ALGORITHMS FOR MINIMIZING DIFFERENCES OF CONVEX FUNCTIONS AND APPLICATIONS

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Outline

1. Minimizing Differences of Convex Functions-The DCA
2. Nesterov's Smoothing Technique via Convex Analysis
3. The DCA and Nesterov's Smoothing Technique for Weighted Fermat-Torricelli Problems
4. The DCA and Nesterov's Smoothing Technique for Multifacility Location

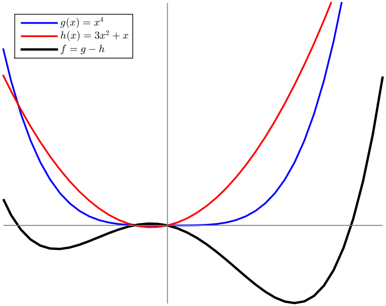
Differences of Convex Functions

Consider the problem

$$\text{minimize} \qquad f(x) := g(x) - h(x)\ , \qquad x \in \mathbb{R}^n$$

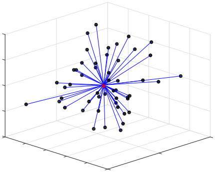
where $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^n \to \mathbb{R}$ are convex.

We call $g - h$ a **DC decomposition** of f.



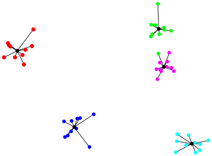
Examples of DC Programming

Fermat-Torricelli



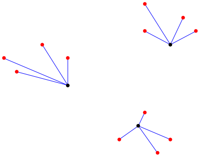
$f(x) = \sum_{i=1}^{m} c_i \|x - a_i\|$

Clustering



$f(x_1, \ldots, x_\ell) = \sum_{i=1}^{m} \min_{\ell=1}^{k} \|x_\ell - a_i\|^2$

Multifacility Location



$f(x_1, \ldots, x_\ell) = \sum_{i=1}^{m} \min_{\ell=1}^{k} \|x_\ell - a_i\|$

Subgradients and Fenchel Conjugates of Convex Functions

Definition

Let $f\colon \mathbb{R}^n \to (-\infty, \infty]$ be a convex function and let $\bar{x} \in \mathrm{dom}(f) := \{x \in \mathbb{R}^n \mid f(x) < \infty\}$. A **subgradient** of f at $\bar{x}$ is any $v \in \mathbb{R}^n$ such that

$$\langle v, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) \text{ for all } x \in \mathbb{R}^n.$$

The **subdifferential** $\partial f(\bar{x})$ of f at $\bar{x}$ is the set of all subgradients of f at $\bar{x}$.

Definition

Let $f\colon \mathbb{R}^n \to (-\infty, \infty]$ be a function. The **Fenchel conjugate** of f is defined by

$$f^*(x) = \sup_{u \in \mathbb{R}^n} \{\langle x, u\rangle - g(u)\}, \; x \in \mathbb{R}^n.$$

DC Algorithm-The DCA

Consider the problem

$$\text{minimize} \qquad f(x) := g(x) - h(x) \,, \qquad x \in \mathbb{R}^n$$

where $g\colon \mathbb{R}^n \to \mathbb{R}$ and $h\colon \mathbb{R}^n \to \mathbb{R}$ are convex.

The DCA[1]**.**

INPUT: $x_1 \in \mathbb{R}^n$, $N \in \mathbb{N}$
for $k = 1, \ldots, N$ **do**
  Find $y_k \in \partial h(x_k)$
  Find $x_{k+1} \in \partial g^*(y_k)$
end for
OUTPUT: x_{N+1}

[1] P.D. Tao, L.T.H. An, A d.c. optimization algorithm for solving the trust-region subproblem, SIAM J. Optim. 8 (1998), 476–505.

DC Algorithm-The DCA

Theorem

Let $g, h\colon \mathbb{R}^n \to (-\infty, \infty]$ be proper lower semicontinuous convex functions. Then $v \in \partial g^*(y)$ if and only if

$$v \in \operatorname{argmin}\left\{g(x) - \langle y, x\rangle \mid x \in \mathbb{R}^n\right\}.$$

Moreover, $w \in \partial h(x)$ if and only if

$$w \in \operatorname{argmin}\left\{h^*(y) - \langle y, x\rangle \mid y \in \mathbb{R}^n\right\}.$$

DC Algorithm-The DCA

The DCA.

