Data Assimilation: New Challenges in Random and Stochastic Dynamical Systems

Daniel Sanz-Alonso & Andrew Stuart

D Blömker (Augsburg), D Kelly (NYU), KJH Law (KAUST), A. Shukla (Warwick), KC Zygalakis (Southampton)

SIAM Dynamical Systems 2015
Snowbird, Utah, May 18^{th} 2015

Funded by EPSRC, ERC and ONR

Enabling Quantification of EQUIP Uncertainty for Inverse Problems

THE UNIVERSITY OF WARWICK

Outline

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Signal: Unpredictability

Consider the following map with random initial conditions:

$$v_{j+1} = \Psi(v_j), \qquad v_0 \sim \mu_0.$$

Example – Lorenz '63 $(a, b, r) = (10, 8/3, 28)$.

$$\frac{dv^{(1)}}{dt} = -a(v^{(1)} - v^{(2)}),$$

$$\frac{dv^{(2)}}{dt} = -av^{(1)} - v^{(2)} - v^{(1)}v^{(3)},$$

$$\frac{dv^{(3)}}{dt} = -bv^{(3)} + v^{(1)}v^{(2)} - b(r + a).$$

Set $v = (v^{(1)}, v^{(2)}, v^{(3)})$, $v_j = v(jh)$ to put in framework above.

Signal and Observation: Control Unpredictability?

Signal, **deterministic** (chaotic) dynamics on Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|)$:

Signal Process

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Observations, **partial and noisy**, $P : \mathcal{H} \to \mathcal{H}$ is a **projection**.

Observation Process

$$y_{j+1} = Pv_{j+1} + \epsilon\xi_{j+1}, \qquad \mathbb{E}\xi_j = 0, \quad \mathbb{E}|\xi_j|^2 = 1, \quad \text{i.i.d.}$$

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Filter: probability distribution of v_j given observations to time j:

Filter

$$\mu_j(A) = \mathbb{P}(v_j \in A|\mathcal{F}_j), \quad \mathcal{F}_j = \sigma(y_1, \dots, y_j).$$

Goal (Cerou [3], SIAM J. Cont. Opt. 2000)

Key Question: For which Ψ and P does the filter μ^j concentrate on the true signal, up to error ϵ, in the large-time limit?

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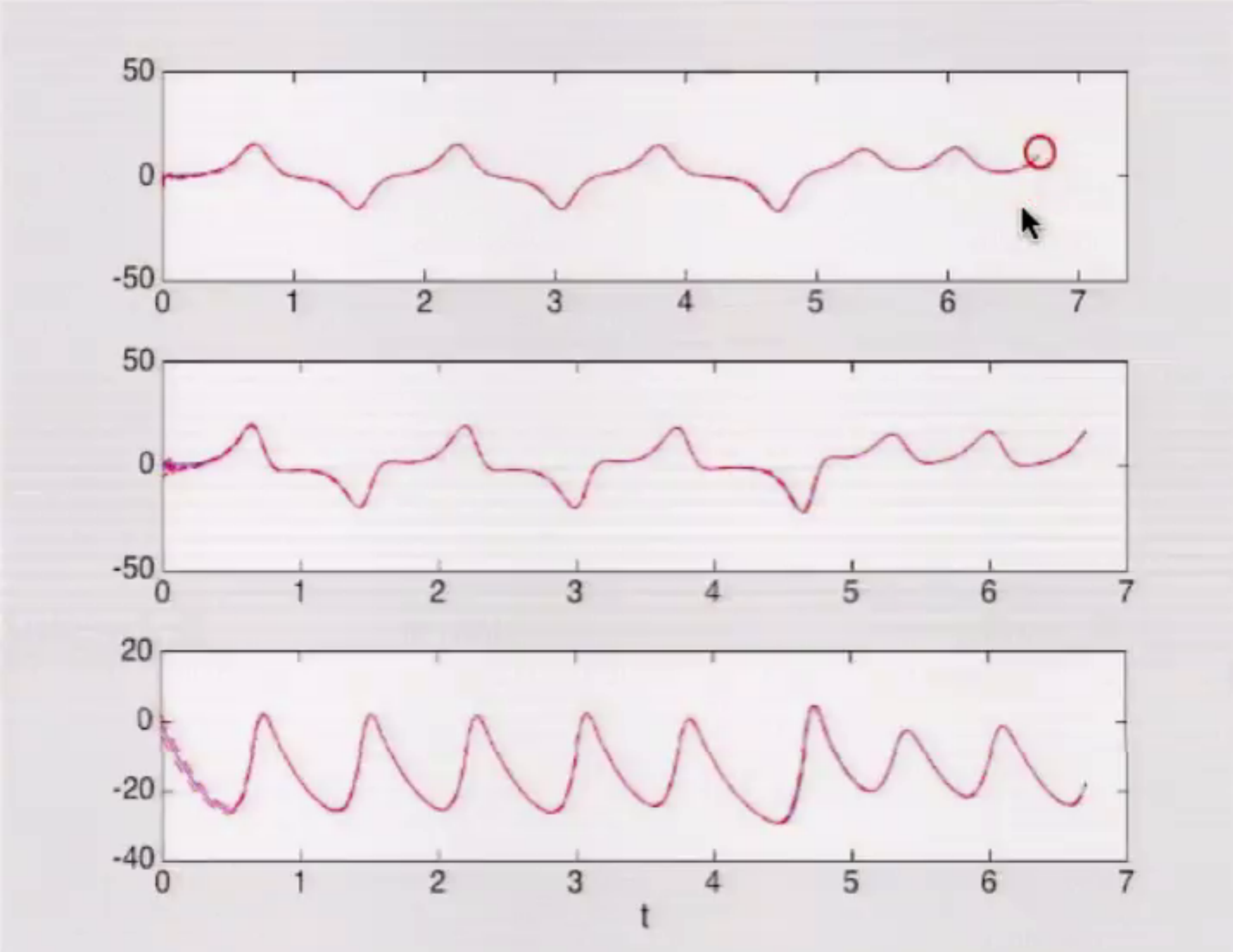
Key Problem: Ψ may expand

Define $Q = I - P$.

