SINGULAR PERTURBATIONS IN NOISY DYNAMICAL SYSTEMS

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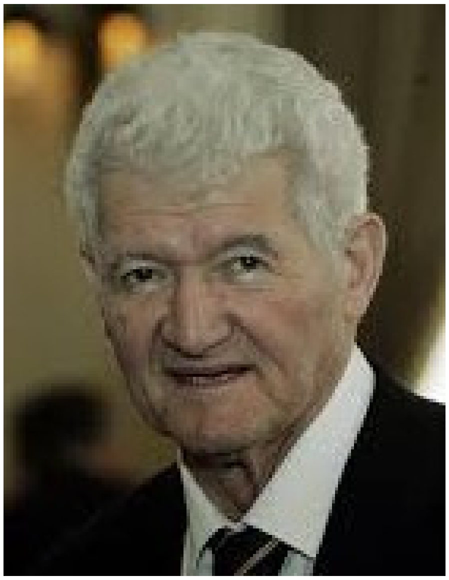
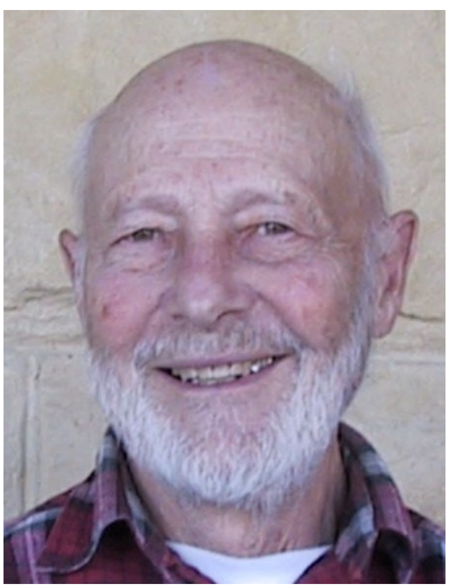
Four previous awardees were my teachers and inspirations

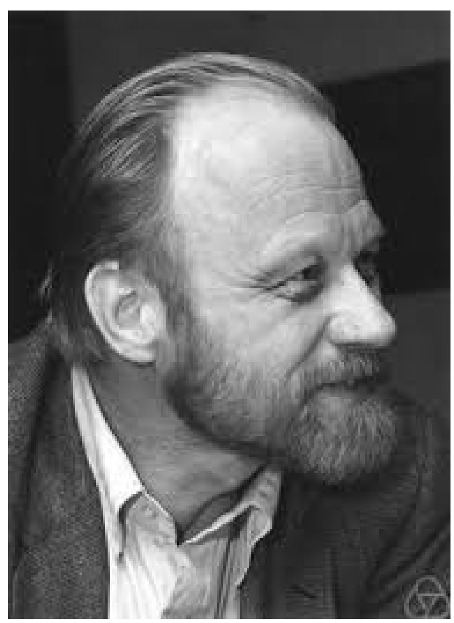


Peter Lax, 1968

Kurt Friedrichs, 1979



Joe Keller, 1983



Jurgen Moser, 1984

Two spoke on Asymptotics and Applications
I follow their footsteps, speak on same topic

Singular Perturbations in in
Noisy Dynamical Systems

- Asymptotics studies the local behavior of functions
  - Functions may be known *a-priori*
  - For some functions we may only have hints e.g., satisfy DEs + BCs or ICs
  - Perturbation Theory
    - Regular perturbations
    - Singular perturbations
- Asy series often divergent
- Abel: invention of the devil
- Diff. bet. convergent and asymptotic
- Asymptotic often superior
- Abel comment not relevant

Regular Perturbation

- Small changes in model lead to small changes in behavior
- Results little noted nor long remembered

Singular Perturbation (SP)

- Small changes in model lead to large changes in behavior
- Results deeper and more interesting
- Can occur if perturbation random

