Weak resolution of singularities for integrable free boundary problems in the plane

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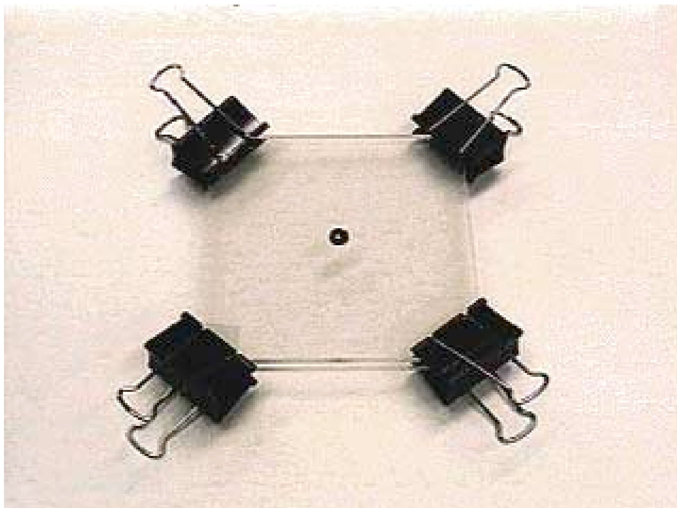
The many guises and uses of the Darcy Law

(H. Darcy 1856, H.H.S. Hele-Shaw 1898)



- Incompressible, immiscible fluids with very large, resp. small viscosity, small Reynolds numbers;

- Averaging the Navier-Stokes equations along one dimension ("small vertical extent" of size $b \to 0$), and neglecting gravity, surface tension:

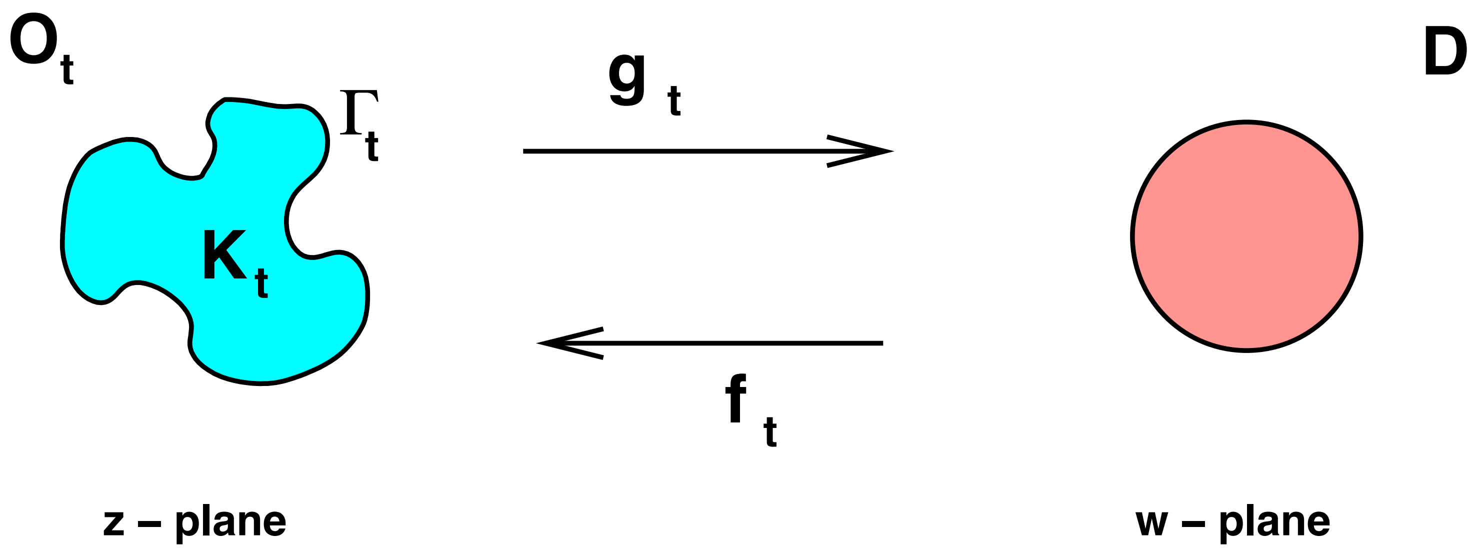
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}$$

