Approximating the Fractal Spectral Measure of a Koopman Operator

MS48 – Koopman Operator Techniques in Dynamical Systems: Theory

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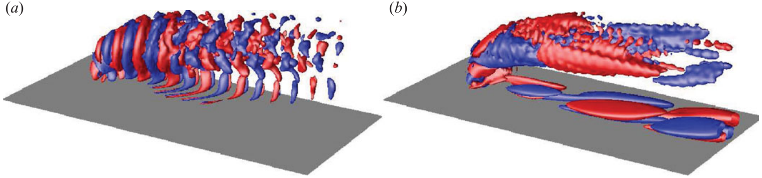
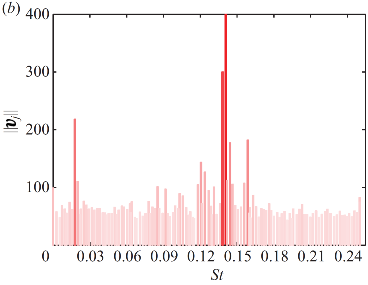
Koopman Mode Analysis

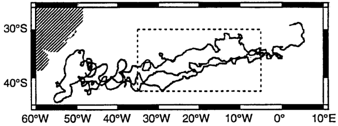




- (Quasi)periodic features $\leftrightarrow$ Eigenvalues
- Eigenvalues $\leftrightarrow$ atomic part of spectral measure
- Dynamic Mode Decomposition approximates features associated with eigenvalues.

[1] Rowley CW, Mezić I, Bagheri S, Schlatter P, Henningson DS (2009) Spectral analysis of nonlinear flows. Journal of Fluid Mechanics, 641:115–127.

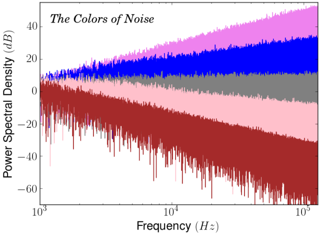
Mixing behavior modeled by stochastic terms

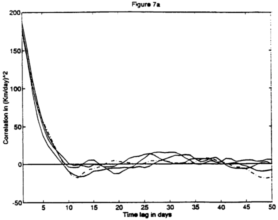


$$dx = vdt = (U + u)dt$$

$$du = -\theta u dt + \sigma\sqrt{2\theta}dW$$

PSDs of noise models (image: Wikipedia)



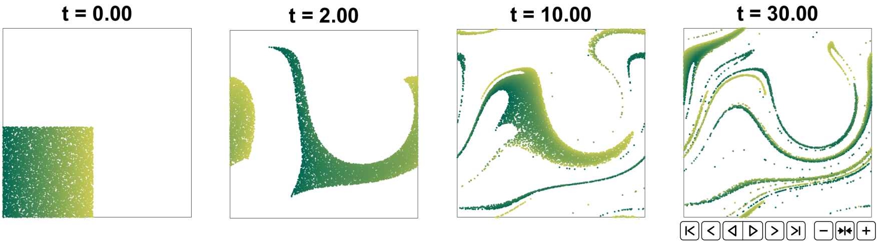


Turbulent transport ↔ decay of correlations

- Mixing (turbulent) transport
  ↔ power spectrum density (PSD)
- PSD ↔ **abs. continuous** part of spectral measure
- modeled as stochastic terms

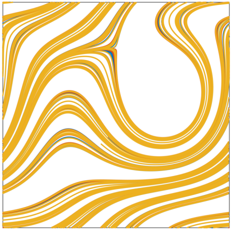
[1]Griffa A, Owens K, Piterbarg L, Rozovskii B (1995) *Estimates of turbulence parameters from Lagrangian data using a stochastic particle model.* Journal of Marine Research, 53(3):371–401.

Between: non-mixing, non-regular dynamics



Viscous steady flow past a lattice of obstacles results in **anomalous transport** (faster than diffusive, but slower than mixing).

Goal: Model for Koopman spectral measure for anomalous transport.



Attractor

[1] Zaks, M. A. *Fractal Fourier spectra of Cherry flows*. Physica D 149, 237–247 (2001).