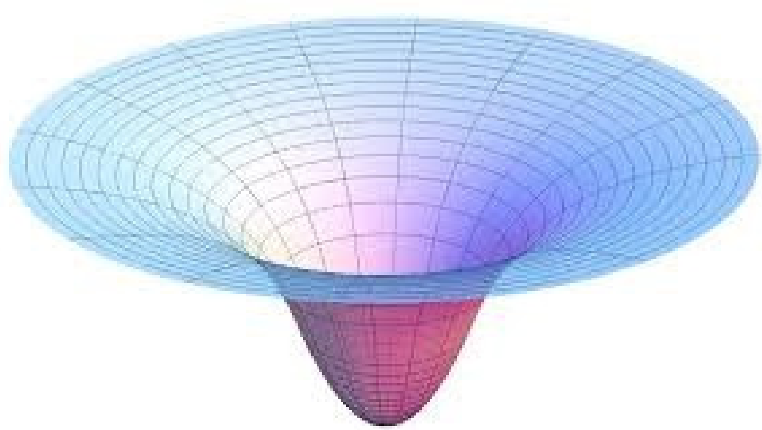
We employ singular perturbation methods in noisy dynamical systems

- The method of matched asymptotic expansions (MAE)

The Exit Problem + Applications

The Exit Problem

- Deterministic dynamical system perturbed by white noise
  - ”Derivative” of Brownian motion
  - Brownian motion is nowhere differentiable
- Example: particle in a potential well
  - Deterministic problem has a ‘stable equilibrium
  - Particle suffers random collisions with smaller, lighter particles of medium
    - ∗ Particles exit the well
    - ∗ Rare event (not low probability, low frequency)



Potential well

Questions

1. How long to exit?
   Mean Free Passage Time (MFPT)
2. From where on the boundary (rim)
   does exit occur?

- Each quantity satisfies a deterministic BVP (Kolmogorov backward equation)
- When noise is small, the resulting BVP is a singular perturbation problem
- Solve the BVP by
  singular perturbation methods (MAE)

L. Prandtl - Boundary Layer Theory

- 1904 Prandtl ICM Talk In Heidelberg
  – revolutionized fluid mechanics
- Pre-Prandtl few solutions of Navier Stokes were known
- Low viscosity flow over solid
  - Ignore viscosity away from the boundary
  - Consider Euler equations, not Navier Stokes
  - Viscosity important only in thin layer near the boundary (boundary layer) where the solution varies rapidly

  Upon hearing talk, Felix Klein arranged position in Gottingen, mecca of Math., Sci.

  Prandtl undoubtedly great F.M.
  flawed human being, apologist for nazi regime
- Boundary layer theory was later generalized and systematized: Friedrichs, Wasow, MAE; later by others

Idea of Matched Asymptotic Expansions

- Outer expansion

$$\sum a_j(x)\epsilon^j$$

- Stretching Transformation

$$\xi = \frac{x - x_0}{\epsilon^\alpha}$$

x_0 layer location, α layer width

- Boundary layer Expansion

$$\sum b_j(\xi)\epsilon^j$$

rapidly varying

- Matching inner and outer expansion: smooth connection

MAE

- Successful for many problems & applications
- Not always - “Failure of MAE”
  - Problem exhibits
    boundary layer resonance
  - “Spurious Solutions”
- MAE not successful on the exit problem
- Caused some to claim ”failure of MAE”

Here we present a physical and four
mathematical arguments which
modify or augment MAE
so it is successful for the Exit Problem

We restrict 1D linear DEs and
limit technical detail
though extensions to higher dims,
limit cycle escape, different noise,
nonlinear problems

Brownian Motion

- 1827 Robert Brown: Pollen grains in water agitated, irregular motion
- 1785 Jan Ingenhausz: Carbon dust in alcohol, less systematic, possible
  Stigler Law of Eponomy, which states
  "No Discovery Named For
  its Original Discoverer"

Brownian Motion

- 1905 Albert Einstein: Explanation of Brownian motion; 1906 Smoluchowski independently same result:
  - Motion due to collisions with smaller, lighter particles in which they're suspended
  - Probabilistic description $O(10^{21})$ collisions/sec can't observe collisions, nor path
  - Beginning of stochastic modeling
  - Two forces: collisions + viscous drag
  - Process is diffusive:

    $$p_t = Dp_{xx}, \quad D = \frac{kT}{6\pi\eta a}$$

    collisions modeled by diffusion
  - Confirmed existence of atoms then topic of debate
  - 1908 Perrin, later Nobel Prize experimental confirmation

- 1908 Langevin
  first stochastic differential equation – (SDE)
  $$m\ddot{x} + 6\pi\eta a\dot{x} + R, \quad D = \frac{kT}{6\pi\eta a}$$
  SDE solution only known statistically.
  x is the particle position, and R is a
  random force modeling the collisions