Key Idea: $Q\Psi$ should contract

Filtering Lorenz '63 (Observe First Component)



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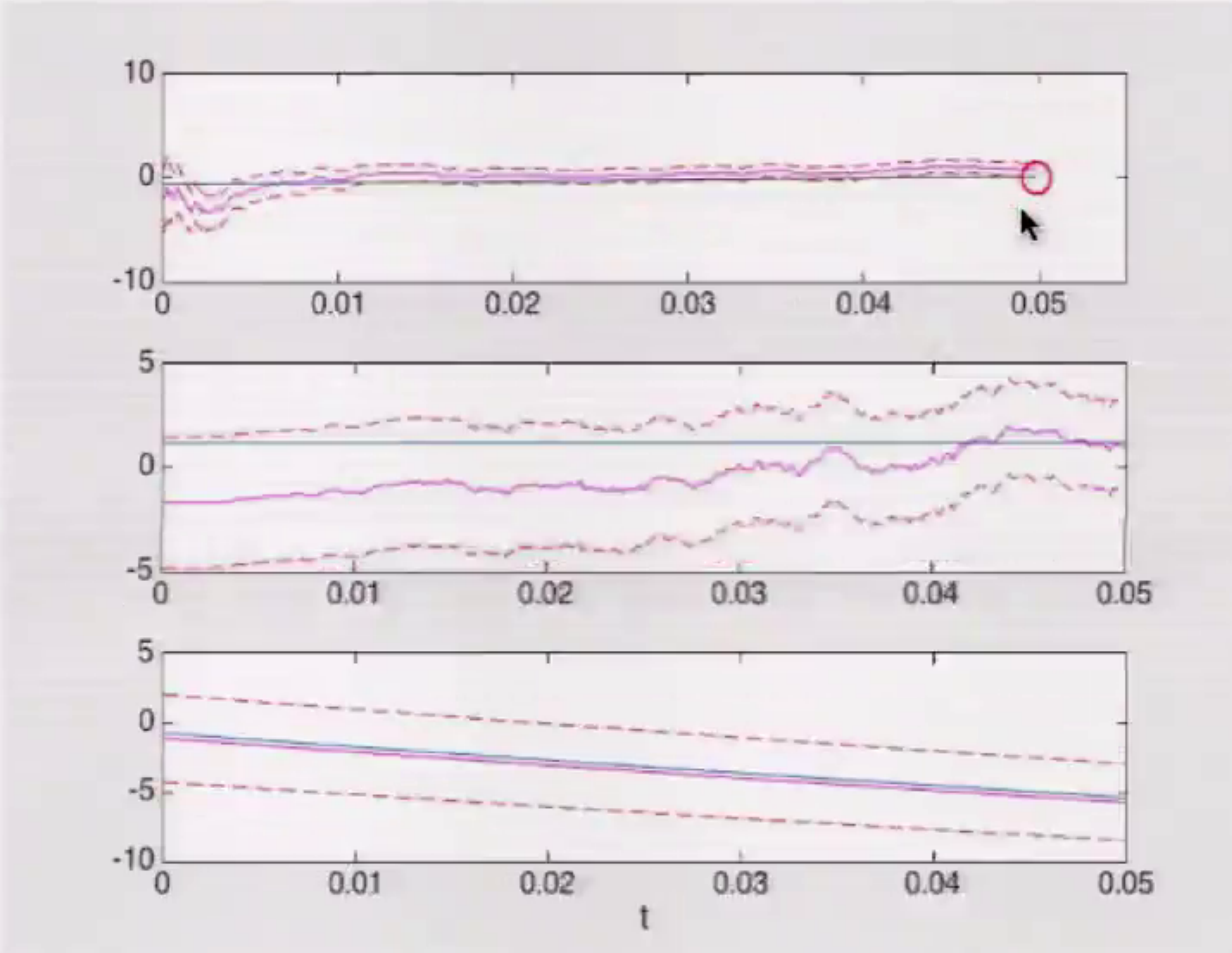

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Running Example: Geophysical Applications

Consider the ODE for $v(\cdot) \in \mathcal{H}$, given $u, f \in \mathcal{H}$:

$$\frac{dv}{dt} + Av + B(v, v) = f, \quad v(0) = u.$$

Assume dissipative with energy-conserving nonlinearity i.e. $\exists \lambda > 0$:

$$\langle Av, v \rangle \geq \lambda |v|^2, \quad \langle B(v, v), v \rangle = 0.$$