INPUT: $x_1 \in \mathbb{R}^n$, $N \in \mathbb{N}$

for $k = 1, \ldots, N$ **do**

Find $y_k \in \partial h(x_k)$ or find y_k approximately by solving:

$$\text{minimize } \psi_k(y) := h^*(y) - \langle x_k, y \rangle, \; y \in \mathbb{R}^n.$$

Find $x_{k+1} \in \partial g^*(y_k)$ or find x_{k+1} approximately by solving:

$$\text{minimize } \phi_k(x) := g(x) - \langle x, y_k \rangle, \; x \in \mathbb{R}^n.$$

end for

OUTPUT: x_{N+1}

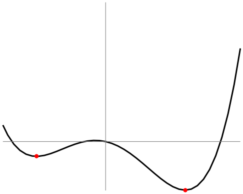
An Example of the DCA

$$\text{minimize } f(x) = x^4 - 3x^2 - x, x \in \mathbb{R}$$

Then $f(x) = g(x) - h(x)$, where $g(x) = x^4$ and $h(x) = 3x^2 + x$. We have

$$\partial h(x) = \{6x + 1\}, \partial g^*(y) = \left\{ \underset{x \in \mathbb{R}^n}{\text{argmin}} \left(x^4 - yx\right) \right\} = \left\{ \sqrt[3]{\frac{y}{4}} \right\}$$

$$x_{k+1} = \sqrt[3]{\frac{6x_k + 1}{4}}$$



DC Algorithm-The DCA

Definition

A function $h\colon \mathbb{R}^n \to (-\infty, \infty]$ is called γ-convex ($\gamma \geq 0$) if there exists $\gamma \geq 0$ such that the function defined by $k(x) := h(x) - \frac{\gamma}{2}\|x\|^2$, $x \in \mathbb{R}^n$, is convex. If there exists $\gamma > 0$ such that h is $\gamma-$convex, then h is called strongly convex with parameter γ.

Theorem

Consider the sequence $\{x_k\}$ generated by the DCA. Suppose that g is γ_1-convex and h is γ_2-convex. Then

$$f(x_k) - f(x_{k+1}) \geq \frac{\gamma_1 + \gamma_2}{2}\|x_{k+1} - x_k\|^2 \text{ for all } k \in \mathbb{N}.$$

DC Algorithm-The DCA

Definition

We say that an element $\bar{x} \in \mathbb{R}^n$ is a stationary point of the function $f = g - h$ if $\partial g(\bar{x}) \cap \partial h(\bar{x}) \neq \emptyset$. In the case where g and h are differentiable, $\bar{x}$ is a stationary point of f if and only if $\nabla f(\bar{x}) = \nabla g(\bar{x}) - \nabla h(\bar{x}) = 0$.

Theorem

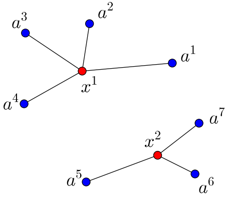
Consider sequence $\{x_k\}$ generated by the DCA. Then $\{f(x_k)\}$ is a decreasing sequence. Suppose further that f is bounded from below and that g is γ_1-convex and h is γ_2-convex with $\gamma_1 + \gamma_2 > 0$. If $\{x_k\}$ is bounded, then every subsequential limit of the sequence $\{x_k\}$ is a stationary point of f.

The DCA for Clustering

- Problem formulation[2]: Let a_i for $i = 1, ..., m$ be target points in $\mathbb{R}^n$.

$$\text{Minimize} \quad f(x_1, \ldots, x_\ell) := \sum_{i=1}^{m} \min\{\|x_l - a_i\|^2 \ : \ l = 1, ..., k\}$$

$$\text{over } x_l \in \mathbb{R}^n, \ l = 1, ..., k.$$



[2]L.T.H. An, M.T. Belghiti, P.D. Tao, A new efficient algorithm based on DC programming and DCA for clustering, J. Glob. Optim., 27 (2007), 503–608.

K-Mean Clustering

Let $x_1, x_2, \ldots, x_m$ be the data points and let $c_1, \ldots, c_k$ denote the centers.

- Randomly select k cluster centers.
- Assign each data point to the nearest center.
- Find the average of the data points assigned to each center.
- Repeat the second step with the obtained new centers in the third step until the centroids no longer move.

Although $k-$mean clustering is effective in many situations, it also has some disadvantages.

- The k-means algorithm does not necessarily find the optimal solution
- The algorithm is sensitive to the initial selected cluster centers

DCA for Clustering and K-Mean[3]

- Both DCA1 and DCA2 are better than K-means: the objective values given by DCA1 and DCA2 are much smaller than that computed by K-means.
- DCA2 is the best among the three algorithms: it provides the best solution with the shortest time. DCA2 is very fast and can then handle large-scale problems.

$$f(x_1,\ldots,x_k)=\sum_{i=1}^{m}\min_{\ell=1,\ldots,k}\|x_\ell-a_i\|_1$$

This is a nonsmooth nonconvex program for which there are rarely efficient solution algorithms, especially in the large scale setting.