1D Random Walk Collision Model

- Particle at x, jump right $r(x)$, jump left $\ell(x)$, no jump $1 - r(x) - \ell(x)$
- Jump size ϵ, jump time δt small
- $p(x, y, t)$ probability to reach $x(t) = y$ given $x(0) = x$
- $$p_\mathcal{T} = L^* p = \frac{\epsilon}{2}[(r + \ell)p]_{yy} - [(r - \ell)p]_y$$
  $$\tau = \epsilon t \qquad \text{(long time scale)}$$
  – hardly any motion on shorter scales
- If $r = \ell$, no drift – pure diffusion
  $$p_\mathcal{T} = (rp)_{yy}$$
- Intimate connection between probability & partial differential equations

N. Van Kampen asked: Why Do Stochastic Processes Enter Physics?

He answers: Many phenomena which evolve in time in an extremely complicated way, well beyond the possibility of calculation or observation, have some average properties that can be observed and obey simple laws. The use of probability is justified by our ignorance of the precise microscopic state. Nevertheless macroscopic variables are observable and can be calculated.

- Process goes from y at time s to x at time t
- $p(x, t|y, s)$ satisfies $p_t = L^* p = d p_{xx}$
- p describes the time evolution of a probability density function

- x, t forward variables (to where it's going) y, s backward variables (from where coming)
- pure Brownian motion $p_t = L^* p = p_{xx}$ forward Kolmogorov eq. $p_t = Lp$ backward Kolmogorov eq.
- 1923 Wiener formalized mathematical theory of Brownian motion
  - Wiener process w "derivative" dw (white noise)
- deterministic dynamical system

$$\dot{x} = b(x)$$

perturbed by small white noise

SDE

$$dx = b(x)dt + \sqrt{2\epsilon}dw,$$

SIE - Ito, Stratonovich

- Kolmogorov forward operator

$$L^*p = \epsilon p_{xx} - (bp)_x$$

Kolmogorov backward operator

$$Lp = \epsilon p_{yy} + bp_y$$

L, L^* are adjoints

- We'll use boundary value problems for Lp to compute MFPT & distribution of exit points in the exit problem

- Deterministic force derived from potential

$$V(x) = \frac{x^2}{2}$$

$$\text{force} = -V' = -x,$$

$$D = (-a, b), \quad a, b > 0$$

small random perturbation (white noise)

- MFPT τ free Brownian particle

$$L\tau = \epsilon\tau'' = -1 \quad \text{in } D$$

$$\tau = 0 \quad \text{on } \partial D$$

- Follows from Ito's formula

$$dx_\epsilon = b(x_\epsilon)dt + \sqrt{2\epsilon}dw$$

- $f(x_\epsilon) = f(x) + \int_0^t Lf ds + \int_0^t Mf dw$
  L is the backward operator,

$$Mf = \frac{\partial f}{\partial x}, \text{any } f$$

  Last term is a stochastic integral

- MFPT satisfies

$$Lv = \epsilon v'' - xv' = -1 \text{ in } D$$

$$v = 0 \text{ on } \partial D$$

- Set $f = v$, $t = T$, T is first passage time to ∂D

$$v(x_\epsilon(T)) = v(x) - T + \int_0^T Mv dw$$

  Take expectation,
  Use $E(\text{Stochastic Integral}) = 0$ and BC
  $v(x) = \tau$

- Similarly, $u(x)$ satisfies

$$Lu = \epsilon u'' - xu' = 0 \text{ in } D$$

$$u = \phi \text{ on } \partial D$$

- Set $f = u$, $t = T$,

$$u(x_\epsilon(T)) = u(x) + \int_0^T Mudw$$

E(Stochastic integral) $= 0$ and BC

$$u(x) = E(x_\epsilon(T))$$

$$u(x) = \int_{\partial D} \phi(y)\rho(x,y)dy$$

- ρ is probability density of exit points
  $=$ Green's function of Dirichlet problem

- MFPT for free Brownian particle

$$\tau = \frac{(a+x)(b-x)}{\epsilon}$$

algebraically large in ϵ

- Brownian particle in force field

$$L\tau = \epsilon\tau'' - x\tau' = -1 \text{ in } D$$

$$\tau = 0 \text{ on } \partial D$$

  – Can show

  $$\tau = O\left(e^{\frac{1}{\epsilon}}\right)$$

  exponentially large in ϵ

  – Takes longer time to overcome potential barrier

- Probability distribution of exit points

$$u(x) = \int_{\partial D} \phi(y)\rho(x,y)dy$$

- In our two point boundary value problem

$$u = P_{-a}\alpha + P_b\beta$$

P_{-a}, P_b probabilities to exit at $-a, b$

- Use MAE to find
uniform asymptotic solution
  - Reduced problem ($\epsilon = 0$)
  - Cannot satisfy both boundary conditions, boundary layer(s) necessary

$$u \sim c_0 + (\alpha - c_0)e^{-a\xi} + (\beta - c_0)e^{-b\eta}$$

$$\xi = \frac{x+a}{\epsilon}, \quad \eta = \frac{b-x}{\epsilon}$$

- But what is c_0?