$$\vec{v}(x,y,z) = \frac{3}{2}\left[1 - \left(\frac{2z}{b}\right)^2\right] \widetilde{\vec{v}}(x,y)$$

$$p(x,y,z) \to \tilde{p}(x,y) = \frac{1}{b}\int_{-b/2}^{b/2} p(x,y,z)dz$$

$$\text{Hele-Shaw Flow:} \quad \begin{cases} \nabla^2 p = 0 & \text{on } D_{out} = O_t, \\ p = 0 & \text{on } D_{in} = K_t \\ \boxed{V_n = -\nabla_n p} & \text{on } \partial K_t = \Gamma_t \\ p \to -\log[x^2 + y^2], & x, y \to \infty \end{cases}$$

Classical (free)-boundary value problem



$$p(t,z) = -\frac{1}{2\pi}\log|g_t(z)|, \quad V_n \sim |g_t'(t,z)|_{z\in\Gamma_t}$$

where

$$f_t : \overline{\mathbb{D}}^c \to O_t, \quad g_t = f_t^{-1}, \quad f_t(\infty) = \infty, \quad f_t'(\infty) = r(t) \in \mathbb{R}_+$$

$$\text{Harmonic moments : } t_k = -\frac{1}{k\pi}\int_{O_t} z^{-k}\mathrm{d}A, \;\; v_k = \frac{1}{k\pi}\int_{K_t} z^k \mathrm{d}A,$$

$$\text{Normalized area : } t = \frac{1}{\pi}\int_{K_t} \mathrm{d}A.$$

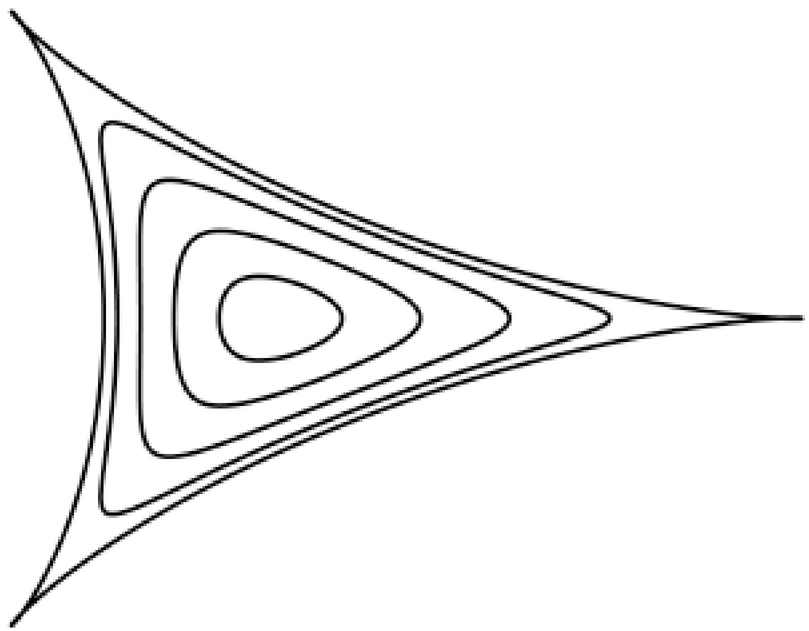
Richardson (1972) - Integrability, Poisson structure: $\boxed{\frac{dt_k}{dt} = 0, \quad k \geq 1.}$

Problem 1 (hard): *Find the support K_t of uniform measure with* <u>*unit*</u> *density and total mass t, whose exterior harmonic moments are $\{t_k, \bar{t}_k\}_{k\geq 1}$*

Problem 2 (easier): *Find the (non-signed) measures $\{\mu_t\}_{t>0}$ with total mass t and the same* <u>*interior*</u> *harmonic moments $\{v_k, \bar{v}_k\}_{k\geq 1}$* as $\mathbb{I}_{K_t}(z)$, i.e.

$$\int_{K_t} f(z)\mathrm{d}A = \int_{K_t} f(z)\mathrm{d}\mu, \quad \forall\, f(z) \in L^1(K_t), \text{ analytic in } K_t$$

Generalization to "domain subordination" chains



Problems 1', 2' *(Relaxing local growth law): Find a chain of domains* $\{K_t\}_{t\in[0,T]},\ K_s \subset K_t\ \forall s < t$, *where* $t =$*Area*(K_t)*, and whose exterior harmonic moments* $\{t_k, \bar{t}_k\}_{k\geq 1}$ are fixed, for T arbitrarily large/$T \to \infty$.

