University of Strathclyde

2015 SIAM Conference on Applied Linear Algebra
Atlanta, Wednesday, October 28

On Growth and Form of Networks

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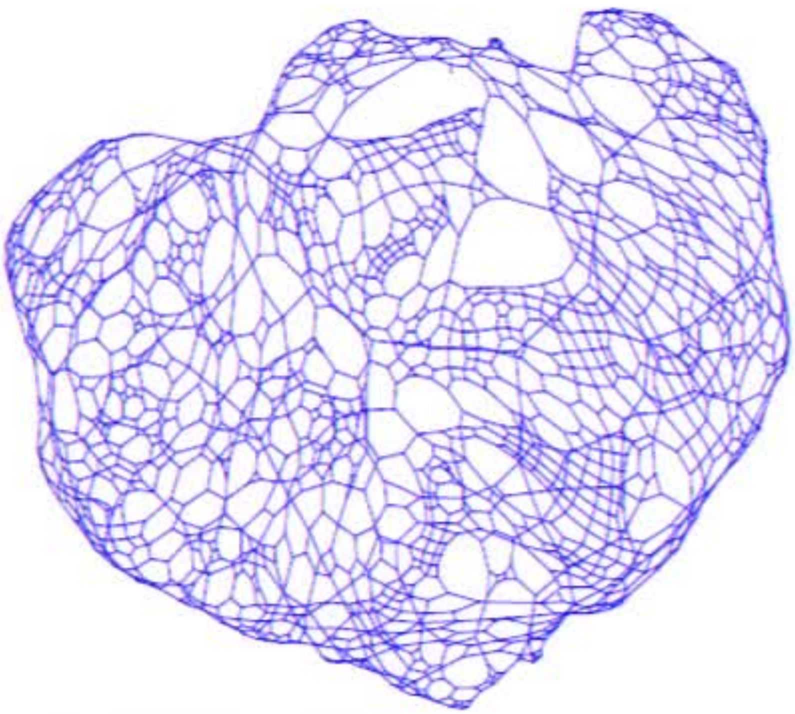
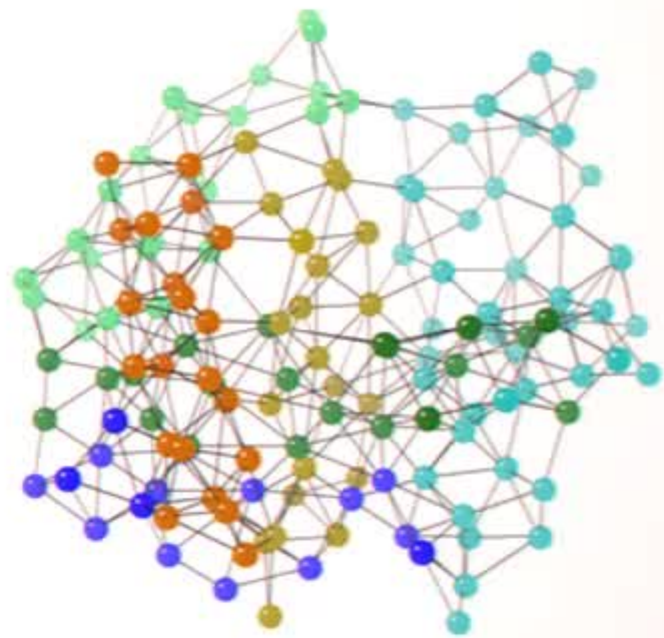
Motivation

How efficiently geometrically-embedded networks use the available space?

Urban street network



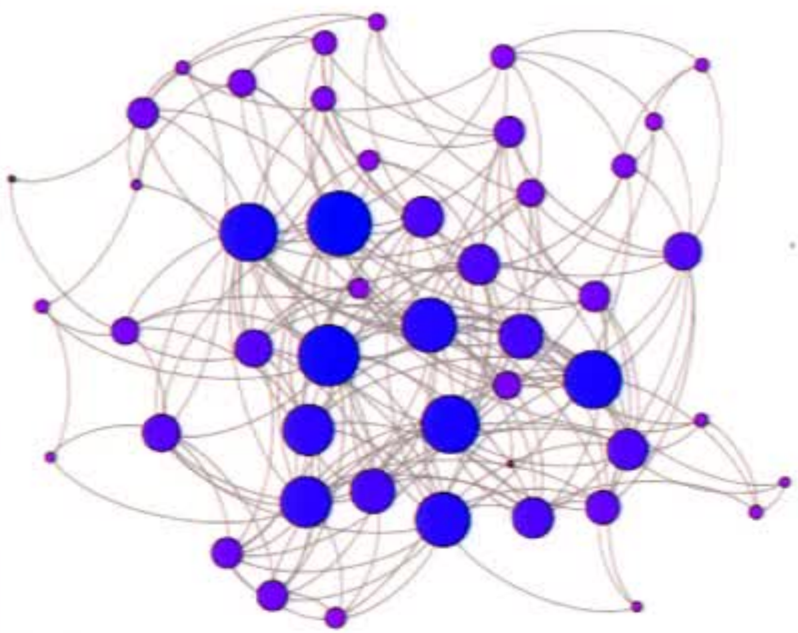
Protein residue network



Motivation

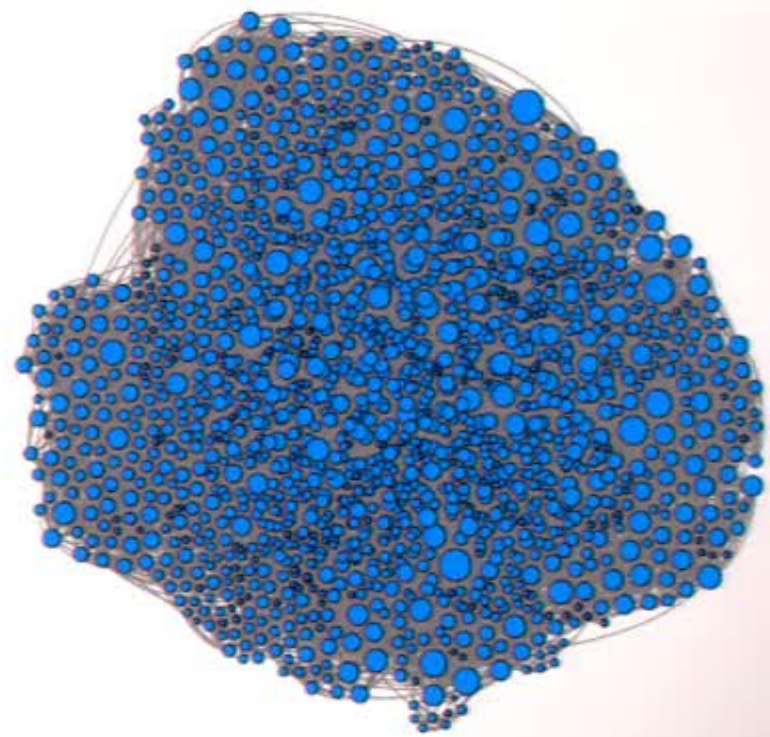
What about networks not embedded geometrically?

Food web



Ecological space?

Social network



Social space?

What is 'Spatial Efficiency'?

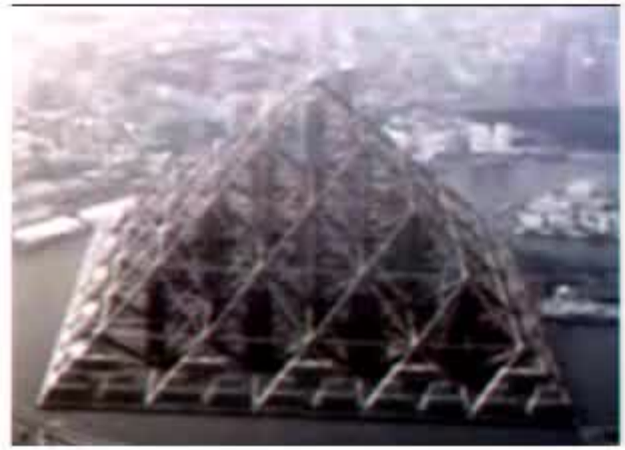
"describes how much time, effort and cost a given arrangement produces (…) as compared to alternative arrangements".

"refers to the organization of physical assets (…) which structure the transportation, communication, (…) within the region and beyond".

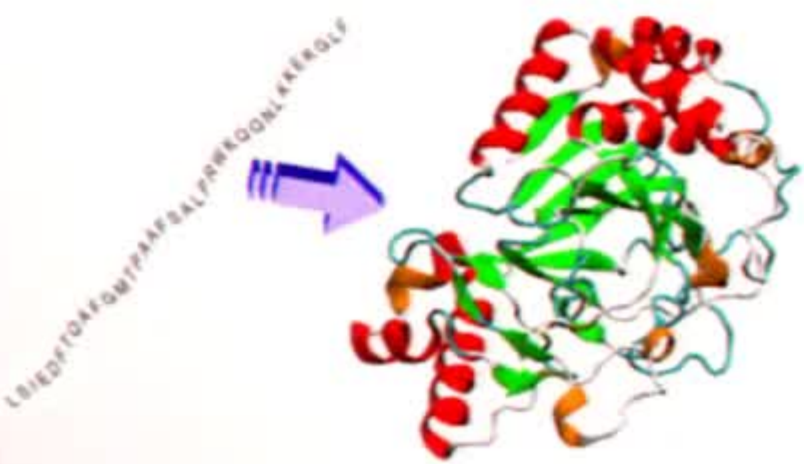
Sarzynski & Levy, *Spatial Efficiency and Regional Prosperity: A Literature Review and Policy*. http://www.gwu.edu/~gwipp/SpatialEfficiencyWPAug16.pdf.