- Uniform expansion consists of outer + BL
  - Outer $O(1)$
  - Boundary layer goes from $O(1)$ to exponentially small
- Appropriate to ask:
  enough functions to represent solution?
  i.e., enough to span the solution space?
- If not, need to add more functions

- All MAE conditions employed, no answer
- No help from h.o.t.
- Though solution is unique
  asymptotic solution not unique
- 1 parameter family of possible
  asymptotic solutions,
  some called "Spurious solutions"
- Some declared "Failure Of MAE"
- Goal – Rescue (modify or augment)

We Present Intuitive Argument and 4 Mathematical Arguments To Rescue MAE

Intuitive Argument

- Exit path should be shortest to exit point
  - 1 exit point, probability 1
  - N exit points, probability $\frac{1}{N}$
- Thus,

$$a < b, \quad c_0 = \alpha, P_{-a} = 1, P_b = 0$$

(No left boundary layer)

$$b < a, \quad c_0 = \beta, P_{-a} = 0, P_b = 1$$

(No right boundary layer)

$$a = b, \quad c_0 = \frac{\alpha + \beta}{2}, \quad P_{-a} = P_b = \frac{1}{2}$$

(2 boundary layers)

- However, intuition is not conclusive
- We next present 4 different mathematical arguments to show these results are correct

(I): Modify (Matkowsky 1975)

- Replace standard MAE boundary layers

$$(\alpha - c_0)e^{\frac{-a(x+a)}{\epsilon}}, \quad (\beta - c_0)e^{\frac{-b(b-x)}{\epsilon}}$$

by JWKB boundary layer function

$$A(x)e^{\frac{\phi(x)}{\epsilon}}$$

- ϕ satisfies Eikonal equation

$$(\phi')^2 + x\phi' = 0$$

- A satisfies transport equation

$$xA' + A = 0$$

- Two solutions

$$\phi' = 0, \quad \text{(outer)}$$

$$\phi' = -x$$

so $\phi = \frac{K^2 - x^2}{2}$

– ϕ quadratic - not linear

– Want $\phi \geq 0, \phi = 0$ at boundaries

– Choose $K = \max(a, b)$

- $A(x) = \dfrac{a_0}{x}$, a_0 constant
- Note:

$$\phi \to a(x+a) \text{ as } x \to -a,$$

$$\phi \to b(b-x) \text{ as } x \to b$$

  reduces to standard MAE construction
- Note: single boundary layer function describes multiple boundary layers
- Note: apparent singularity gone, no effect outside boundary layers & using Friedrichs mollifier
- Results same as intuitive argument

(II): Augment (Grasman, Matkowsky 1977)

- Introduce variational problem whose Euler Lagrange equation is given DE
- Use MAE family as admissable functions
- Set first variation to zero
- Same result as intuitive & (I)

(III): Augment (Matkowsky, Schuss 1977)

- Replace variational condition
  by orthogonality condition
- $(p^s, Lu) = 0$, $(f, g) = \int_{-a}^{b} fg\,dx$
- Stationary Kolmogorov forward equation

$$L^* p^s = 0$$

  so $p^s = Ce^{-\frac{x^2}{2\epsilon}}$, C normalization constant
- Variational condition (Ritz)
- Orthogonality condition (Galerkin)
- Same result as intuitive, (I), (II)

(IV): Augment (Chapman, Matkowsky 2013)

- Asymptotics beyond all orders,
  aka exponential asymptotics
- Reason: unable to determine c_0.
  Not enough terms in outer expansion to span solution space
- Add exponentially small terms to outer expansion (construct by JWKB)

$$c_0 = \frac{a\alpha e^{\frac{-a^2}{2\epsilon}} + b\beta e^{\frac{-b^2}{2\epsilon}}}{a e^{\frac{-a^2}{2\epsilon}} + b e^{\frac{-b^2}{2\epsilon}}}.$$