Blow-up of strong solutions: finite-time singularities

Non-trivial example: $t_3 \neq 0$, all others vanish:

$$f_t(w) = rw + 3t_3 r^2 w^{-2}, \quad t = r^2 - 18|t_3|^2 r^4, \quad t \leq t_c = \frac{1}{2} \cdot \frac{1}{36|t_3|^2}$$

$$\frac{\mathrm{d}t}{\mathrm{d}r} = 0, \quad \text{at } t = t_c$$
$$\frac{\mathrm{d}f_t}{\mathrm{d}w} = 0, \quad \text{at } w = 1.$$

$$\boxed{g'_{t_c}(z) \to \infty, \quad z = f_{t_c}(1) \in \Gamma_{t_c}}$$

Remark: True for any $n-$vertex hypotrochoid $t_k \sim \delta_{k,n}, n > 1$.

Advances in Mathematical Fluid Mechanics

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Alexander Vasil'ev

Classical and Stochastic Laplacian Growth

Classes of domains and their Hele-Shaw evolution

Theorem 1. [M. Sakai, Howison et. al.] *If $\Gamma_{t=0}$ is real-algebraic, then it remains so up to the boundary singularity formation time, $t_c > 0$.*

Theorem 2. [Gustafsson, Putinar] *If $K_{t=0}$ is a quadrature domain, then it remains a quadrature domain up to the boundary singularity formation time (critical time), $t_c > 0$.*

Theorem 3. [M. Sakai] *If $K_{t=0}$ is a quadrature domain, then the boundary singularity which forms at $t \to t_c$ is a cusp singularity.*

Theorem 4. [Khavinson, Mineev, Putinar, T.] *If $\Gamma_{t=0}$ is a polynomial leminscate level set, then it is instantly destroyed by the Hele-Shaw flow.*

Theorem 5. [M. Sakai] *If the cusp singularity is of type $(2, 4k-1), k \in \mathbb{N}$, then there is no classical solution beyond $t = t_c$*

$$(Y - Y_0)^2 = (X - X_0)^{4k-1} + O((X - X_0)^{4k-1+\epsilon}), \epsilon > 0.$$

Asymptotic analysis of infinite-time solutions

Definition 6. *Extremal domain: smooth bounded domain Ω in a Riemannian manifold $\mathcal{M}_g$ with metric g, such that the first eigenvalue λ_1 of the Laplace-Beltrami operator on Ω has a corresponding real, positive eigenfunction u_1 satisfying $u_1 = 0, \frac{\partial u_1}{\partial n} = 1$ on $\partial\Omega$.*
Exceptional domain: sequence of extremal domains $\{\Omega_t\}$ with increasing volumes, such that the limit domain $\Omega = \Omega_{t\to\infty}$ is unbounded, and its first eigenvalue vanishes as $t \to \infty$. Roof function: limit $(u_{1,t})_{t\to\infty} \to u$ is a positive, harmonic function on Ω solving the overdetermined boundary value problem with null Dirichlet data and constant Neumann data.

Theorem 7. [Khavinson, Lundberg, T., 2012] *Assume $\Omega \subset \mathbb{C}$ is an exceptional domain. Then Ω can only be one of the following: $\mathbb{H}$, $\overline{\mathbb{D}}^c$, or the image of the strip $|\Im\zeta| \le \pi/2$ under the conformal map $g(\zeta) = \zeta + \sinh(\zeta)$.*

Only known infinite-time solution for Hele-Shaw flow: (elliptical domains)^c.

Generic situation: Whitham hierarchy and equations of hydrodynamic type

Dubrovin and Novikov (1989) “Hydrodynamics of weakly-deformed soliton lattices” and refs. therein.

Whitham averaging of integrable structure $\to$ modulated equations.
Famous examples: (incompressible) Euler equation.

Solutions for modulated (averaged, homogenized) equations break down in “extreme” cases: large initial data, vanishing temporal/spatial scales, high gradients, etc.

Typically regularized by restoring the “averaged” effects: e.g., dispersive regularization (Gurevich-Pitaevskii, as in Hopf $\to$ KdV).

How to proceed when “underlying model” not known?
Integrability-preserving regularization/singularity resolution.

(Surface tension (Tanveer et. al.) destroys integrability.)