Dissipative semigroup Ψ denoting time h flow $\big(v_j = v(jh)\big)$:

$$v_{j+1} = \Psi(v_j), \quad v_0 = u.$$

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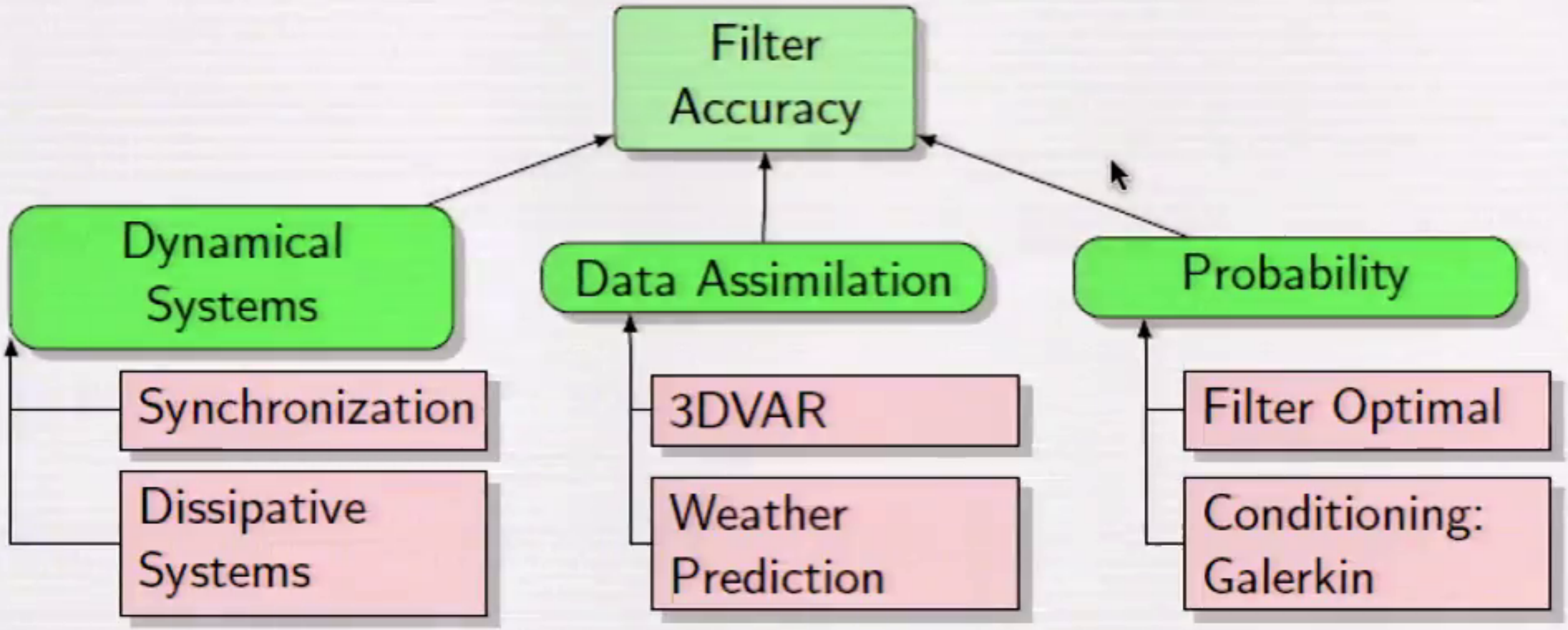
Filter Accuracy
Dynamical Systems
Data Assimilation
Probability
Synchronization
Dissipative Systems
3DVAR
Weather Prediction
Filter Optimal
Conditioning: Galerkin

Idea 1: Synchronization (Foias and Prodi [4], RSM Padova 1967 Pecora and Carroll [10], PRL 1990.)

Truth $v^\dagger = (p^\dagger, q^\dagger)$	Synchronization Filter $m = (p, q)$
$p^\dagger_{j+1} = P\Psi(p^\dagger_j, q^\dagger_j),$	$p_{j+1} = p^\dagger_{j+1},$
$q^\dagger_{j+1} = Q\Psi(p^\dagger_j, q^\dagger_j),$	$q_{j+1} = Q\Psi(p^\dagger_j, q_j);$
$---$	$---$
$v^\dagger_{j+1} = \Psi(v^\dagger_j),$	$m_{j+1} = Q\Psi(m_j) + p^\dagger_{j+1}.$

Idea 1: Synchronization (Foias and Prodi [4], RSM Padova 1967 Pecora and Carroll [10], PRL 1990.)

Truth $v^\dagger = (p^\dagger, q^\dagger)$ — **Synchronization Filter $m = (p, q)$**

$$p^\dagger_{j+1} = P\Psi(p^\dagger_j, q^\dagger_j), \qquad p_{j+1} = p^\dagger_{j+1},$$

$$q^\dagger_{j+1} = Q\Psi(p^\dagger_j, q^\dagger_j), \qquad q_{j+1} = Q\Psi(p^\dagger_j, q_j);$$

— — —

$$v^\dagger_{j+1} = \Psi(v^\dagger_j), \qquad m_{j+1} = Q\Psi(m_j) + p^\dagger_{j+1}.$$

Synchronization for various chaotic dynamical systems [5, 10, 11]:

$$|m_j - v^\dagger_j| \to 0, \text{ as } j \to \infty.$$

Idea 2: 3DVAR (Lorenc [9] Q. J. R. Met. Soc 1986)

3DVAR Filter. $|\cdot|_C = |C^{-\frac{1}{2}}\cdot|$.

$$m_{j+1} = \operatorname{argmin}_{m\in\mathcal{H}}\{|m-\Psi(m_j)|_C^2 + \epsilon^{-2}|y_{j+1} - Pm|_\Gamma^2\}.$$

Solve Variational Equations (with $C = \epsilon^2(\eta^{-2}\Gamma P + Q)$)

$$m_{j+1} = (I-K)\Psi(m_j) + Ky_{j+1}, \quad K = (1+\eta^2)^{-1}P,$$

Variance Inflation (from weather prediction) $\eta \ll 1$

$m_{j+1} = Q\Psi(m_j) + Py_{j+1}, \quad \eta = 0.$ **Synchronization Filter.**

Idea 3: Filter Optimality (Folklore, but see e.g. Williams ⋯)

Recall $\mathcal{F}_j = \sigma(y_1, \ldots, y_j)$ and define the mean of the filter:

$$\hat{v}_j := \mathbb{E}(v_j|\mathcal{F}_j) = \mathbb{E}^{\mu_j}(v_j).$$

Use Galerkin orthogonality wrt conditional expectation

For any $\mathcal{F}_j$ measurable m_j :

$$\mathbb{E}|v_j - \hat{v}_j|^2 \le \mathbb{E}|v_j - m_j|^2.$$

Take m_j from **3DVAR** to get bounds on the mean of the filter.
Similar bounds apply to the variance of the filter. (Not shown.)