- Consider the 3 cases

  1. $a < b \quad \rightarrow \quad c_0 = \alpha$
  2. $a > b \quad \rightarrow \quad c_0 = \beta$
  3. $a = b \quad \rightarrow \quad c_0 = \frac{\alpha+\beta}{2}$

- Same result as intuitive & (I) (II) (III)

- Again apparent singularity gone as before
  - No effect on interior
  - Only important to match boundary layer
- Solution should depend continuously on data
  - Does not: discontinuous at $a = b$
- Reason: only considered $b - a = O(1)$
- Now consider $b - a = \epsilon d$

$$c_0 = \frac{\alpha + e^{-ad}}{1 + e^{-ad}}$$

$d \to \infty, \quad c_0 \to \alpha, \quad d \to -\infty, \quad c_0 \to \beta,$
$d = 0, \quad c_0 = \frac{\alpha+\beta}{2}$

- Solution depends continuously on data & bridges gap between results
- Result indicates exit doesn't occur at isolated value $c_0 = \frac{\alpha+\beta}{2}$, but in a thin layer about that value

KRAMERS MODEL OF CHEMICAL REACTION RATES
BROWNIAN PARTICLE IN FIELD OF FORCE

1940 KRAMERS: FIELD = POTENTIAL WELL

REACTION OCCURS WHEN PARTICLE OVERCOMES POTENTIAL BARRIER
E = HT. OF WELL, κ = RATE

$\kappa = \frac{1}{2\tau}$
τ is MFPT

FACTOR 1/2; AFTER REACHING RIM EQUALLY LIKELY TO EXIT-RETURN

ARRHENIUS LAW
$\kappa = Ae^{-\frac{E}{kT}}$
E=V(b)-V(a)=ACTIVATION ENERGY, T=TEMP
A=PREEXPONENTIAL FACTOR

MERELY STATES τ is $O(e^{\frac{1}{\epsilon}})$ $\quad \epsilon = \frac{kT}{E}$

κ INCREASES WITH T
MILK SOURS FASTER IN ROOM THAN FRIDGE

SIMILAR APPLICATIONS IN:
ATOMIC MIGRATION IN CRYSTALS
IONIC CONDUCTIVITY IN CRYSTALS
TRANSITIONS BET. EQUIL. STATES
IN JOSEPHSON JUNCTIONS
TO NAME BUT A FEW

κ EMPLOYED IN

$$\frac{dC}{dt} = -\kappa C$$

C = CONCENTRATION OF
REACTION COMPONENT

DIFFUSION APPROXIMATION IN NEUTRON TRANSPORT THEORY

ABOVE, CONSIDERED SP
IN NOISY SYSTEMS
NOISE MODELED COLLISIONS
BY DIFFUSION
NOW, CONSIDER SP FOR SYSTEM
NOT MODELED BY NOISE
YET, DIFFUSION EQ. RESULTS

NEUTRON TRANSPORT THEORY
STUDIES NEUTRON POPULATIONS
NEUTRONS MAY COLLIDE, ANNIHILATED
NEW NEUTRONS BORN BY FISSION
IMPT IN NUCLEAR REACTOR DESIGN

MODEL IS LINEAR
INTEGRODIFFERENTIAL EQ (LBE)
FEW SOLUTIONS AVAILABLE
DESIRE SIMPLER MODEL
AMENABLE TO ANALYSIS

DIFFUSION "APPROXIMATIONS"
PREVIOUS ATTEMPTS UNSATISFACTORY
WE USE SP FOR DIFFUSION APPROX.

NUCLEAR AGE BEGAN WITH
1932: CHADWICK DISCOVERED NEUTRONS
1939: 1939: HAHN, MEITNER -FISSION
1942: FERMI et. al. - NUCLEAR REACTOR

MANY NEUTRONS, $O(10^7)$ - CONTINUUM
FAR MORE NUCLEI $O(10^{23})$ IN MEDIUM,
ONLY NEUTRON-NUCLEAR INTERACTION
LINEAR INTEGRODIFFERENTIAL EQ (LBE)

COLLISIONS CHANGE
$\mu \to \mu'$
$\mu = cos\theta$
FISSION, LARGE ENERGY RELEASED
ENERGY USED GENERATE ELECTRICITY
LINEAR BOLTZMANN EQ (LBE)

$$\frac{1}{v}\Psi_\tau + \mu\Psi_x + \sigma(x)\Psi - \frac{\sigma(x)c(x)}{2}\int_{-1}^{1}\Psi(x,\mu',\tau)d\mu' = 0,$$

BOUNDARY CONDITIONS

$$\Psi(x=0) = f_1(\mu,\tau) \quad for \quad \mu > 0$$
$$\Psi(x=d) = f_2(\mu,\tau) \quad for \quad \mu < 0,$$

+ INITIAL CONDITION.