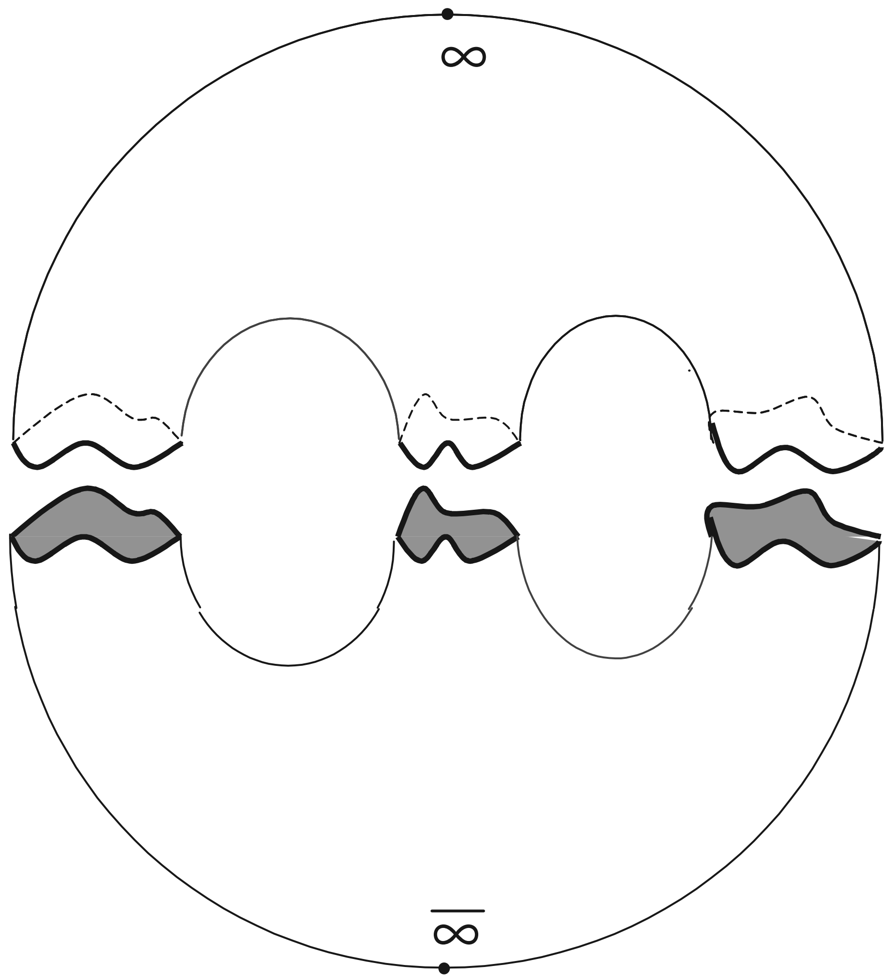
Shottky doubles

Theorem 8. *Let* $\Omega \subset \mathbb{C}$ *be a bounded domain. The following are equivalent:*
(i) Ω *is a (classical) quadrature domain.*
(ii) *The exterior part* $S_{-}(z)$ *of the Cauchy transform of* Ω *is a rational function.*
(iii) *There exists a meromorphic function* $S(z)$ *in* Ω, *continuous up to* $\partial\Omega$, *such that* $S(z) = \bar{z} \,\mathrm{on}\, \partial\Omega$.
If Ω *is simply connected, a further equivalent property is:*
(iv) *Any Riemann mapping function* $f : \mathbb{D} \to \Omega$ *is a rational function.*

There exists a (nontrivial) polynomial in $\mathbb{C}^2$, $Q(z, w)$, such that

$$Q(z, S(z)) = 0 \quad (z \in \Omega) \Rightarrow Q(z, \bar{z}) = 0 \quad \text{for } z \in \partial\Omega,$$

i.e., $\partial\Omega$ is an algebraic curve. $Q(z, w) = 0$ in $\mathbb{C}^2$ is a Shottky double.