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Assumptions

There are two equivalent Hilbert spaces $(\mathcal{H}, \langle\cdot,\cdot\rangle, |\cdot|)$ and $(\mathcal{V}, \langle\cdot,\cdot\rangle_{\mathcal{V}}, \|\cdot\|)$:

Assumption 1: Absorbing Ball Property

There is $R_0 > 0$ such that:

- for $B(R_0) := \{x \in \mathcal{H} : |x| \le R_0\}$, $\Psi(B(R_0)) \subset B(R_0)$;
- for any bounded set $S \subset \mathcal{H}$ $\exists J = J(S) : \Psi^J(S) \subset B(R_0)$.

Assumption 2: Squeezing Property

There is $\alpha(R_0) \in (0,1)$ such that, for all $u, v \in B(R_0)$,

$$\|Q(\Psi(u) - \Psi(v))\|^2 \le \alpha(\mathbf{R_0})\|u - v\|^2.$$

Theorem (Sanz-Alonso and S, 2014, [12])

Let Assumptions 1,2 hold. Then there is a constant $c > 0$ independent of the noise strength ϵ such that

$$\limsup_{j\to\infty} \mathbb{E}|v_j - \hat{v}_j|^2 \le c\epsilon^2$$

.

Idea of proof:

- Fix $m_0 \in B(R_0)$ and let $\mathcal{P}$ denote the $\mathcal{H}-$projection onto $B(R_0)$. Define the **modified 3DVAR**:

$$m_{j+1} = \mathcal{P}\big(Q\Psi(m_j) + y_{j+1}\big).$$

- Prove

$$\limsup_{j\to\infty} \mathbb{E}|v_j - m_j|^2 \le c\epsilon^2.$$

- Use the L^2 optimality of the filtering distribution.

Idea of proof (sketch, Ψ globally Lipschitz):

$$
\begin{aligned}
m_{j+1} &= Q\Psi(m_j) && + \overbrace{P\Psi(v_j) + \epsilon\xi_{j+1}}^{\mathbf{y_{j+1}}}, \\
v_{j+1} &= Q\Psi(v_j) && + P\Psi(v_j).
\end{aligned}
$$

Subtract and use independence plus contractivity of $Q\Psi$:

$$
\begin{aligned}
\mathbb{E}\|v_{j+1} - m_{j+1}\|^2 &= \mathbb{E}\|Q\left(\Psi(v_j) - \Psi(m_j)\right) - \epsilon\xi_{j+1}\|^2 \\
&\le \mathbb{E}\|Q\left(\Psi(v_j) - \Psi(m_j)\right)\|^2 + \epsilon^2\mathbb{E}\|\xi_{j+1}\|^2 \\
&\le \alpha\mathbb{E}\|v_j - m_j\|^2 + \epsilon^2\mathbb{E}\|\xi_{j+1}\|^2.
\end{aligned}
$$

Lorenz '63 (Hayden, Olson and Titi [5], Physica D 2011.)

$$\begin{aligned}
\frac{dv^{(1)}}{dt} + a(v^{(1)} - v^{(2)}) &= 0\\
\frac{dv^{(2)}}{dt} + av^{(1)} + v^{(2)} + v^{(1)}v^{(3)} &= 0\\
\underbrace{\frac{dv^{(3)}}{dt}}_{\frac{\mathbf{dv}}{\mathbf{dt}}} + \underbrace{bv^{(3)}}_{\mathbf{Av}} \underbrace{- v^{(1)}v^{(2)}}_{\mathbf{B(v,v)}} &= \underbrace{-b(r+a)}_{\mathbf{f}}
\end{aligned}$$

Standard parameter values: $(a, b, r) = (10, 8/3, 28)$.

Observation matrix

$$P := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Theory applicable with $\|\cdot\|^2 := |P\cdot|^2 + |\cdot|^2$: [5], [8].

Lorenz '96 (Law, Sanz-Alonso, Shukla and S [11], arXiv 2014.)

Consider the following system, subject to the periodicity boundary conditions $v_0 = v_{3J}, v_{-1} = v_{3J-1}, v_{3J+1} = v_1$:

$$\underbrace{\frac{dv^{(j)}}{dt}}_{\frac{\mathbf{dv}}{\mathbf{dt}}} + \underbrace{v^{(j)}}_{\mathbf{Av}} + \underbrace{v^{(j-1)}(v^{(j+1)} - v^{(j-2)})}_{\mathbf{B(v,v)}} = \underbrace{F}_{\mathbf{f}}, \quad j = 1, 2, \dots, 3J.$$

Observation matrix P: observe 2 out of every 3 points. Theory applicable with $\|\cdot\|^2 := |P\cdot|^2 + |\cdot|^2$: [11].