Spatial Efficiency Ingredients

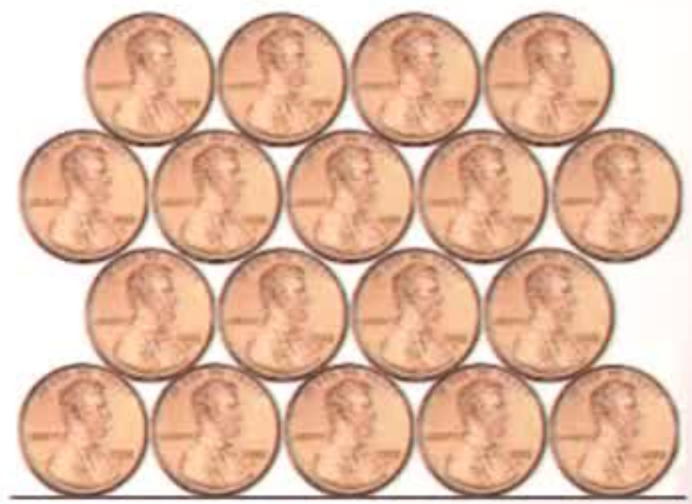
1) Planarity



2) Foldeness



3) Packing



Network 'Spatial Efficiency'

Definition 1. *The spatial efficiency of a network is the efficiency in the communication between the nodes produced by a given spatial embedding of the network.*

Definition 2. *The communication efficiency of a network is the ratio of the amount of information 'successfully transferred' between every pair of nodes in the network to the amount of information 'lost' in the communication process.*

Network Communicability

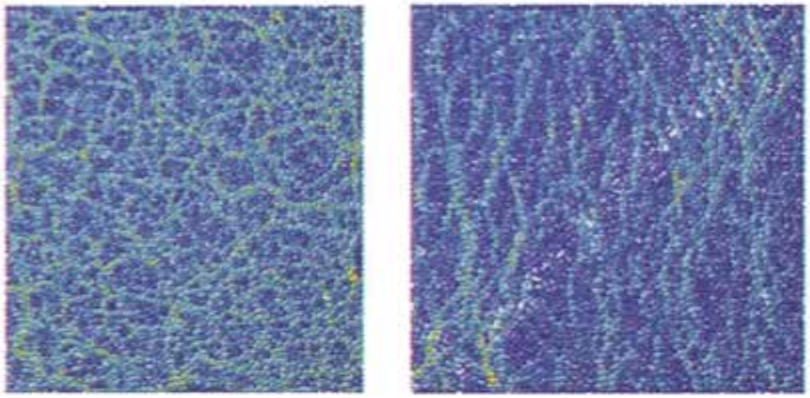
Definition 3. *The communicability between two nodes in a network is defined as a function of the total number of walks connecting them, giving more weights to the shorter than to the longer ones.*

$$G_{pq} = \sum_{l=0}^{\infty} \frac{\left(A^l\right)_{pq}}{l!} = \left(e^A\right)_{pq} = \sum_{j=1}^{n} \psi_{j,p}\psi_{j,q} e^{\lambda_j}$$

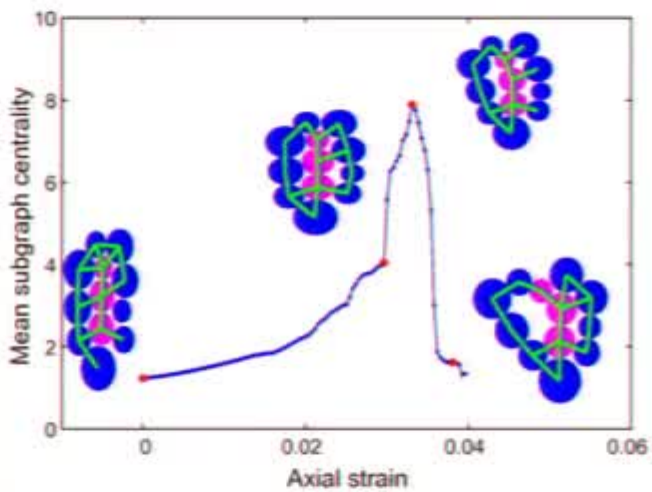
Estrada & Hatano: *Phys. Rev. E* 77, **2008**, 036111
Estrada, Hatano, Benzi, *Phys. Rep.* 514 **2012**, 89-119

Communicability Applications

Granular materials

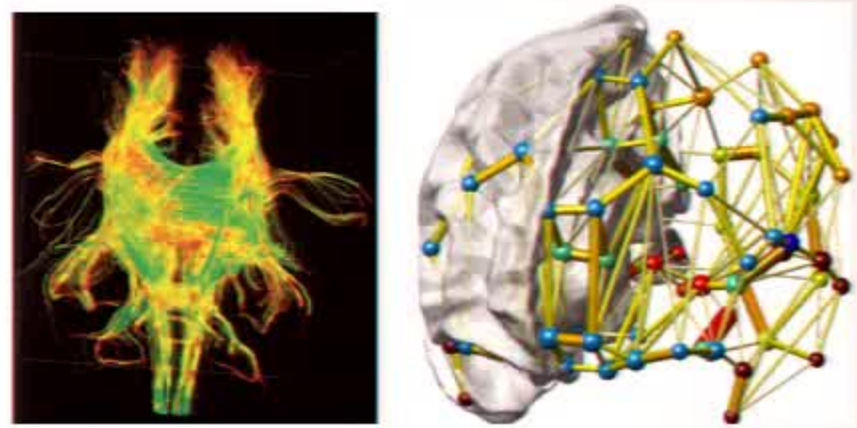


Effects of axial strain

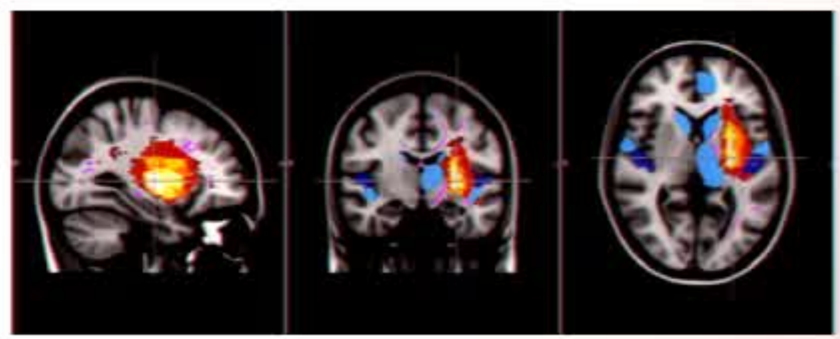


Walker, Tordesillas.:
Int. J. Sol. Struct. 47 (2010) 624-629.

Brain networks



Effects of brain strokes



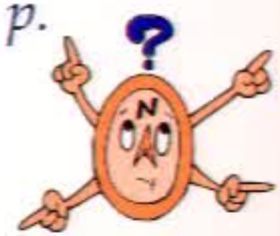
Crofts et al.:
NeuroImage 54 (2011) 161-169.

Communication Efficiency

Consider that two nodes p and q are trying to communicate with each other by sending information both ways through the network.

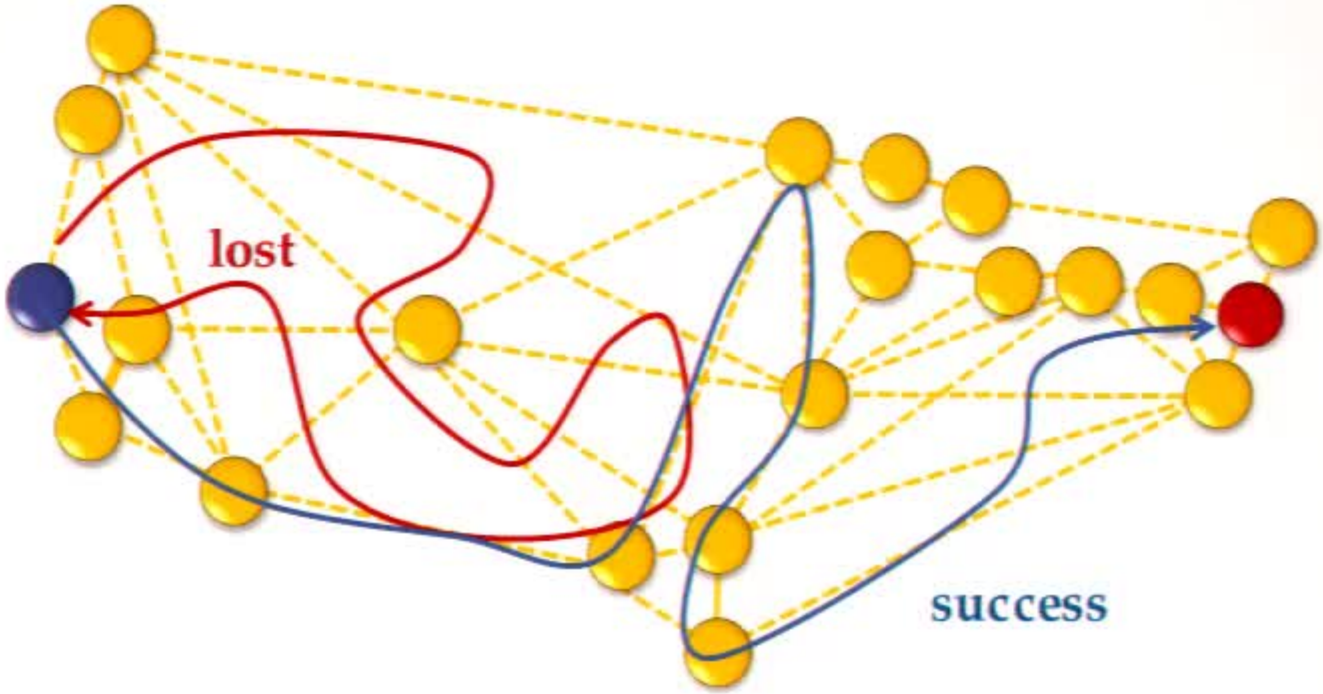
We know that:

G_{pp} — *Quantifies the amount of information that is sent by p, wanders around the network, and returns to the origin, the node p. i.e., it is the amount of lost information.*



G_{pq} — *Quantifies the amount of information that is sent by p, wanders around the network, and arrives at its destination, the node q, i.e., it is the amount of successfully delivered information.*

Communication Efficiency



$$\xi_{pq} \overset{def}{=} G_{pp} + G_{qq} - 2G_{pq}$$

$$\gamma_{pq} \overset{def}{=} \frac{G_{pq}}{\sqrt{G_{pp}G_{qq}}}$$

Communicability Geometry

Theorem 1. *The function* ξ_{pq} *is a squared Euclidean distance between the nodes p and q in the network.*

Theorem 2. *The function* γ_{pq} *is the cosine of the Euclidean angle spanned by the position vectors of the nodes p and q.*

Estrada: *Lin. Algebra Appl*. 436 **2012** 4317-4328
Estrada & Hatano: *SIAM Rev* **2015** accepted, *Arxiv* (2014) 1412.7388.