The entire talk in 3 sentences.

- Koopman operator is a linear representation of nonlinear dynamics.
- In steady state, its spectral measure decomposes into:
  - atomic spectrum (regular components),
  - spectral density (mixing, chaotic components), and
  - fractal parts (weak anomalous transport, intermittently correlated).
- We propose to model the fractal spectral measure by Affine Iterated Function Systems (AIFS).

The Koopman Operator

Nonlinear dynamics:
Time-T map on invariant set $\mathcal{A}$:
$\Phi : \mathcal{A} \to \mathcal{A}$

$$x_{n+1} = \Phi(x_n)$$

Φ typically nonlinear,
$\mathcal{A}$ (in)finite-dimensional, compact.

Koopman operator:
For $f \in L^2(\mathcal{A})$,
$\mathbb{K} : L^2(\mathcal{A} \to L^2(\mathcal{A})$

$$\mathbb{K} f(x) = f \circ \Phi(x)$$

$\mathbb{K}$ linear without truncations,
$L^2(\mathcal{A})$ (in)finite-dimensional.

Spectral Decomposition of the Koopman Operator

$$\mathbb{K}^n f = \int_{-\pi}^{\pi} e^{in\omega} d[\mathbb{E}(\omega) f] = \underbrace{\sum_k e^{in\omega_k} \mathbb{P}_k f}_{\text{atomic}} + \underbrace{\int_{-\pi}^{\pi} e^{in\omega} d[\mathbb{E}_c(\omega) f]}_{\text{continuous}} .$$

[1] Budišić, M., Mohr, R. M. & Mezić, I. *Applied Koopmanism*. Chaos 22, 047510–1–33 (2012).

Spectral measure and the autocorrelation function.

Operator-valued $\mathbb{E}(\omega)$ applied to $f \perp \mathbf{1} \Rightarrow$ scalar-valued $\sigma_f(\omega)$ for erg. dynamics:

$$\int_{-\pi}^{\pi} e^{in\omega} \overbrace{\langle d\mathbb{E}(\omega) f, f \rangle}^{d\sigma_f(\omega)} = \langle \mathbb{K}^n f, f \rangle = \underbrace{\lim_{K \to \infty} \frac{1}{K} \sum_{k=0}^{K-1} [\mathbb{K}^n f](x_k) f(x_k) = C_f(n)}_{\text{autocorrelation function}}.$$

Autocorrelation Function $\leftrightarrow$ Spectral Measure

Autocorrelation of $f(x_k)$ is the Fourier tfm. of the spectral measure.

Note:

- Single observable $\leftrightarrow$ Fourier spectral measure
- For some systems, Fourier s.m. = Koopman s.m.
- For others, this analysis needs to be extended.

Detection of components of spectral measure

Autocorrelation:

$$C_f(n) = \frac{\langle \mathbb{K}^n f, f \rangle - \langle \mathbb{K}^n f \rangle^2}{\langle (\mathbb{K}^n f)^2 \rangle - \langle \mathbb{K}^n f \rangle^2}$$

Mean-squared Autocorrelation:

$$\overline{C}_f(n) = \frac{1}{n} \sum_{k=0}^{n-1} |C_f(k)|^2$$

Detection of fractal spectral measure[1]

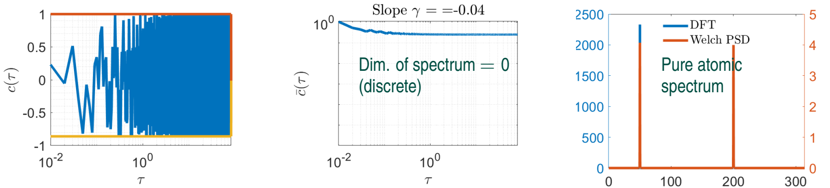
- $C_f(n) \to 0 \quad \Leftrightarrow$ only spectral density
- $\overline{C}_f(n) \to 0 \quad \Rightarrow$ no non-trivial eigenvalues[2]
- $\overline{C}_f(n) \sim n^{-D} \quad \Rightarrow$ D is the fractal dimension of the spectral measure[3]