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\frac{dv^{(1)}}{dt} &+ a(v^{(1)} - v^{(2)}) && = 0\\
\frac{dv^{(2)}}{dt} &+ \quad av^{(1)} + v^{(2)} + v^{(1)}v^{(3)} && = 0\\
\underbrace{\frac{dv^{(3)}}{dt}}_{\frac{dv}{dt}} &+ \underbrace{\quad bv^{(3)}\quad}_{Av} \underbrace{- v^{(1)}v^{(2)}}_{B(v,v)} && = \underbrace{-b(r+a)}_{f}
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Navier-Stokes Equation on a 2D Torus

(Hayden, Olson and Titi [5], Physica D 2011.)

P_{leray} denotes the Leray projector:

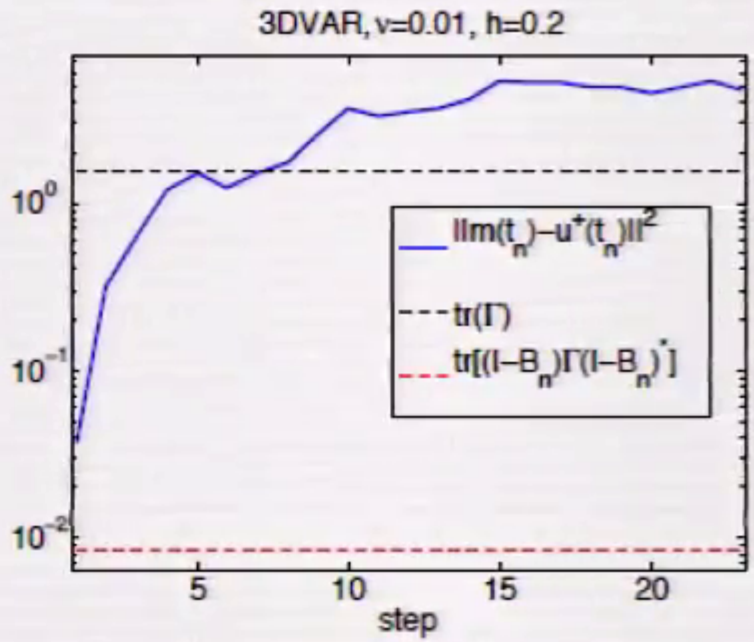
$$Au = -\nu P_{\text{leray}} \Delta u, \qquad B(u,v) = \frac{1}{2} P_{\text{leray}}[u \cdot \nabla v] + \frac{1}{2} P_{\text{leray}}[v \cdot \nabla u].$$

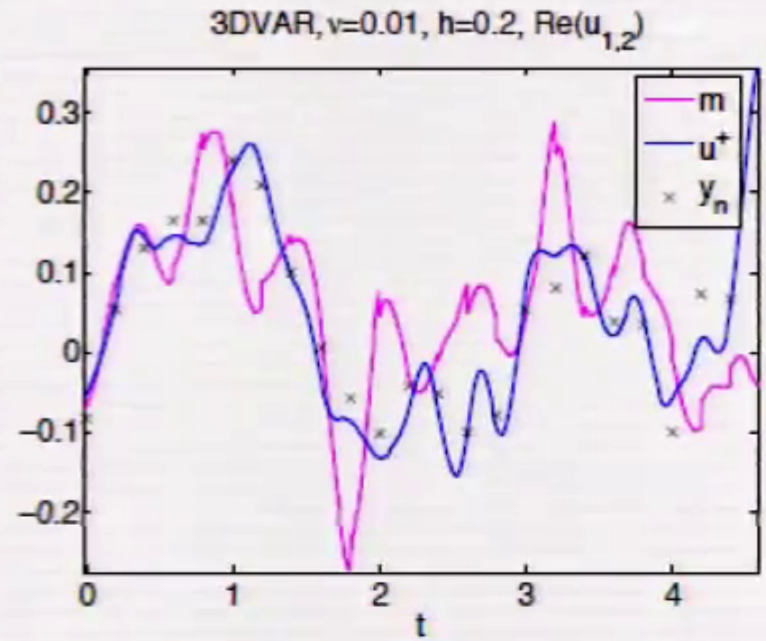
Observation operator in (divergence-free) Fourier space:

$$Pu = \sum_{|k| \leq k_{\max}} u_k e^{ikx}.$$

Theory applicable with $\mathcal{H} = \mathcal{V} := H^1_{\text{div}}(\mathbb{T}^2)$ and $k_{\max}$ sufficiently large/h sufficiently small: [2], [5].

Inaccurate: η too large. (NSE torus) Law and S [7], Monthly Weather Review, 2012





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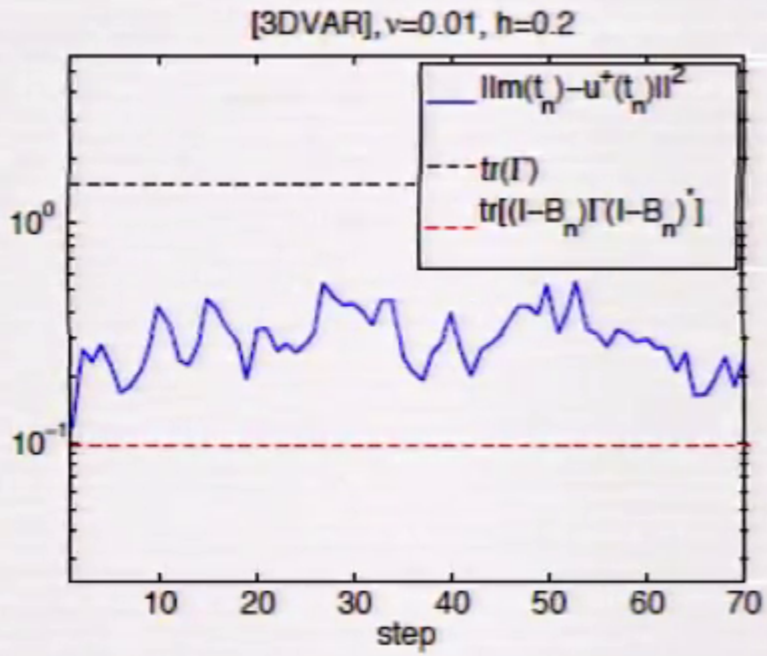