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Communicability Geometry

Theorem 3. *The communicability distance induces an embedding of a network into an (n-1)-dimensional Euclidean sphere of radius:*

$$R^2 = \frac{1}{4}\left(c - \frac{(2-b)^2}{a} \right)$$

$$a = \vec{1}^T e^{-A} \vec{1} \qquad b = \vec{s}^T e^{-A} \vec{1} \qquad c = \vec{s}^T e^{-A} \vec{s}$$

Remark. *The communicability distance matrix C is circum-Euclidean:*

$$C = \vec{s}\vec{1}^T + \vec{1}\vec{s}^T - 2e^A \qquad \vec{s} = diag\left(e^A\right)$$

E. Estrada, et al. *Discr. Appl. Math.* 176 **2014**, 53-77.

Communicability Angle

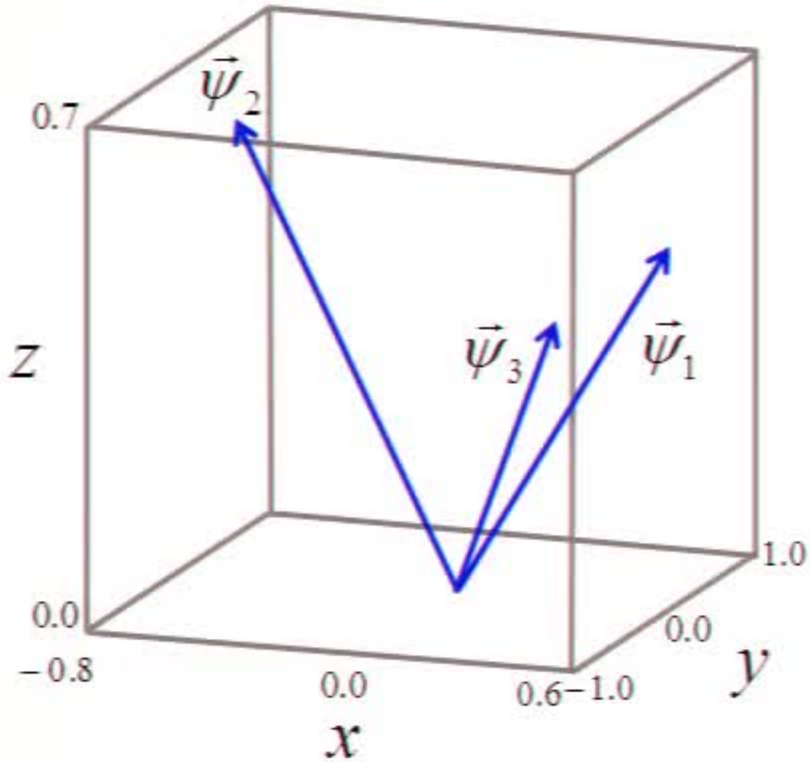
Example: 

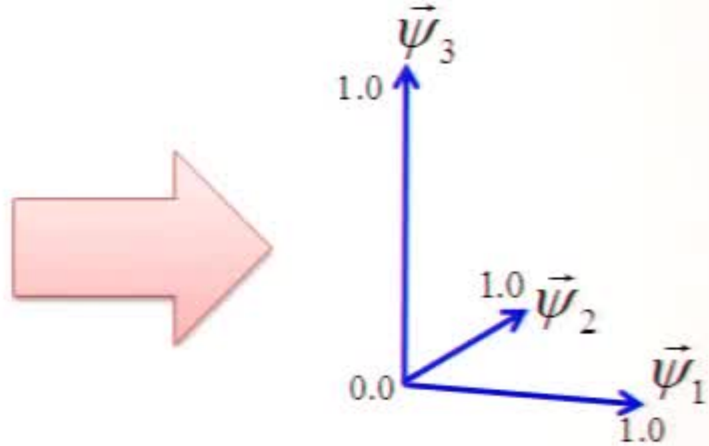
$$A\vec{\psi}_j = \lambda_j \vec{\psi}_j$$

$$\vec{\psi}_1 = \begin{bmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{bmatrix} \quad \vec{\psi}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \quad \vec{\psi}_3 = \begin{bmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{bmatrix}$$

Estrada & Hatano: *SIAM Rev*. (2015), accepted,
Arxiv (2014) 1412.7388.

Communicability Angle





Communicability Angle

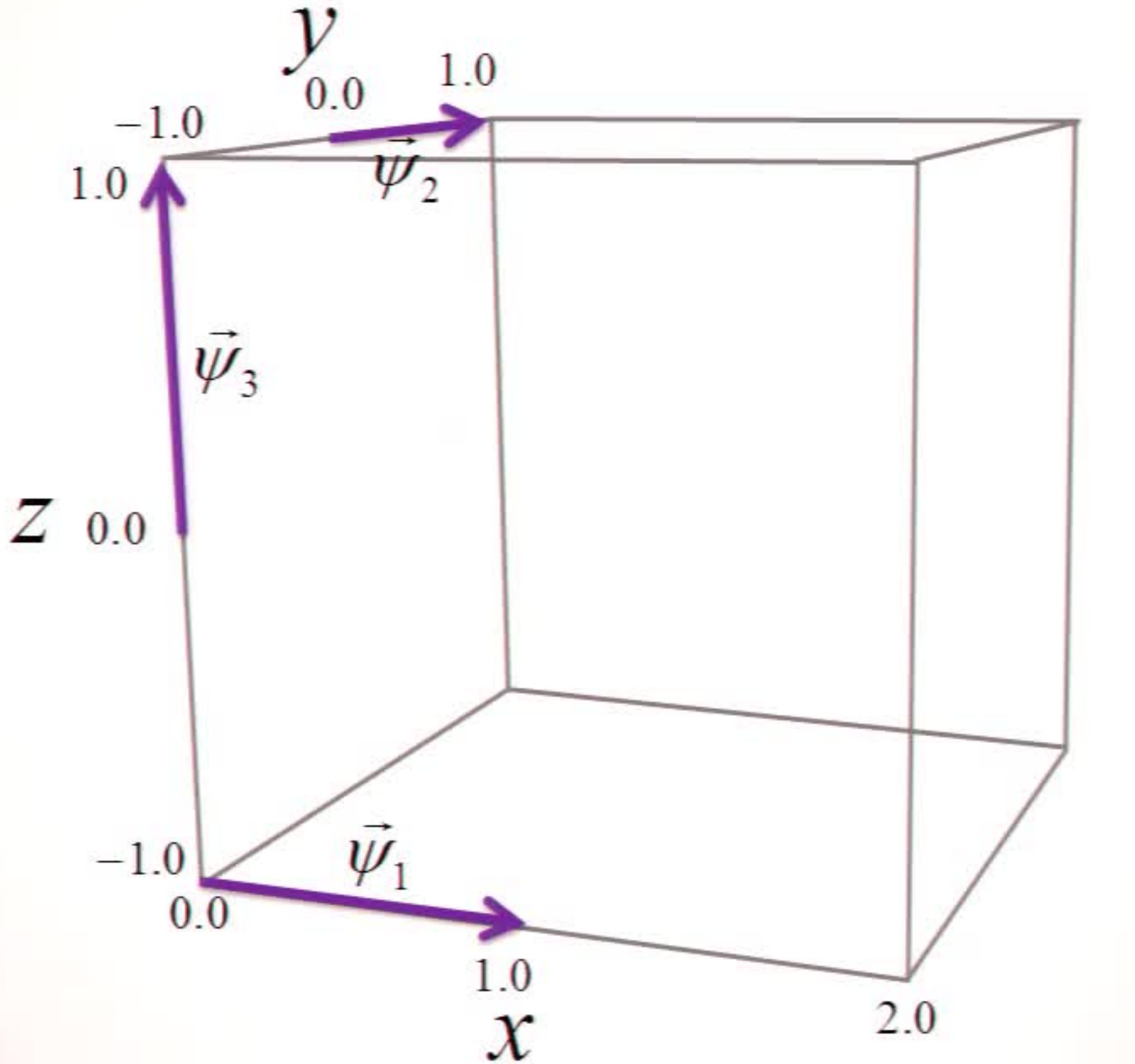
$$A = U \Lambda U^T$$

$$U = \begin{pmatrix} \psi_{1,1} & \psi_{2,1} & \cdots & \psi_{n,1} \\ \psi_{1,2} & \psi_{2,2} & \cdots & \psi_{n,2} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{1,n} & \psi_{2,n} & \cdots & \psi_{n,n} \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\vec{\phi}_p = \left[\psi_{1,p} \cdots \psi_{\mu,p} \cdots \psi_{n,p}\right]^T$$

$$\vec{x}_p = \exp(\Lambda / 2) \vec{\phi}_p$$

Communicability Angle

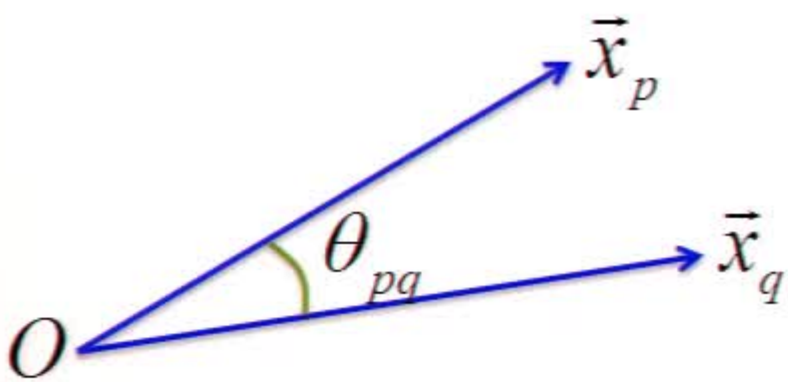


$\langle \theta \rangle$ and Spatial Efficiency

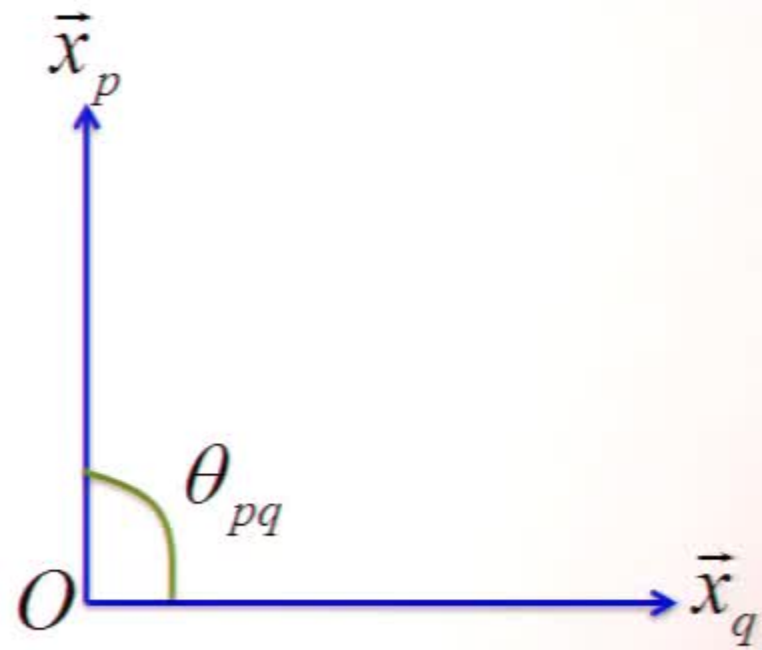
$\langle \theta \rangle$ *: average communicability angle*