∞
$\overline{\infty}$

$$S(z) = \sum_{k>0} k t_k z^{k-1} + \frac{t}{z} + \sum_{p>0} \frac{v_p}{z^p}$$

Darcy law:

$$\partial_t S(z,t) = -\partial_z p(z,t)$$

$$\mathrm{d}W = S\mathrm{d}z + p\mathrm{d}t, \quad \mathrm{d}^2 W = 0, \quad \{t_k\} \text{ fixed.}$$

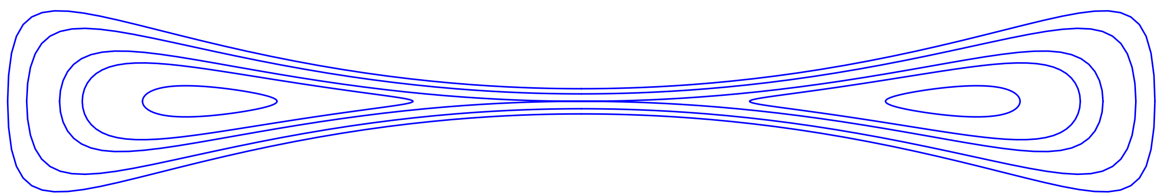
Boutroux-Krichever condition:

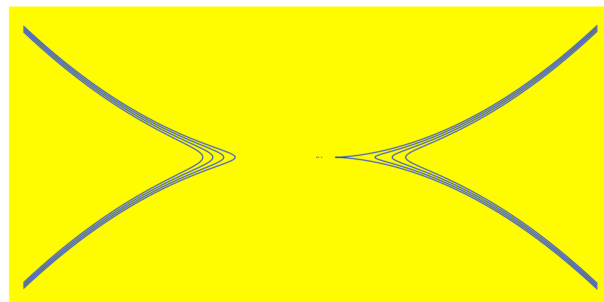
$$\Re \oint_B dW = 0, \quad B - \text{any cycle on the curve.}$$

Open problem: do such curves always exist? (re: dynamics in moduli space of fixed genus)

Pinch-off and merging: hyperelliptic curves with real roots

$$Y^2 = (X - X_1)(X - X_2) \cdot \ldots (X - X_{2k}), \quad k \geq 1$$

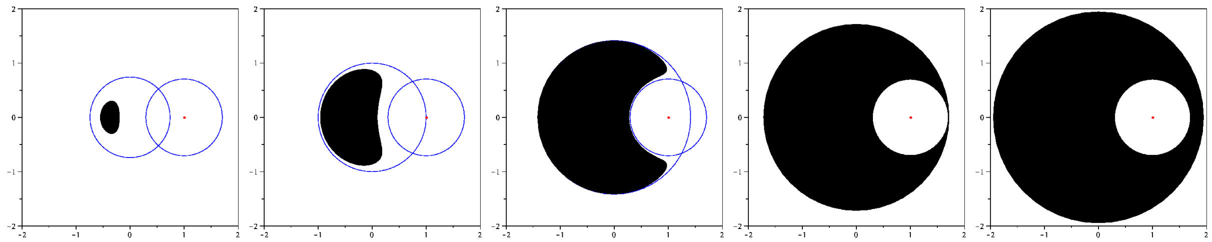


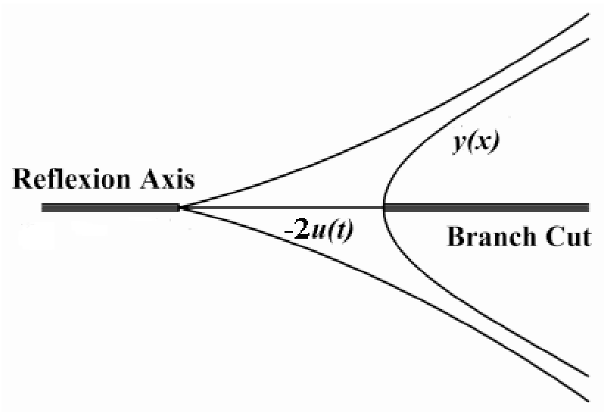


Simplest merger/pinching:

$$Y^2 = (X^2 - \epsilon^2)(X - 1)(X + 1), \quad (2, 4) - \mathrm{cusp}$$

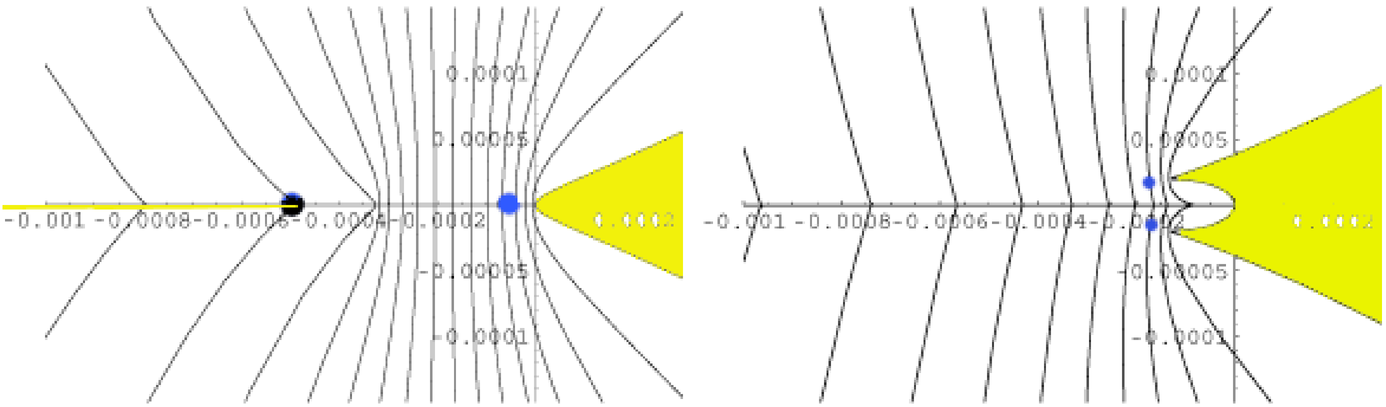
Universal tip-splitting and obstacle problem: complex roots





$$Y^2 = (X - \epsilon)^2 (X + 2\epsilon) \quad (2, 3) - \text{cusp}$$

Perturbing (4, 5)-cusps: higher tip-splitting and tip-merging



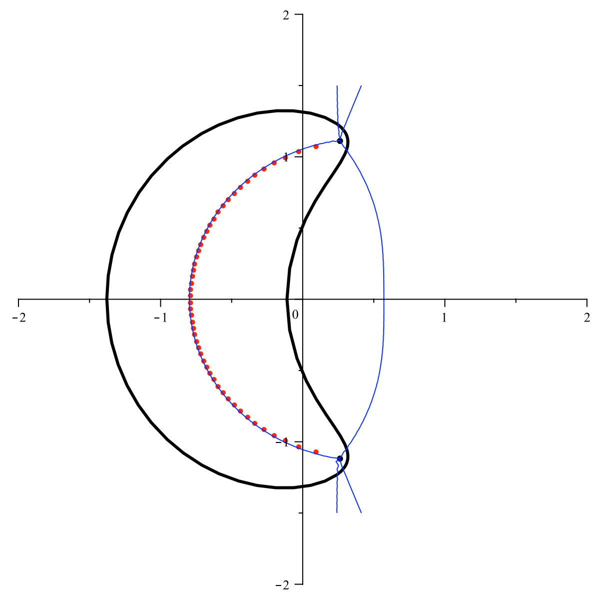
$$Y^4 = (X + 2\epsilon)(X - \epsilon)^2(X - t)^2 \quad (4, 5) - \text{cusp}$$

$$t \neq 0 \Rightarrow (4, 5) \rightarrow (2, 3)$$

Remark: Existence of "higher cusps" not proven for Hele-Shaw flows.

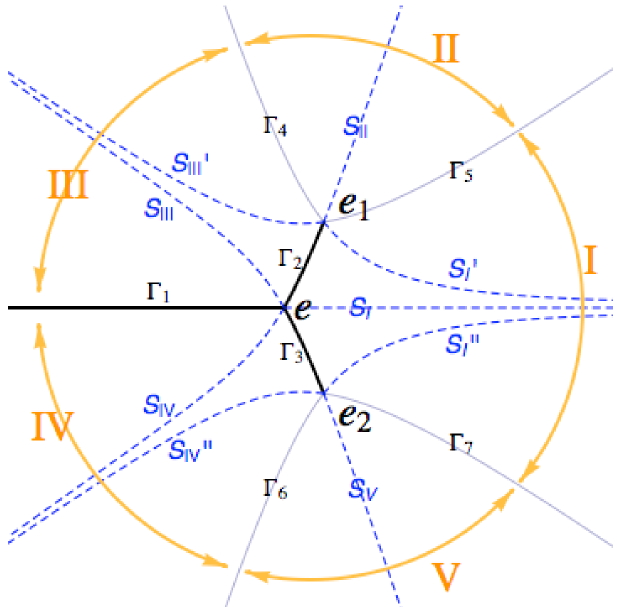
Weak solutions: evolving equilibrium measures

Step 1: any solution to Problem 2' can be continued at $t > t_c$, satisfying the Boutroux-Krichever conditions $\to$ dynamics of support $\to$ Stokes graph/quadratic differential trajectory.
Step 2: find equilibrium measure problem on the support, Disc $\bar{S}(z) = \frac{\mathrm{d}\mu}{\mathrm{d}\ell}$.