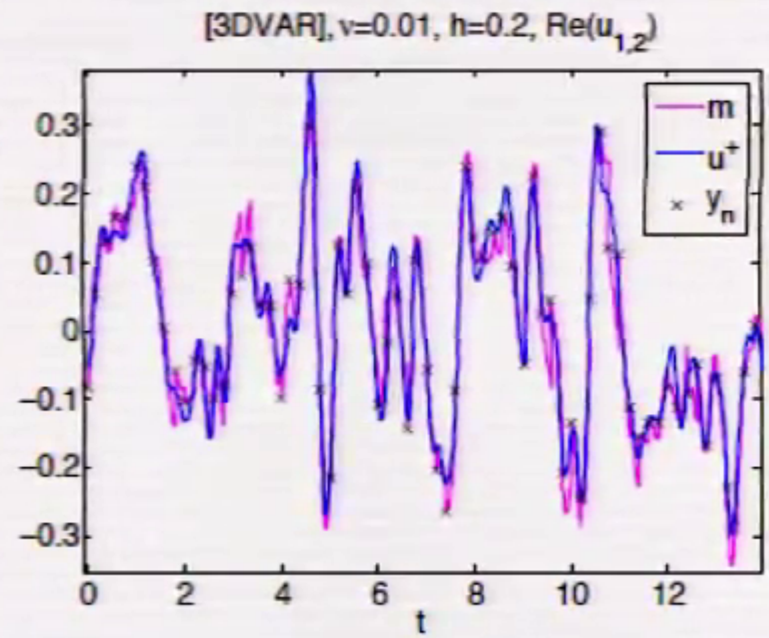

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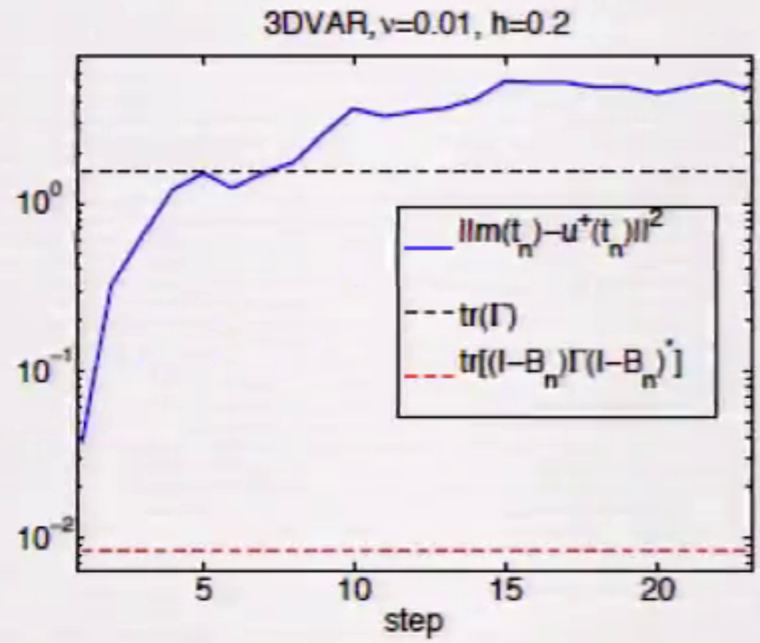