For simple, unweighted, undirected networks:

$$0^\circ \le \theta_{pq} \le 90^\circ$$



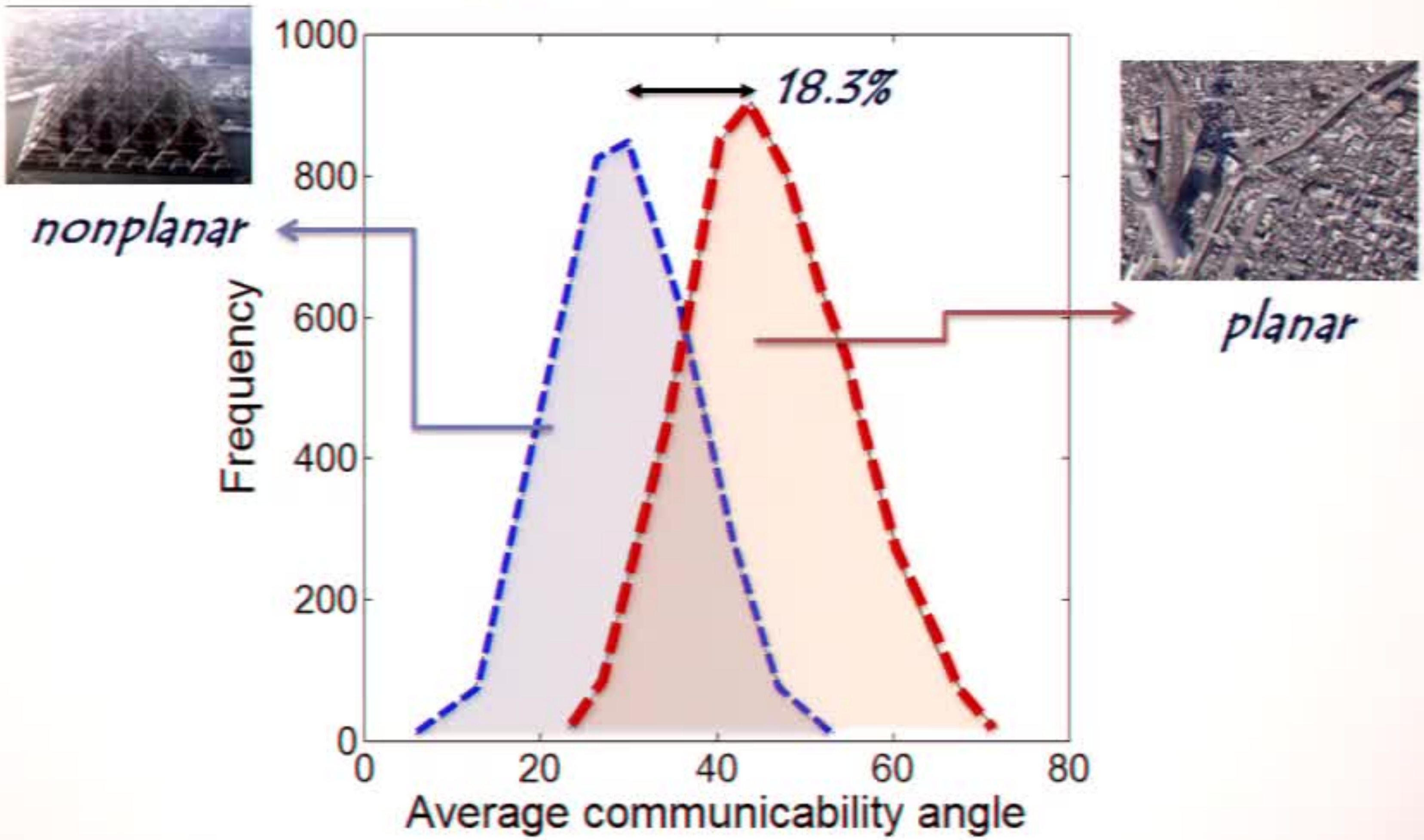
Good spatial efficiency



Poor spatial efficiency

$\langle\theta\rangle$ and Planarity

11,117 connected
graphs with 8 nodes



Estrada & Hatano: *SIAM Rev*. (2015), accepted,
Arxiv (2014) 1412.7388.

$\langle \theta \rangle$ and Planarity

11,117 connected
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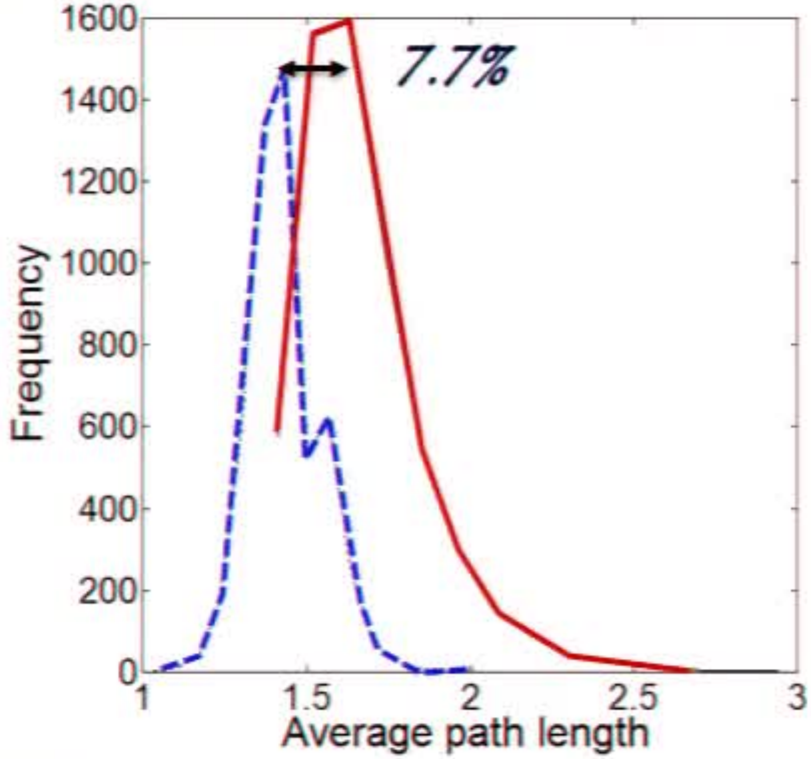
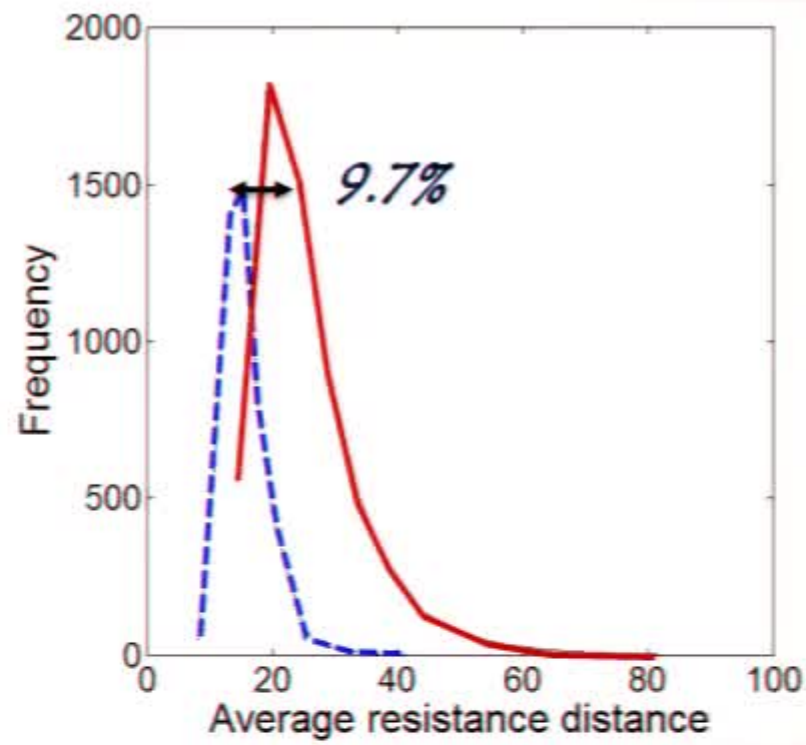




For the 261,080 connected graphs with 9 nodes the percentages of variation are 23.2% for the angles, 9.4% for path length and 10.7% for resistance.

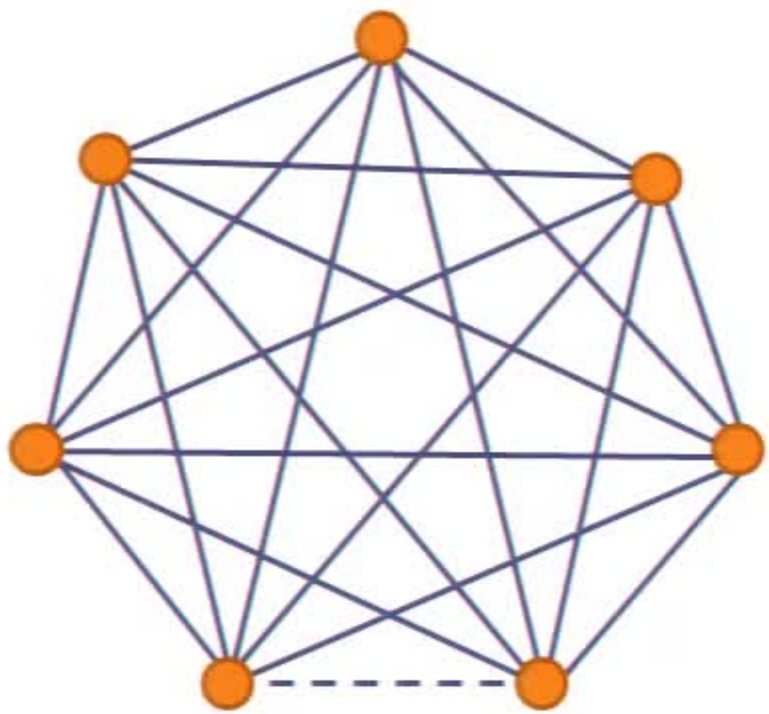
$\langle\theta\rangle$ and Spatial Efficiency

2) Foldeness

P_n

$$\lim_{n\to\infty}\langle\theta(P_n)\rangle = 90^\circ$$

K_n

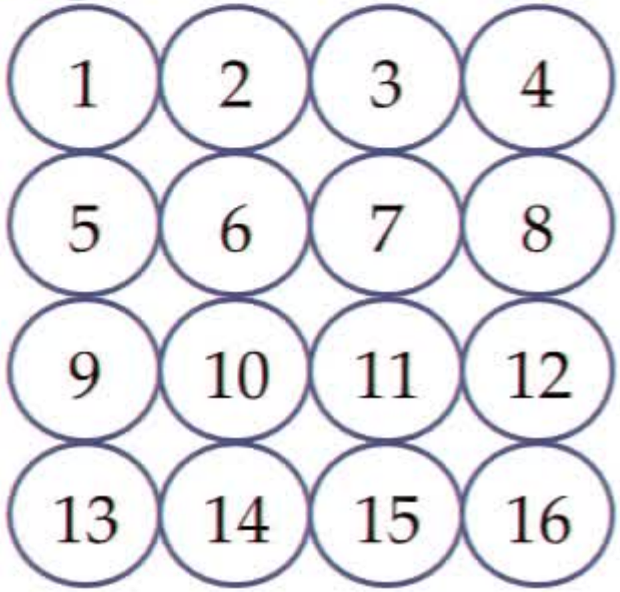


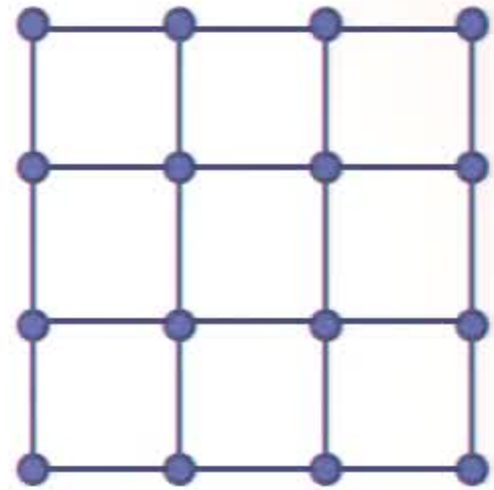
$$\lim_{n\to\infty}\langle\theta(K_n)\rangle = 0^\circ$$

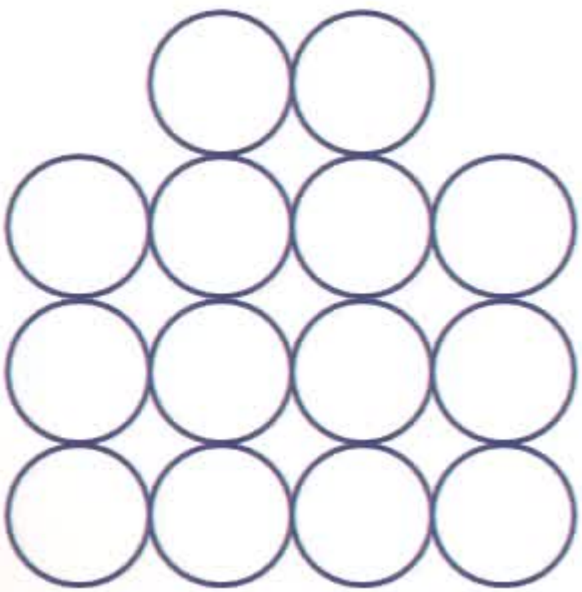
Estrada & Hatano: *SIAM Rev*. (2015), accepted,
Arxiv (2014) 1412.7388.