$\Psi(x,\mu,\tau)$ IS NEUTRON DISTRIBUTION FCTN
IN SLAB GEOMETRY,
PROBABLE NUMBER NEUTRONS AT x, τ,
TRAVELING AT CONST SPEED v
IN DIRECTION $\mu = cos\theta$,
θ IS ANGLE v MAKES WITH HORIZONTAL
$\sigma(x)$ IS SCATTERING CROSS SECTION,
PROB. THAT NEUTRON INCIDENT ON
NUCLEUS RESULTS IN SCATTERING EVENT
(DIRECTION CHANGES μ' to μ)
AVG. INVERSELY PROPORTIONAL TO
MEAN FREE PATH l,
AVG. DIST.TRAVELED BET. COLLISIONS.
$c(x)$ AVG. # SECONDARY NEUTRONS BORN

$c = 1$ (critical) NEUTRON POPULATION
JUST SUSTAINED
$c > 1$ ($c < 1$)
SUPERCRITICAL (SUBCRITICAL),
NEUTRON POPULATION GROWS (DECAYS)
TO CONTROL NEUTRON GROWTH (SAFETY)
CONTROL RODS INSERTED
(ABSORB NEUTRONS).
v MICROSCOPIC VELOCITY

COMPLICATIONS
1/2 BC AT EACH BDRY
PRESCRIBE INCOMING, NOT OUTGOING
CONTINUOUS SPECTRUM
FEW SOLUTIONS KNOWN

DESIRE SIMPLER MODELS
FOR REACTOR DESIGN
MORE AMENABLE TO ANALYSIS
DIFFUSION "APPROXIMATION"
WIDELY USED

PREVIOUS ATTEMPTS
P_1 DIFF., ASY DIFF.
P_1 DIFFUSION
EXPAND IN LEGENDRE POLYNOMIALS $P_n(\mu)$
TRUNCATE AFTER N TERMS; P_N APPROX
IF N SUFF LARGE, CONVERGENCE
TRUNCATE AFTER 2 TERMS
GET DIFFUSION "APPROX"
Q: WHY "APPROX" VALID?

ASY DIFFUSION
REPLACE FINITE BY INFINITE DOMAIN
REPLACE VARIABLE BY CONST COEFFS
CONSIDER SOLUTION AT INFINITY
GET DIFFUSION "APPROX"
Q: WHY "APPROX" VALID?

DIFFUSION EQS HAVE
DIFFERENT COEFFICIENTS
CLOSE IF $c \sim 1$ (NEAR CRITICAL)

DIFFERENT BCS POSTULATED, NOT
DERIVED e.g., MARSHAK, MARK

TO MAKE SENSE AS APPROXIMATION
MUST BE ABLE TO ANSWER 4 QUESTIONS
IN WHAT SENSE APPROX?
CONDITIONS FOR VALIDITY?
HOW GOOD IS APPROX?
HOW PROVIDE CORRECTIONS
(IMPROVEMENT)?

WE ACTUALLY DERIVE
A DIFFUSION APPROXIMATION
AND ANSWER THESE QUESTIONS

RATHER THAN STOCHASTIC APPROACH
WE EMPLOY SCALING
ELEMENTARY CALCULUS, SP (MAE)

NONDIMENSIONALIZATION

$$y \equiv \frac{x}{d}, \qquad t \equiv \frac{\bar{v}\tau}{d},$$

$\bar{v}$ REFERENCE MACROSCOPIC VELOCITY,
NONDIM. SCATTERING CROSS SECTION

$$a(y) = \frac{\sigma}{\bar{\sigma}},$$

$\bar{\sigma}$ REF. SCATTERING CROSS SECTION.
INTRODUCES SMALL PARAMETERS

$$\epsilon \equiv \frac{l}{d} \ll 1, \qquad \delta \equiv \frac{\bar{v}}{v} \ll 1,$$

FORMER
MEAN FREE PATH l << TYPICAL MACRO-SCOPIC LENGTH,
e.g., SIZE OF REACTOR
LATTER
MACRO VELOCITY << MICRO VELOCITY.