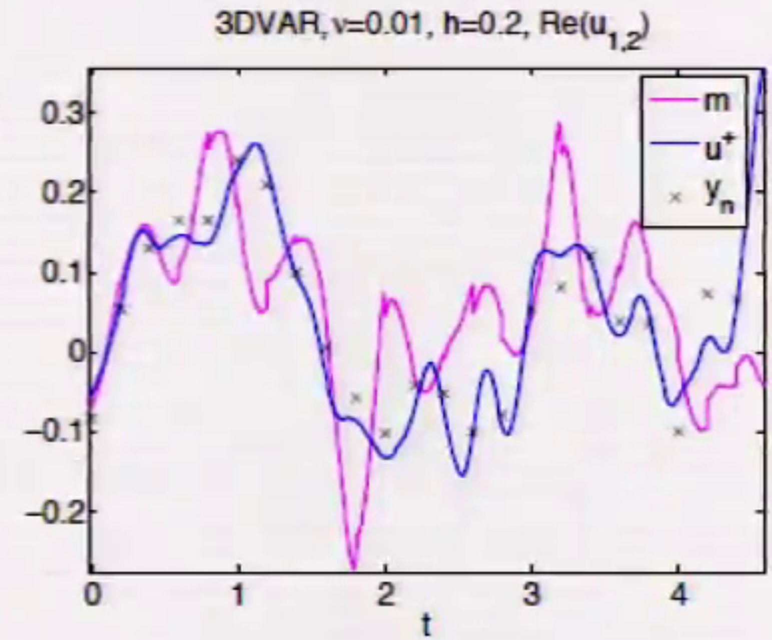

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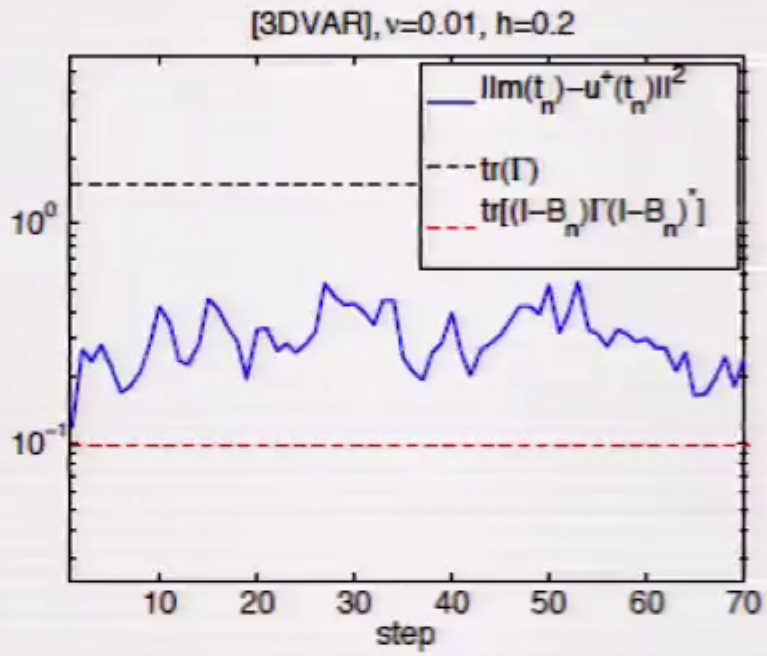

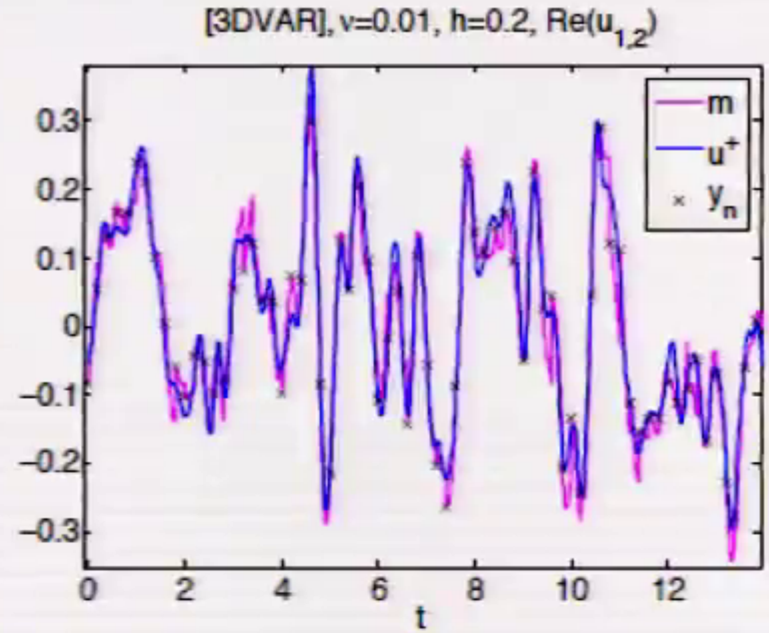

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S(P)DE Limits with Blömker 2012 [1], Kelly 2014 [6]; also Tong, Majda, Kelly 2015 [13] .

High Frequency Data Limit – 3DVAR

$$\frac{dm}{dt} + Am + B(m,m) + CP^*\Gamma^{-1}\Big(P(m-v) + \Gamma^{\frac{1}{2}}\frac{dW}{dt}\Big) = f$$

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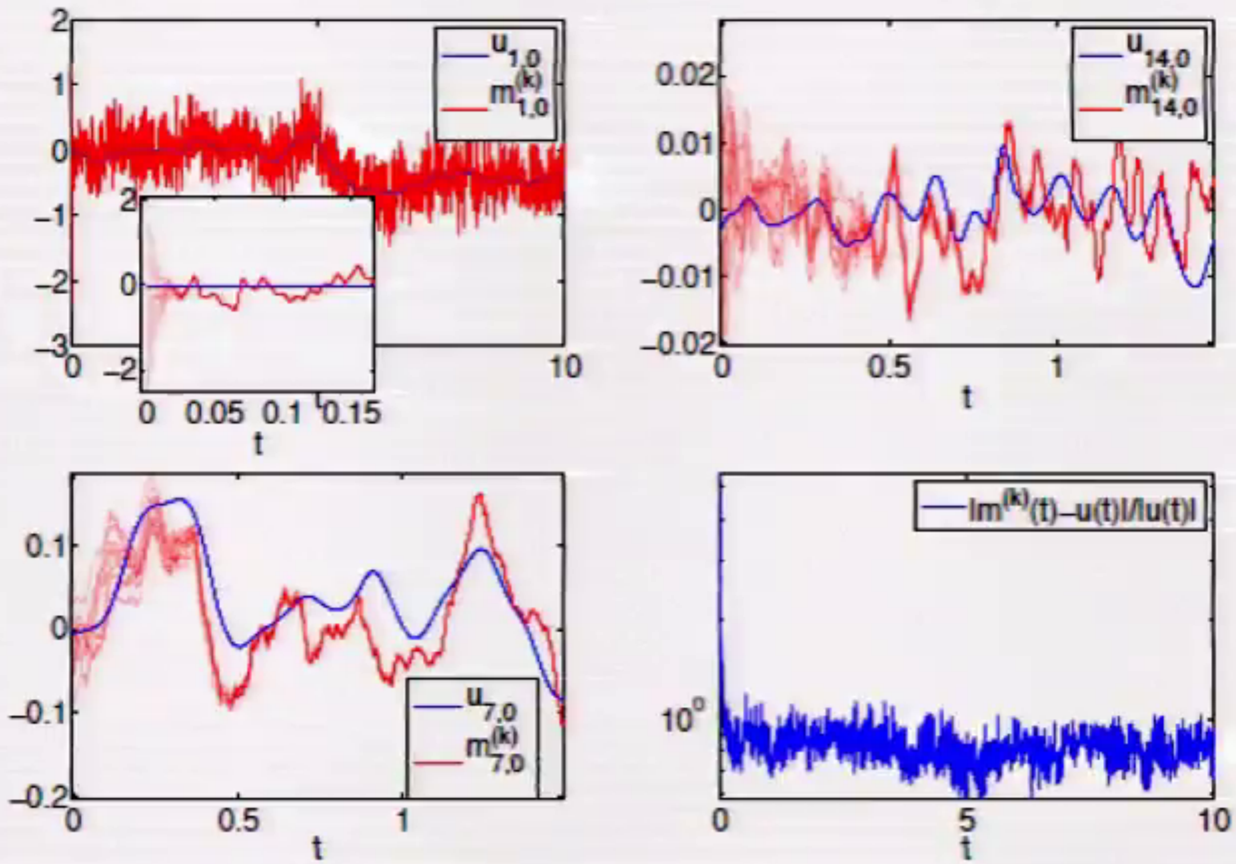
High Frequency Data Limit – Ensemble Kalman Filter