[1] Pikovsky, et al. *Singular continuous spectra in dissipative dynamics*. PRE 52, (1995)

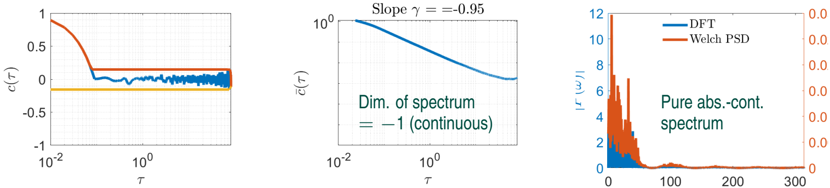
[2] Wiener's Lemma.

[3] Knill O (1998) *Singular continuous spectrum and quantitative rates of weak mixing.* Discrete and Continuous Dynamical Systems, 4(1):33–42.

Regular time series: $\alpha(t) = \sin(50t) + \cos(200t)$



Filtered Gaussian noise: $\beta(t) = \mathcal{N}(0,1) \star \chi(t)$

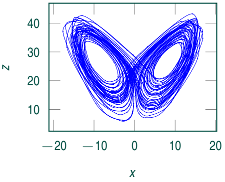


Extended Lorenz system ($D \neq 0$, $A \neq 0$):

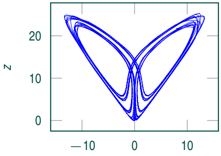
$$\dot{x} = S(y - x) + SDy(z - R)$$
$$\dot{y} = Rx - y - xz$$
$$\dot{z} = xy - bz + Ax$$

- S – Prandtl no.[3]
- R – Rayleigh no.[3]
- B – geometric parameter[3]
- **D – vibrational parameter**[2]
- **A – symmetry br. parameter**[2]

Lorenz'63[3]



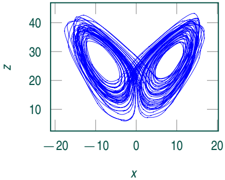
Pikovsky'95[1]



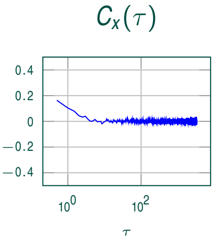
[1] Pikovsky, et al. *Singular continuous spectra in dissipative dynamics*. PRE 52, (1995)

[2] Lyubimov DV, Zaks MA (1983) *Two mechanisms of the transition to chaos in finite-dimensional models of convection*. Physica D: Nonlinear Phenomena, 9(1):52–64.

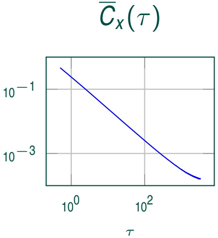
[3] Lorenz E (1963) *Deterministic Nonperiodic Flow*. Journal Of The Atmospheric Sciences, 20(2):130–141.



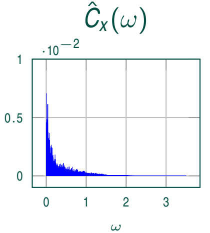
(a) Lorenz'63 attractor



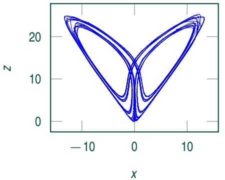
(b) $C_x(\tau) \to 0$: No atomic parts, density exists



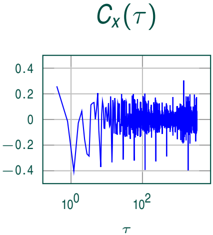
(c) $\overline{C}_x(\tau) \sim \tau^{-1}$: dim. of spectrum $= 1$,



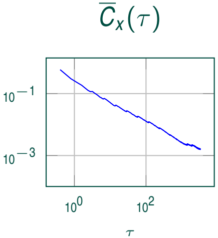
(d) $\hat{C}_x(\omega)$ has a density



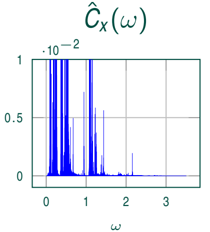
(e) Pikovsky'95 attractor



(f) $C_x(\tau) \not\to 0$: no atomic



(g) $\overline{C}_x(\tau) \sim \tau^{-2/3}$:



(h) $\hat{C}_x(\omega)$ is fractal

Fundamental problem: representation

Goal: Parametric model for the spectral measure.