$\Psi \to \psi$

DEFINITIONS IMPLY
$t = \epsilon\delta\tau$, LONG TIME SCALE.
ASSUME
ϵ, δ SAME ORDER
SET $\epsilon = \delta$, TO GET

$$\epsilon^2\psi_t + \epsilon\mu\psi_y + a(y)\psi - a(y)c(y,\epsilon)\int_{-1}^{1}\psi d\mu' = 0.$$

EQUATION CLEARLY SP TYPE.
EXPAND BOTH ψ, c IN
ASYMPTOTIC SERIES IN ϵ

$$\psi \sim \sum_n \psi^n(y,t,\mu)\epsilon^n, \quad c \sim \sum_n c_n(y)\epsilon^n$$

FOR THE OUTER EXPANSION TO BE
VALID IN INTERIOR OF DOMAIN

BCs OBTAINED BY MATCHING
TO BL EXPANSION

EQUATING COEFF OF EACH
POWER OF ϵ TO ZERO
OBTAIN RECURSIVE EQs FOR COEFFs

$$L\psi^0 \equiv a[\psi^0 - \frac{c_0}{2}\int_{-1}^{1} \psi^0 d\mu'] = 0,$$

LEARN ψ^0 IND. OF $\mu, \psi^0 = \psi^0(x,t)), c_0 = 1$

$$L\psi^1 = -\psi_y^0 + \frac{ac_1}{2}\int_{-1}^{1} \psi^0 d\mu'$$

LEARN ψ^1 LINEAR IN $\mu, c_1 = 0$

$$L\psi^2 = -\psi_y^1 + \frac{ac_2}{2}\int_{-1}^{1} \psi^0 d\mu' + \frac{ac_1}{2}\int_{-1}^{1} \psi^1 d\mu' - \psi_t^0.$$

LEARN ψ^2 QUADRATIC IN μ
RELATIONS BET. COEFFS

COLLECTING RESULTS GET

DIFFUSION EQUATION

$$\psi_t^0 = (\frac{1}{3a}\psi_y^0)_y + ac_2\psi^0.$$

NOTE ANGULAR DEPENDENCE (μ)
DERIVED, NOT ASSUMED AS IN
P_1 DIFFUSION APPROXIMATION.
$c_2 < 0$ (SUBCRITICAL)
SOLUTION DECAYS TO 0,
BOTH R.H.S. TERMS NEGATIVE
REACTION NOT SUSTAINED
$c_2 > 0$ (SUPERCRITICAL)
REACTION CAN BE SUSTAINED

TO COMPLETE DERIVATION , MUST SOLVE
BL PROBLEM NEAR EACH BOUNDARY
THEN, MATCH BL TO OUTER EXPANSION,
TO GET BCs FOR DIFFUSION EQUATION.
WE DON'T CARRY THIS OUT,
INVOLVES TOO MUCH DETAIL
cf. HABETLER, MATKOWSKY PAPER
FOR BL ANALYSIS AND MATCHING

WE SUCCESSFULLY ANSWERED QUESTIONS

Q: IN WHAT SENSE
IS APPROXIMATION APPROXIMATE?

A: AN ASYMPTOTIC APPROXIMATION .

Q: WHEN IS IT VALID?

A: WHEN ϵ IS SMALL
i.e., WHEN MEAN FREE PATH << THAN TYPICAL MACROSCOPIC LENGTH,
e.g., THE SIZE OF THE DOMAIN

Q: HOW GOOD IS THE APPROXIMATION?

A: ERROR IS $O(\epsilon)$

Q: HOW TO IMPROVE APPROXIMATION?

A: INCLUDE HIGHER ORDER TERMS.

NO INITIAL LAYER ANALYSIS NEEDED
MODEL NOT VALID FOR EARLY TIMES (STARTUP)

Dedication This lecture is dedicated to my teacher, role model, colleague and friend, Joe Keller, of blessed memory.

Acknowledgements

It is a pleasure to acknowledge my colleagues; Mark Pinsky (ob'm), who introduced me to the exit problem, Elton Hsu and Mike Miksis for useful discussions. Finally, throughout my career I have been fortunate to collaborate with extremely talented and creative people, who have been, and are, good friends. To each I am most grateful.