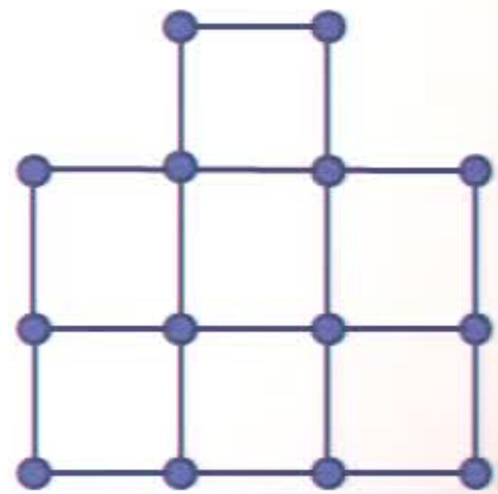
$\langle \theta \rangle$ and Spatial Efficiency

3) Packing

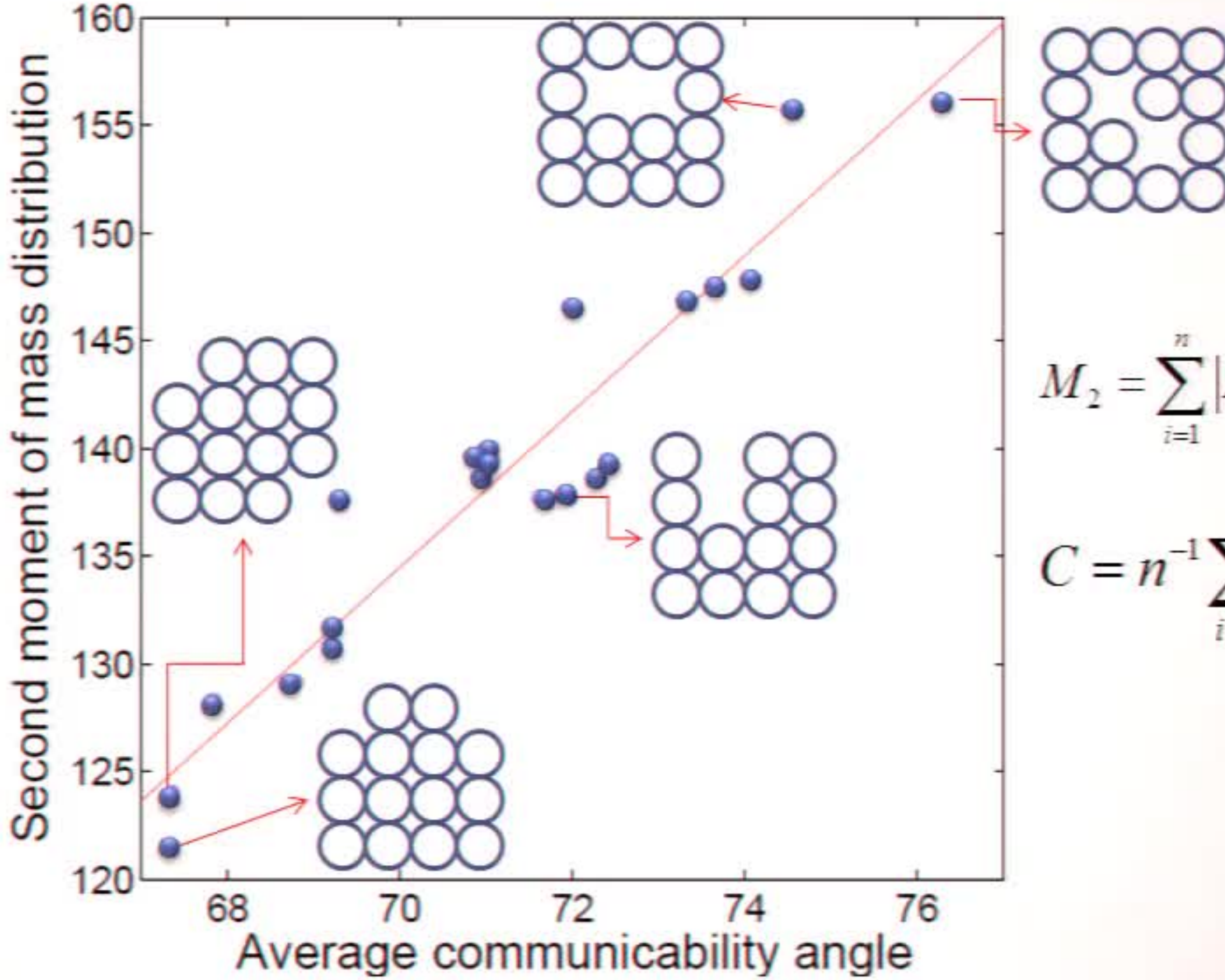








$\langle\theta\rangle$ and Spatial Efficiency

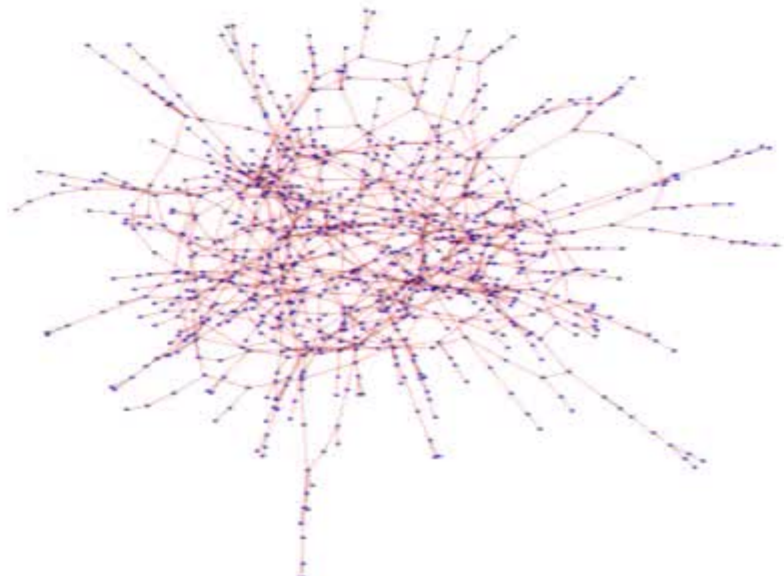


$$M_2 = \sum_{i=1}^{n} |P_i - C|^2$$

$$C = n^{-1} \sum_{i=1}^{n} P_i$$

$\langle\theta\rangle$ in Growing Networks

Erdös-Rényi Random Graphs

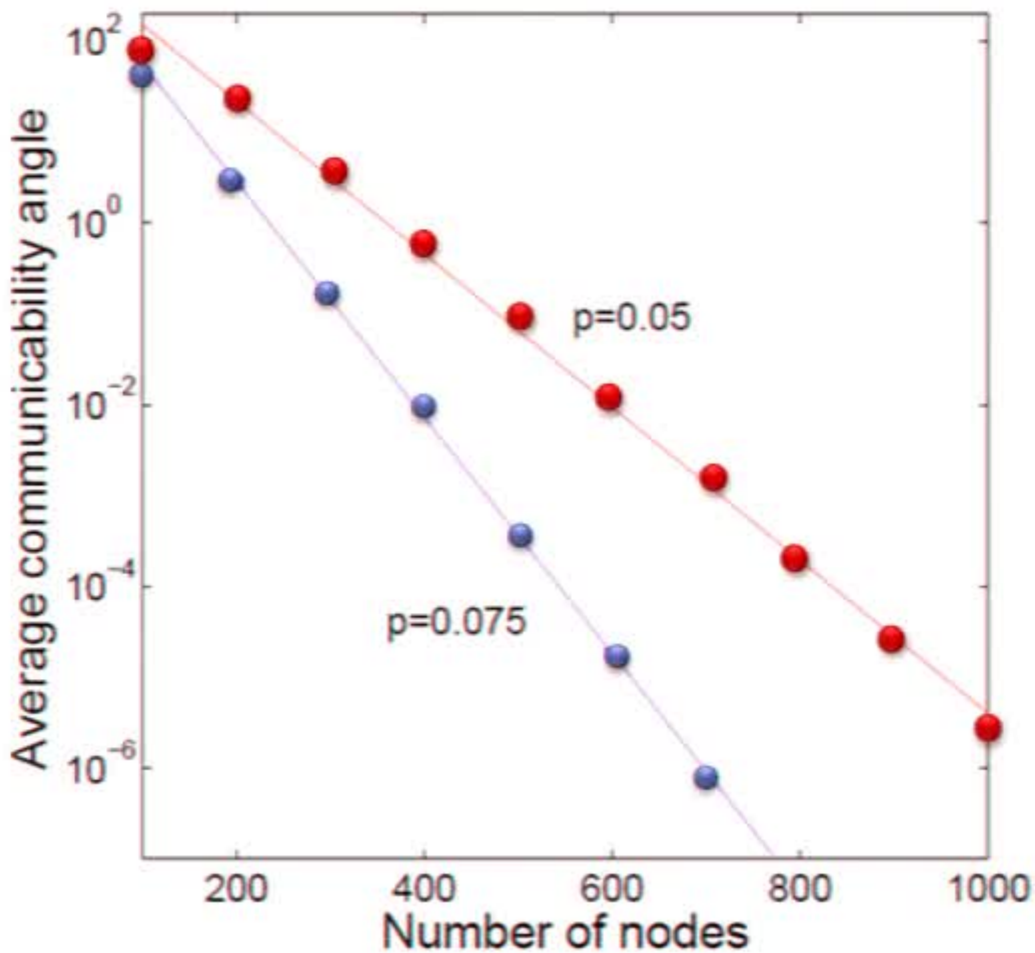


Theorem 4. *Let $G_{n,p}$= G(n,p) be an Erdős-Rényi graph. Let p and q be a pair of nodes in G(n,p). Then,*

$$\lim_{n\to\infty}\left\langle\theta_{pq}\left(G_{n,p}\right)\right\rangle = 0^{\circ}\quad \forall p,q \in V$$