[Lee, Wiegmann and T. (2009, 2011), T. (2014).]

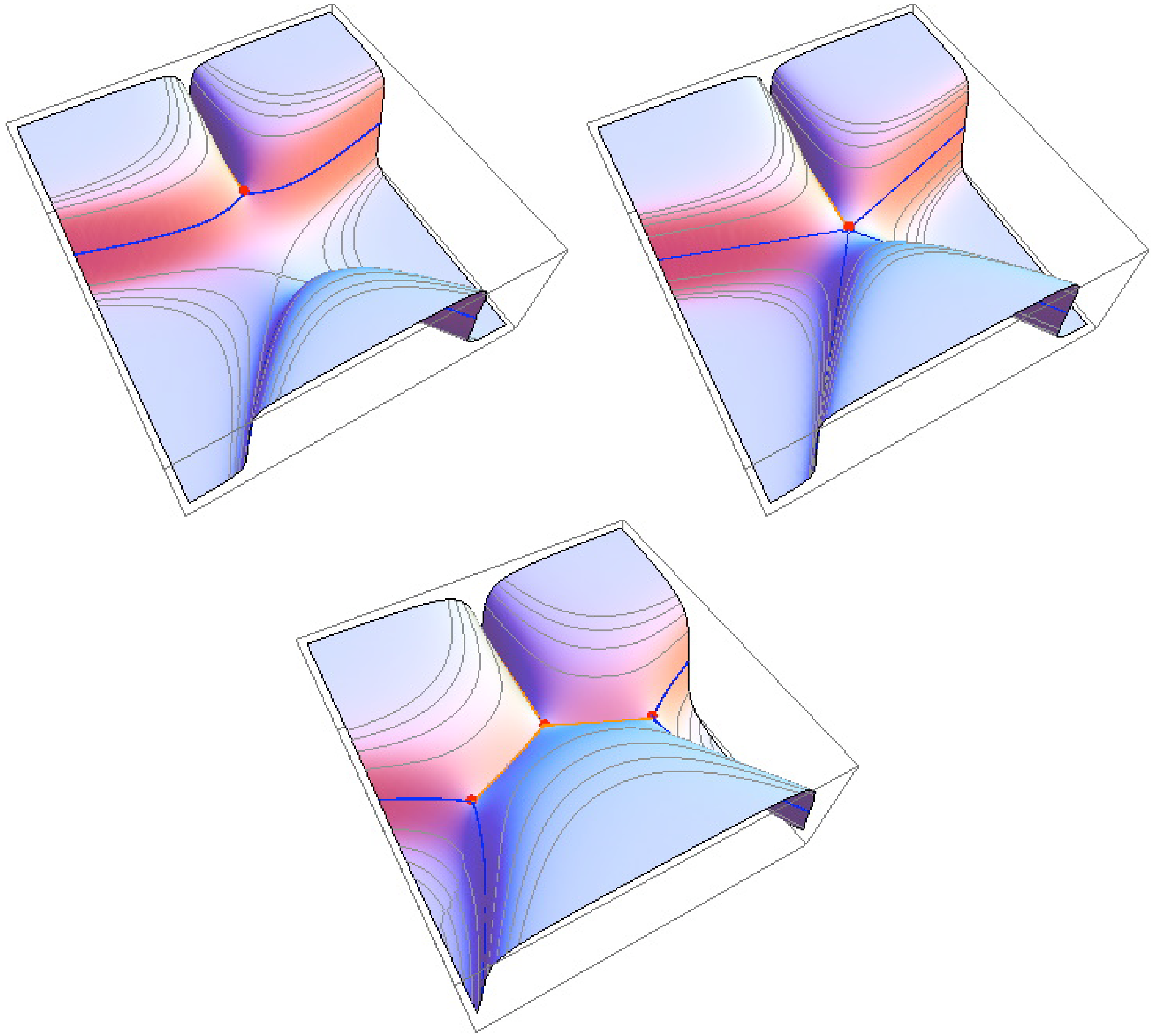
Dynamics past singularities: Stokes graph for associated ODE