Use a fixed (small) number of parameters to represent the spectral measure.

Non-parametric models like FFT and Welch can be difficult to process and interpret.

- atomic part of spec. measure $\rightarrow$ points (eigenvalues)
- a.c. part of spec. measure $\rightarrow$ density function (parametric models)
- s.c. part of spec. measure $\rightarrow$ **self-similar measures?**

Self-similar approximation

Affine Iterated Function System (AIFS)

$$w_k(\omega) = \delta_k \omega + \beta_k, \quad k = 1, \ldots, K$$

with weights p_k.

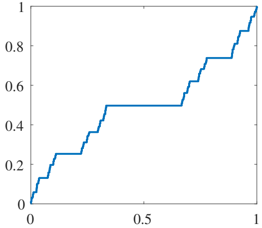
- Invariant measure (detailed balance):

$$\int_{-\pi}^{\pi} g(\omega) d\nu = \sum_k p_k \int_{-\pi}^{\pi} g \circ w_k(\omega) d\nu$$

- Fractal dimension D

$$\sum_k p_k \delta_k^{-D} = 1$$

$$w_1(\omega) = 0.3\dot{3}\omega$$
$$w_2(\omega) = 0.3\dot{3}\omega + 0.6\dot{6}$$
$$p_{1,2} = 0.5$$



$D = 0.63093$

Input:
Moments of Spectral Measure

$$\underbrace{\int_{-\pi}^{\pi} e^{in\omega} d\sigma_f(\omega)}_{\text{Fourier coeff. of spectral measure}} = \underbrace{C_f(n)}_{\text{autocorrelation}} .$$

Output:
AIFS parameters

- Bound on scale δ
- Values of p_k, β_k

Handy–Mantica Algorithm

- Convert moment problem on spectral domain into a moment problem on coefficient space of AIFS.
- Solve the auxiliary moment problem using Padé analysis.

[1] Handy, C. R. & Mantica, G. *Inverse problems in fractal construction: Moment method solution.* Physica D 43, 17–36 (1990).

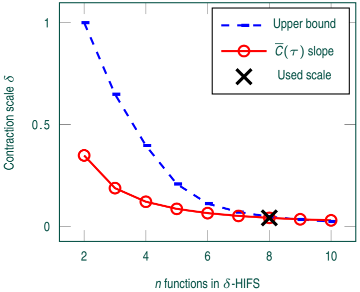


Figure: Estimates of the scale δ (from the slope of $\overline{C}_x(n)$) and upper bounds of δ given by Handy–Mantica algorithm.

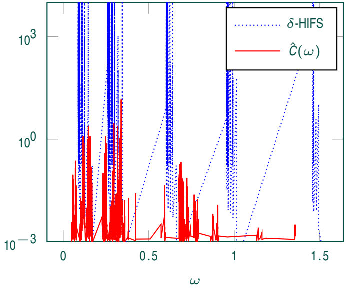
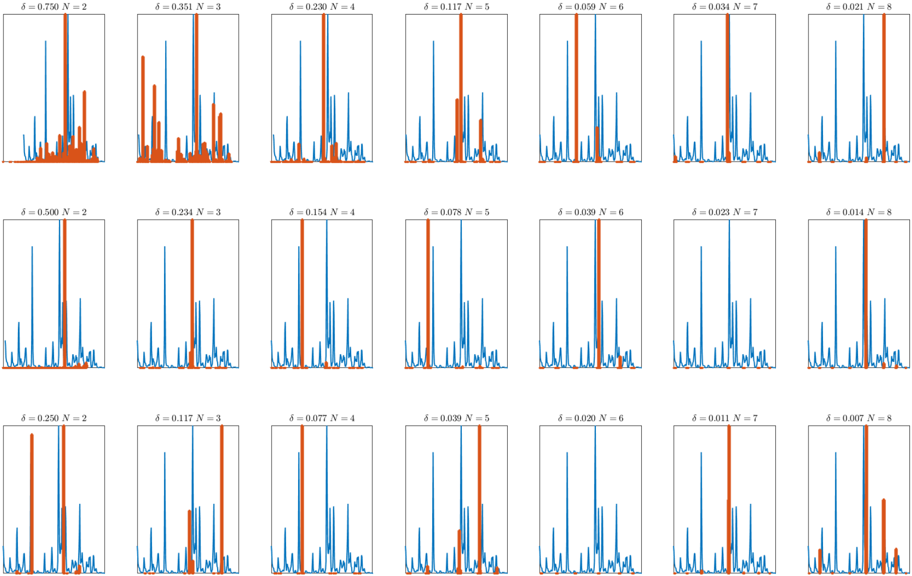