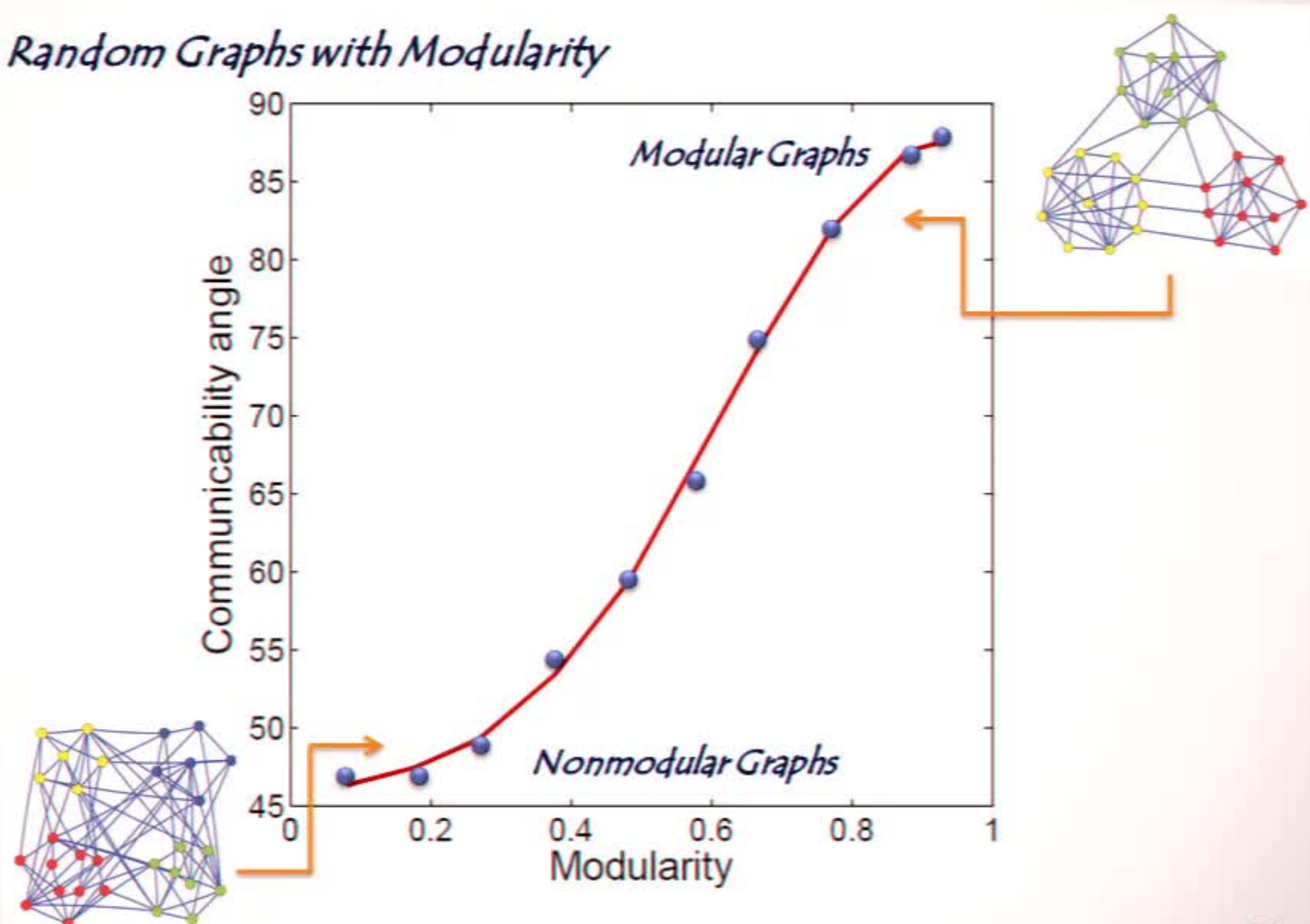
$\langle\theta\rangle$ in Growing Networks

