gamma\,, \quad \textit{referring to } \textbf{modes}: \\ \textit{Mode 1 (Jump)} \qquad \omega(\vartheta) \;=\; \left(\sin\left(\frac{\vartheta}{2}\right) - \frac{5}{3}\sin\left(\frac{3}{2}\vartheta\right)\right) \\ \textit{Mode 2 (Crease)} \qquad w(\vartheta) \;=\; \left(\cos\left(\frac{\vartheta}{2}\right) - \frac{7}{3}\cos\left(\frac{3}{2}\vartheta\right)\right) \end{cases}$$

where $\vartheta \in (-\pi, \pi)$ and constants A, B verify

$$35\,A^2 + 37\,B^2 \;=\; \frac{4\,\alpha}{\pi}\,, \qquad A \neq 0\,.$$