$$\frac{dm^{(j)}}{dt} + Am^{(j)} + B(m^{(j)}, m^{(j)}) + CP^*\Gamma^{-1}\Big(P(m^{(j)} - v) + \Gamma^{\frac{1}{2}}\frac{dW}{dt}\Big) = f,$$

$$\overline{m} = \frac{1}{J}\sum_{j=1}^{J} m^{(j)}, \quad C = C(m) = \frac{1}{J}\sum_{j=1}^{J}(m^{(j)} - \overline{m}) \otimes (m^{(j)} - \overline{m}).$$

SPDE Inaccurate (NSE Torus) (Blömker et al [1])



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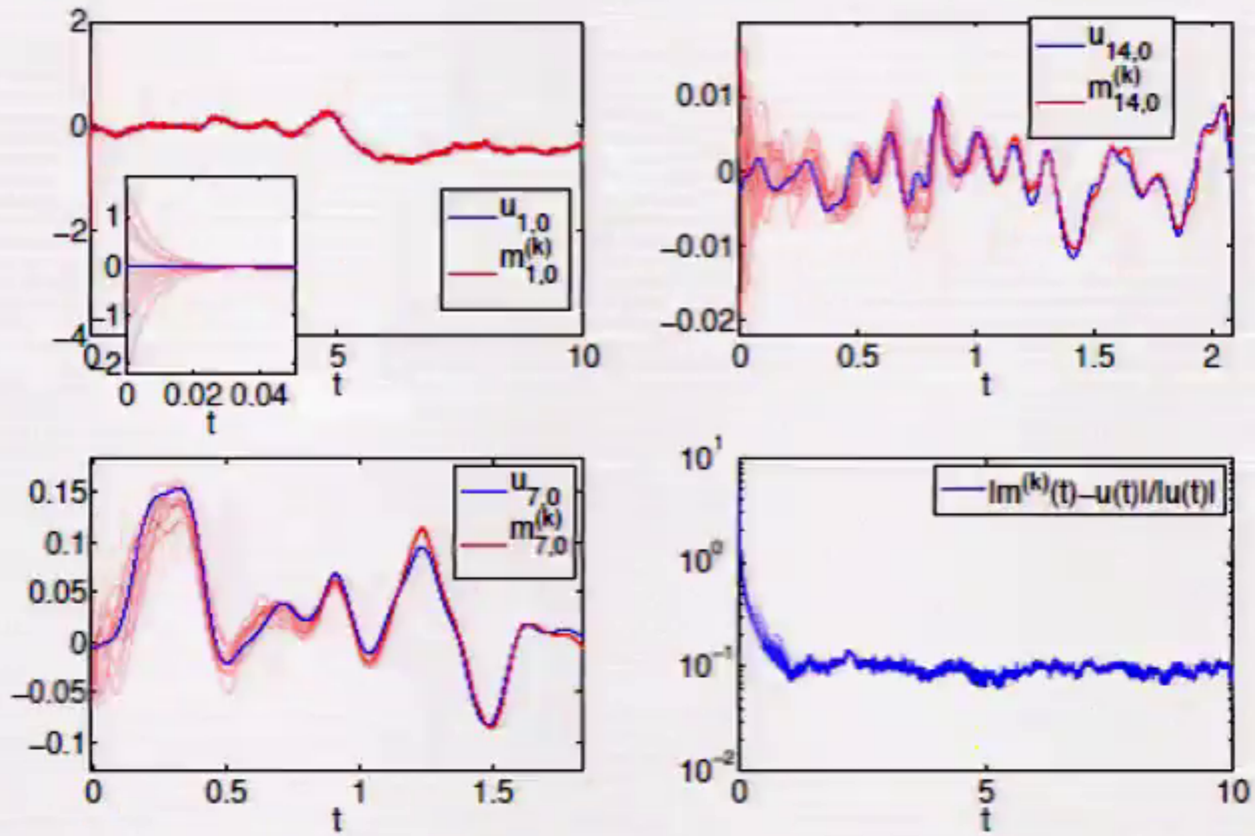

Summary

- **Chaos** – and resulting **unpredictability** – is the enemy in many scientific and engineering applications.
- Its study has led to a great deal of interesting mathematics over the last century.
- **Data** – when combined with **models** – can have a massive positive impact on prediction in all of these scientific and engineering applications.
- The emerging new field, in which **model and data are analyzed simultaneously**, will lead to interesting new mathematics over the next century.
- **Data Assimilation** needs input from **Dynamical Systems**.

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