Erdös-Rényi Random Graphs



$\langle\theta\rangle$ in Growing Networks

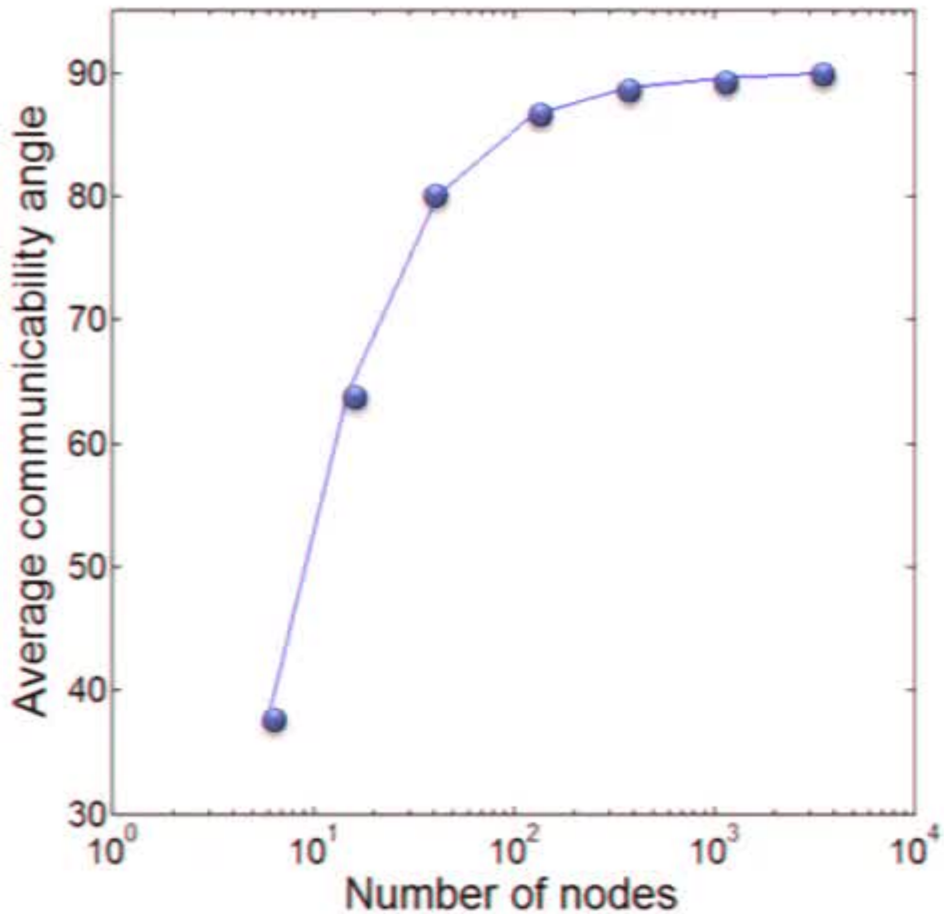
Random Graphs with Modularity



$\langle \theta \rangle$ in Growing Networks

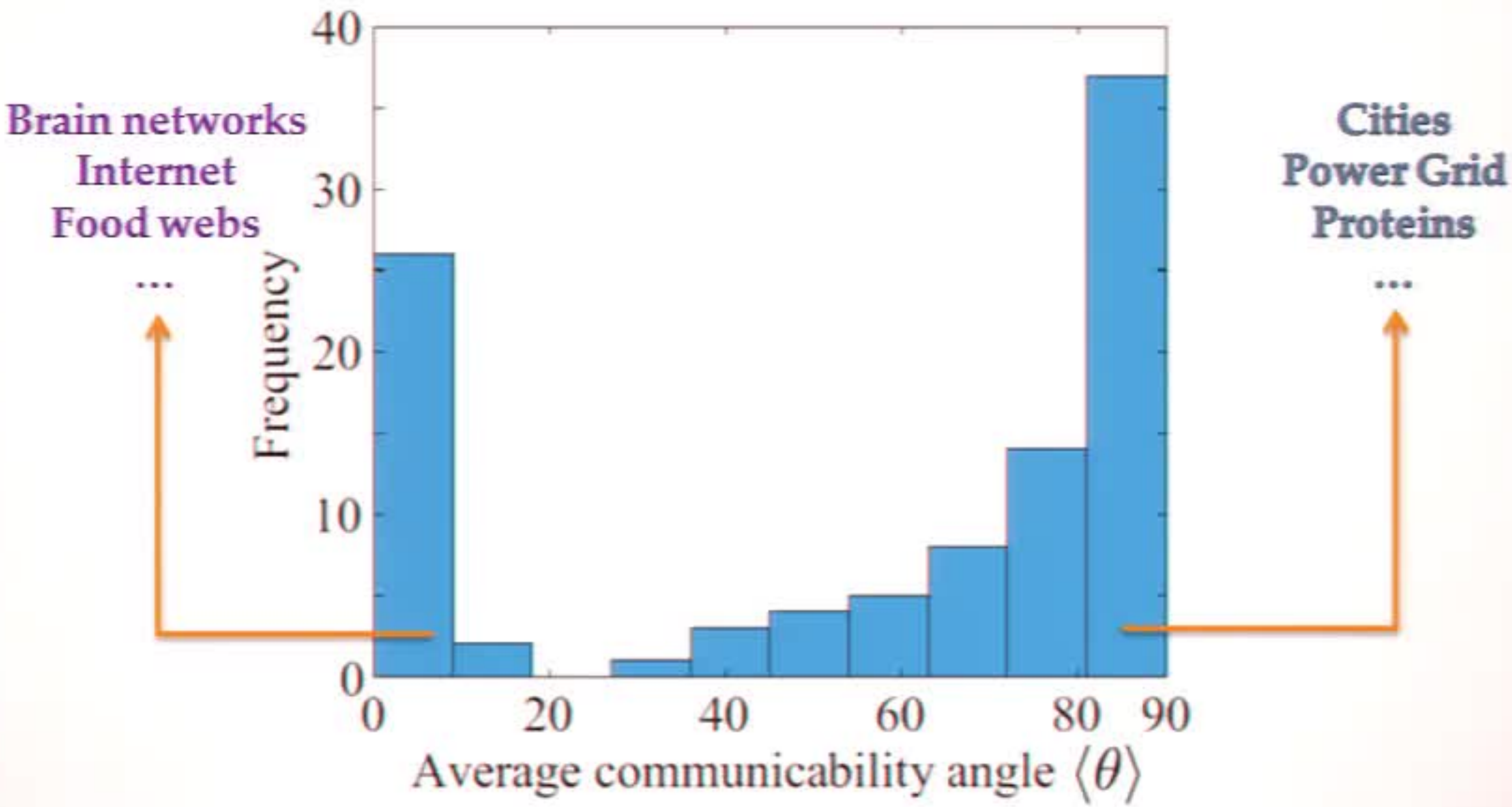
Graphs with Holes

chordless cycle



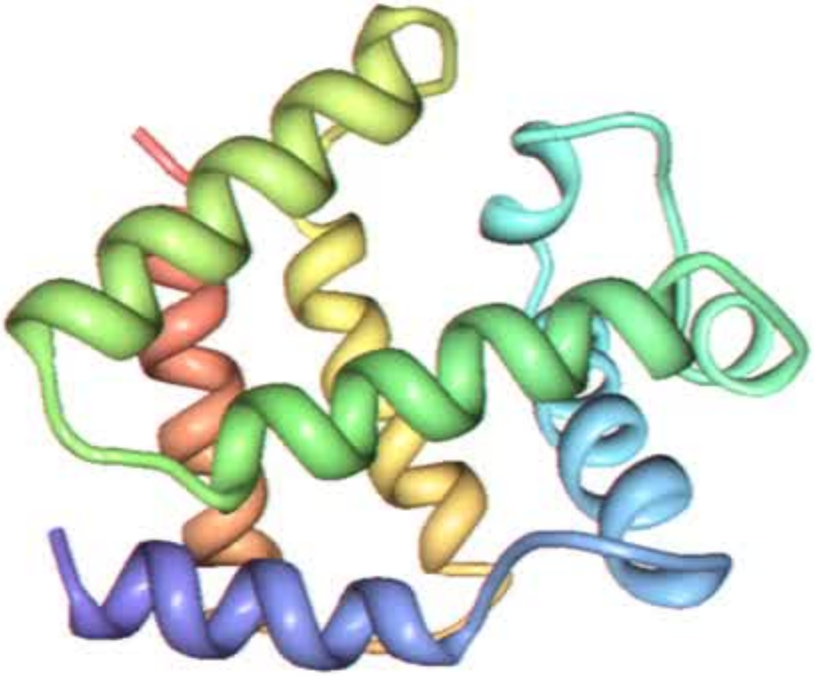
Spatial Efficiency in the Real-World

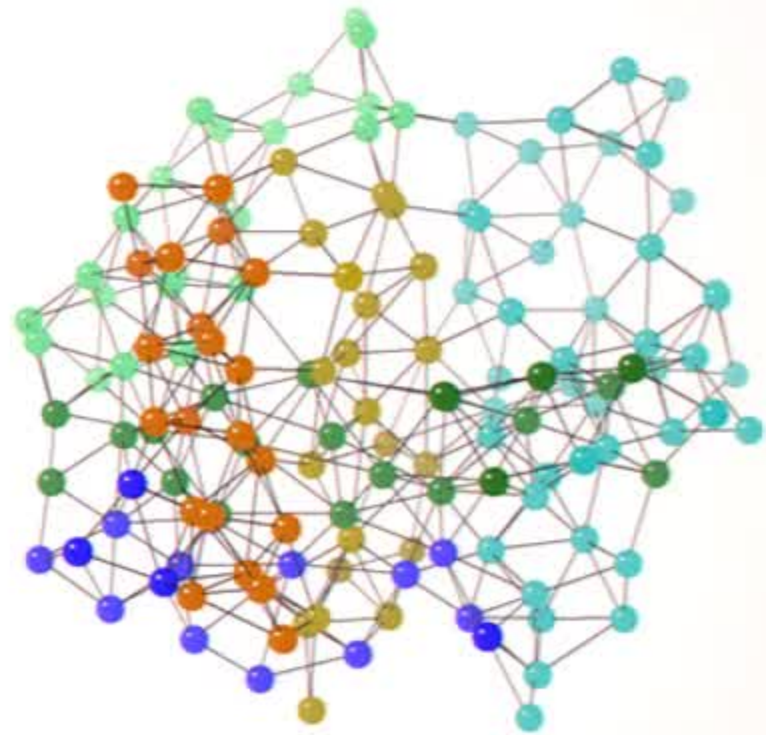
We studied 120 real-world networks representing biological, ecological, infrastructural, social, and technological systems.



Spatial Efficiency in the Real-World

1) Spatially embedded networks





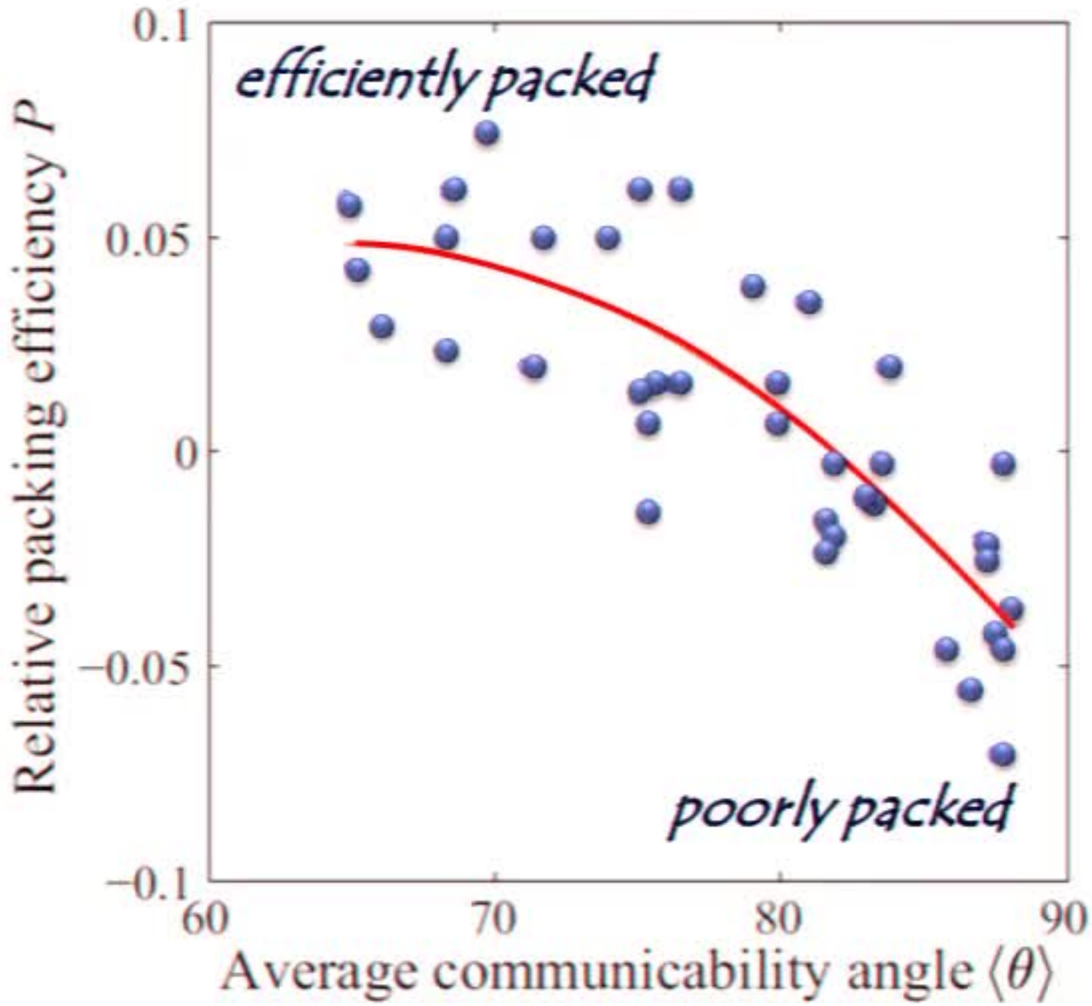
Relative Packing Efficiency

$$P = \frac{V_e - V_o}{V_e}$$

V_e expected volume from ideal 3D structure of the protein

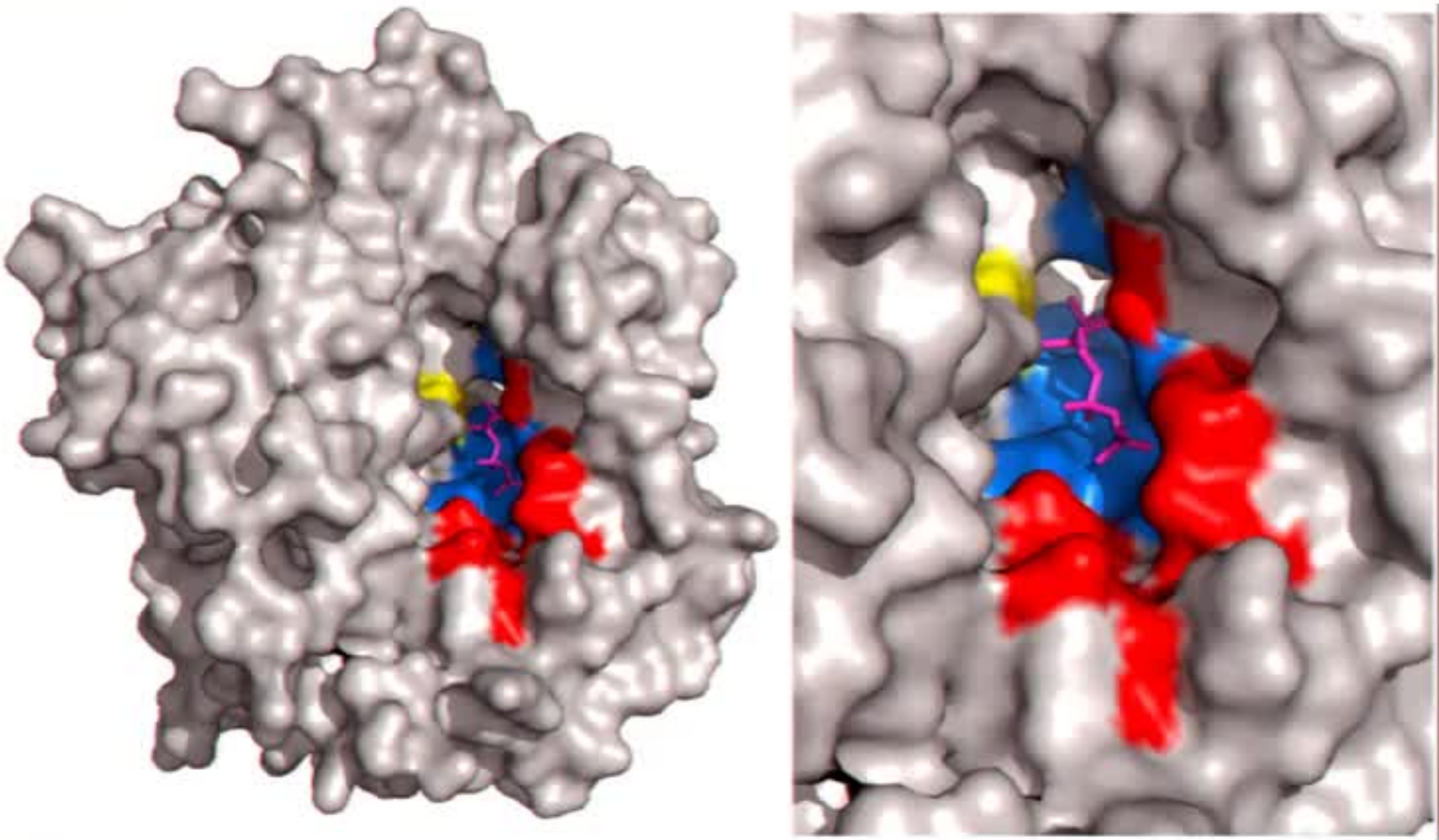
V_o observed volume from X-rays crystal of the protein

Spatial Efficiency in the Real-World



Estrada & Hatano: *Arxiv* (2014) 1412.7388.