Figure: "Density" of δ-HIFS estimated via Chaos Game (blue) and the correlogram $\hat{C}(\omega)$

- 5 functions: **only 1+5+5 values**!
- Matching first 10 power moments.
- Large-scale features reconstructed
- Small scales not: feature or bug?



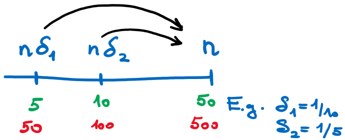
$\delta = 0.750$ $N = 2$
$\delta = 0.351$ $N = 3$
$\delta = 0.230$ $N = 4$
$\delta = 0.117$ $N = 5$
$\delta = 0.059$ $N = 6$
$\delta = 0.034$ $N = 7$
$\delta = 0.021$ $N = 8$
$\delta = 0.500$ $N = 2$
$\delta = 0.234$ $N = 3$
$\delta = 0.154$ $N = 4$
$\delta = 0.078$ $N = 5$
$\delta = 0.039$ $N = 6$
$\delta = 0.023$ $N = 7$
$\delta = 0.014$ $N = 8$
$\delta = 0.250$ $N = 2$
$\delta = 0.117$ $N = 3$
$\delta = 0.077$ $N = 4$
$\delta = 0.039$ $N = 5$
$\delta = 0.020$ $N = 6$
$\delta = 0.011$ $N = 7$
$\delta = 0.007$ $N = 8$

Assume that the Fourier measure $d\sigma_f(\omega) = \langle d\mathbb{E}(\omega)f, f\rangle$ is additionally invariant w.r.t. AIFS $p_k, w_k(\omega) = \delta_k\omega + \beta_k$.
Then it satisfies:

- Spectral theorem $\langle \mathbb{K}^n f, f\rangle = \int_{-\pi}^{\pi} e^{in\omega} \overbrace{\langle d\mathbb{E}(\omega)f, f\rangle}^{d\sigma_f(\omega)}$, $\delta_k \in [0, 1)$.
- (Weak) detailed balance $\int_{-\pi}^{\pi} g(\omega) d\sigma_f = \sum_k p_k \int_{-\pi}^{\pi} g \circ w_k(\omega) d\sigma_f$

Setting $g(\omega) = e^{in\omega}$ we can derive the evolution of autocovariance:

$$\langle \mathbb{K}^n f, f\rangle = \sum_k p_k e^{in\beta_k} \langle \mathbb{K}^{n\delta_k} f, f\rangle$$



Cf. $\langle \mathbb{K}^n(\omega)f, f\rangle = e^{i\omega} \langle \mathbb{K}^{n-1}(\omega)f, f\rangle$.

What's next?

Theory

- Single-observable (Fourier) spectral measures $\rightarrow$ Koopman spectral measure
- Examples of dynamics with s.c. spectrum
- spectral projectors

Computation

- Mixed-type spectral measures
- Numerically-favorable approaches
- Extension to multivariate correlations
- non-homogeneous AIFS
- 2D fractals (off-attractor spectrum)

Analysis

- Numerical analysis of AIFS estimator
- Connections between AIFS attractors and DMD eigenvalues