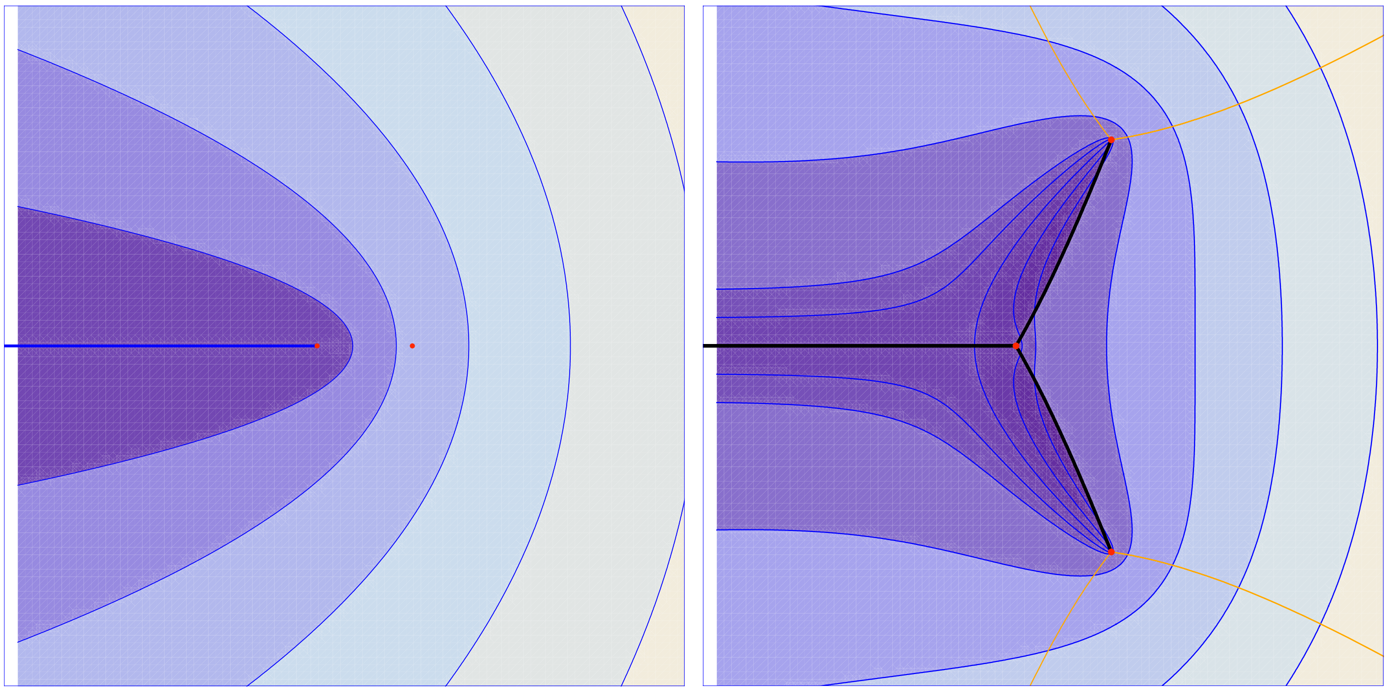
$$Y^2 = 4(X+2\epsilon)(X-\epsilon+i\eta)(X-\epsilon-i\eta), \quad \epsilon, \eta \to 0$$

$$\Re W(e_1) = \Im \left[\int_0^{e_1} \sqrt{4X^3 - g_2 X - g_3}\, dX \right] = 0$$

$$X = \wp(\lambda | g_{2,3}(t)),\ Y = \wp'(\lambda | g_{2,3}(t)) \to \epsilon \frac{d}{dX} \quad (\text{Mumford's } \theta)$$



Tip-splitting mechanism: balayage of equilibrium measure



More connections...

Lax-Oleynik criterion for shock dynamics (enhanced Rankine-Hugoniot condition).

S-curves and complex dynamics (Rakhmanov et al., Shapiro and Solynin)

Critical behavior of Hamiltonian PDE (Dubrovin et al.)

Relation to (Normal) Random Matrix Theory: the R-transform and Burgers-type equations for equilibrium measures (Voiculescu, G. Menon, Blaizot and Nowak).

Vector equilibrium problem and counting measure of orthogonal polynomials (Kuijlaars, Lopez, Its, Bleher).