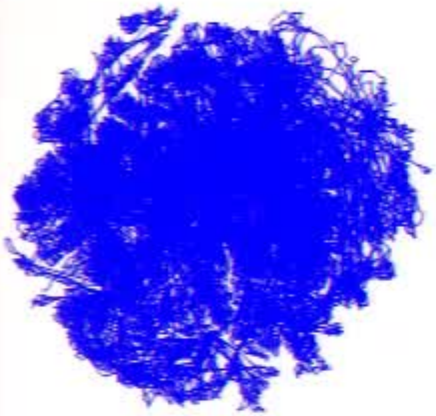
Proteins are Spongy



Spatial Efficiency in the Real-World

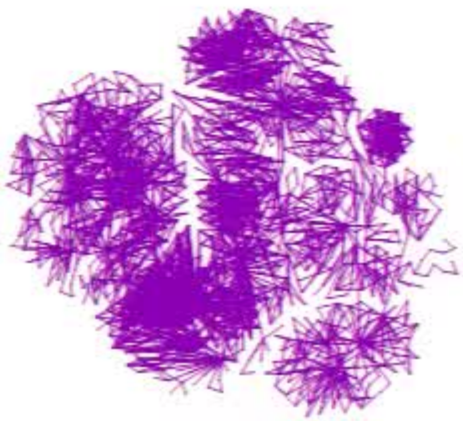
1) Spatially embedded networks

Barcelona



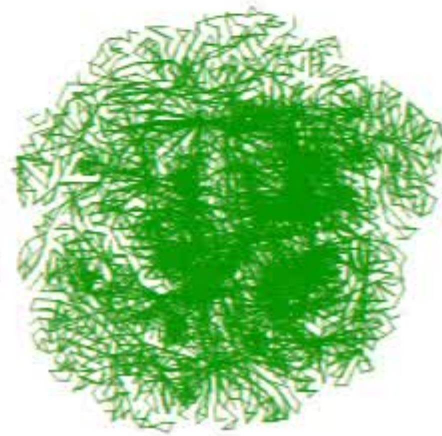
$\langle\theta\rangle = 71.9^\circ$

Rio Grande



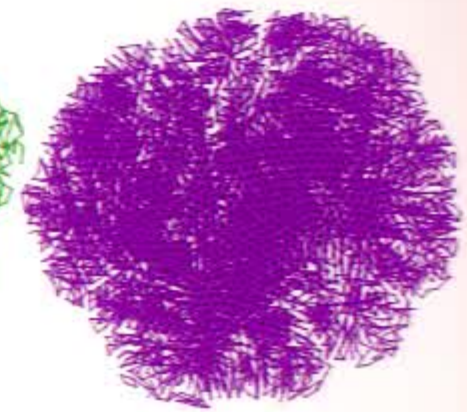
$\langle\theta\rangle = 79.7^\circ$

Atlanta



$\langle\theta\rangle = 86.5^\circ$

Berlin



$\langle\theta\rangle = 88.2^\circ$

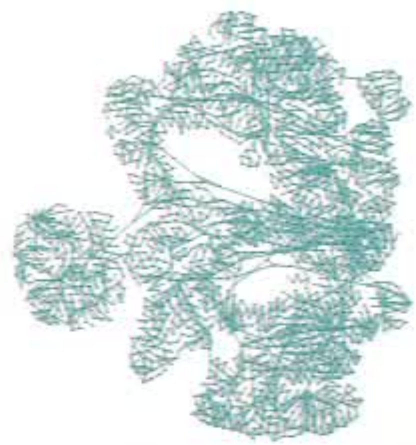
Hong Kong

$\langle\theta\rangle = 88.9^\circ$

Mecca

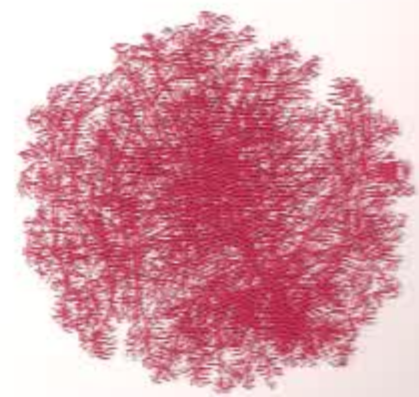
$\langle\theta\rangle = 89.5^\circ$

Oxford



$\langle\theta\rangle = 89.5^\circ$

Milton Keynes



$\langle\theta\rangle = 89.9^\circ$

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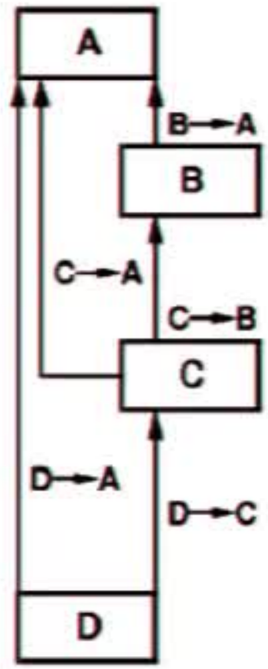
Spatial Efficiency in the Real-World

2) Non geographically embedded networks

Class Collaboration Graph

```
class A {
      // definition of class A
};
class B {
      A* ab;
      // rest of definition of class B
};
class C {
      A* ac;
      B* bc;
      // rest of definition of class C
};
class D: public C {
      A* ad;
      // rest of definition of class D
};
```



Myers: *Phys. Rev. E* **68** (2003) 046116.

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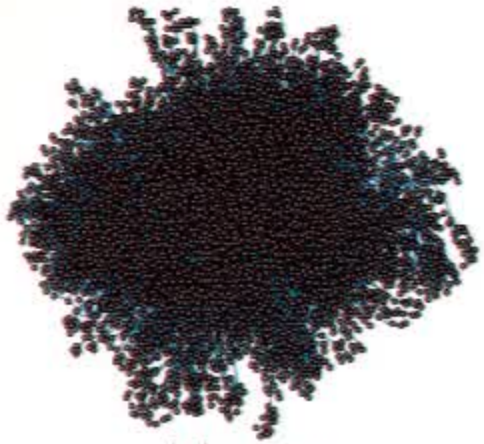
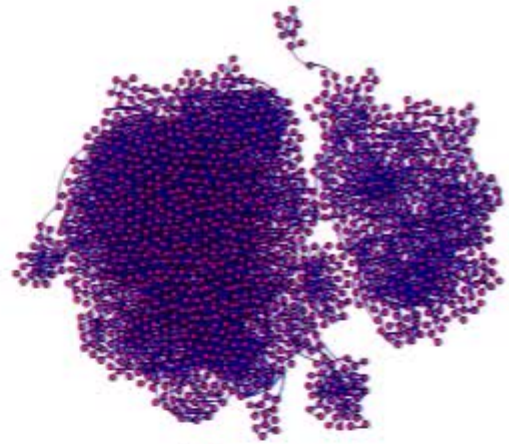
Spatial Efficiency in the Real-World

Linux



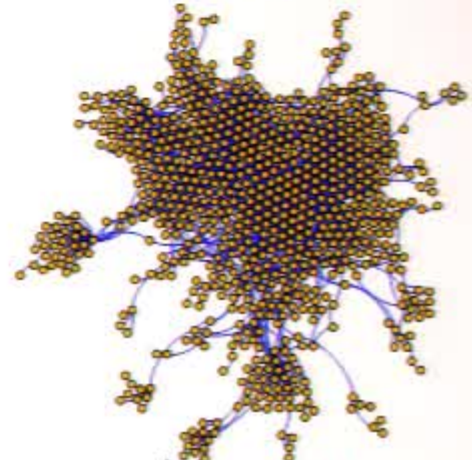
$\langle\theta\rangle = 3.5^\circ$

MySQL



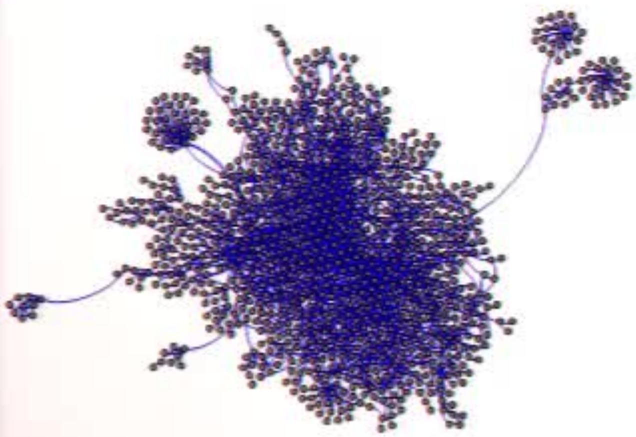
$\langle\theta\rangle = 45.7^\circ$

VTK



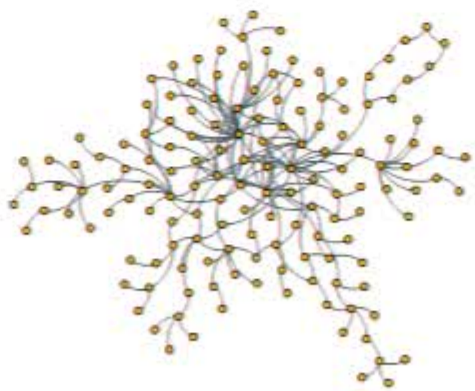
$\langle\theta\rangle = 70.1^\circ$

AbiWord



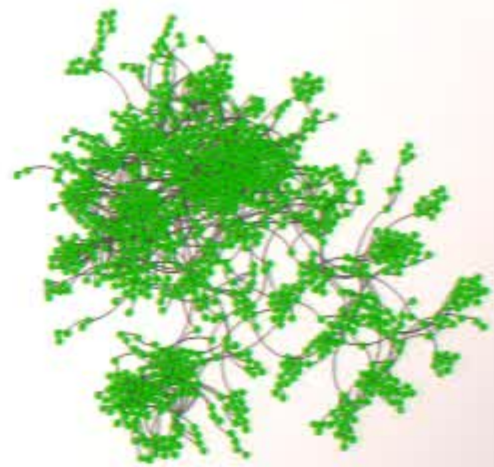
$\langle\theta\rangle = 72.9^\circ$

Digital Material



$\langle\theta\rangle = 81.6^\circ$

XMMS



$\langle\theta\rangle = 84.3^\circ$

Conclusions

- Spatial efficiency is a fundamental (multifactorial) property of complex networks.
- Network spatial efficiency refers to the average quality of communication among the nodes.
- The communication goodness is quantified as the ratio of the amount of information successfully delivered to its destination to the one which is frustrated in its delivery and returned to their originators.
- We have provided analytical and empirical pieces of evidence which reaffirm the idea that the spatial efficiency accounts for fundamental structural characteristics of complex networks.