[3] L.T.H. An, L.H. Minh, P.D. Tao, New and efficient DCA based algorithms for minimum sum-of-squares clustering, Pattern Recognition, 47 (2014), 388–401.

Nesterov's Smoothing Technique via Convex Analysis

Consider the function

$$f_0(x) := \max\{\langle Ax, u\rangle - \phi(u) \mid u \in Q\},$$

where A is an $m \times n-$matrix, Q is a nonempty closed bounded convex subset of $\mathbb{R}^m$, and $\phi\colon \mathbb{R}^m \to \mathbb{R}$ is a convex function.
Define $\|A\| = \sup\{\|Ax\| \mid \|x\| \le 1\}$.
For $\mu > 0$, define

$$f_\mu(x) := \max\{\langle Ax, u\rangle - \phi(u) - \frac{\mu}{2}\|u - u_0\|^2 \mid u \in Q\}, u_0 \in Q.$$

Then f_μ is a C^1 function with $\ell-$Lipschitz gradient where $\ell = \frac{\|A\|^2}{\mu}$ and $\nabla f_\mu(x) = A^T u_\mu(x)$. Here $u_\mu(x) \in Q$ is the element for which the maximum is attained in the definition of $f_\mu(x)$. [4]

[4] Nesterov: Smooth minimization of non-smooth functions. Math.Program., Ser. A 103 , 127-152 (2005).

Nesterov's Smoothing Technique via Convex Analysis

$$f^*(x) = \sup\{\langle x, u\rangle - f(u) \mid u \in \mathbb{R}^n\}.$$

Theorem

If f is $\mu-$strongly convex, then f^* has a Lipschitz continuous gradient with modulus $\frac{1}{\mu}$. Moreover, $\nabla f^*(x) = u(x)$, where $u(x)$ is the unique element of $\mathbb{R}^n$ for which the maximum is attained in the definition of $f^*(x)$.[a]

[a] J. Hiriart-Urruty and C. Lemaréchal. Convex Analysis and Minimization Algorithms I & II. Springer, New York, 1993.

Nesterov's Smoothing Technique via Convex Analysis

Theorem

Let A be an $n \times m$-matrix. Suppose that $\varphi \colon \mathbb{R}^m \to \mathbb{R}$ is a strongly convex function with parameter $\mu > 0$. Then the function $\mu \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) := \max\{\langle Ax, u\rangle - \varphi(u) \mid u \in Q\}$$

is differentiable with $\nabla f(x) = A^T v(x)$, where $v(x)$ is the unique element for which the maximum is attained in the definition of $f(x)$. The gradient is Lipschitz continuous with constant $\ell = \frac{\|A\|^2}{\mu}$.

Nesterov's Smoothing Technique via Convex Analysis

We have
$f(x) = \max\{\langle A^T x, \rangle - [\varphi(u) + \delta(u; Q)] \mid u \in \mathbb{R}^m\} = g^*(A^T x)$,
where $g(u) := \varphi(u) + \delta(u; Q)$.

By the chain rule, $\nabla f(x) = A^T \nabla g^*(A^T x) = A^T u(Ax) = A^T v(x)$.
We also have

$$\begin{aligned}\|\nabla f(x_1) - \nabla f(x_2)\| &= \|A^T u(Ax_1) - A^T u(Ax_2)\| \\ &\le \|A^T\| \|u(Ax_1) - u(Ax_2)\| \\ &\le \|A\| \frac{1}{\mu} \|Ax_1 - Ax_2\| \le \frac{\|A\|^2}{\mu} \|x_1 - x_2\|.\end{aligned}$$

The Minkowski Gauge

Let F be a nonempty closed bounded convex set in $\mathbb{R}^n$ that contains the origin in its interior. Define the *Minkowski gauge* associated with F by

$$\rho_F(x) := \inf\{t > 0 \mid x \in tF\}.$$

Note that if F is the closed unit ball in $\mathbb{R}^n$, then $\rho_F(x) = \|x\|$.

Given a nonempty bounded set K, the *support function* associated with K is given by

$$\sigma_K(x) := \sup\{\langle x, y\rangle \mid y \in K\}.$$

It follows from the definition of the Minkowski function that $\rho_F(x) = \sigma_{F^\circ}(x)$, where

$$F^\circ := \{y \in \mathbb{R}^n \mid \langle x, y\rangle \leq 1 \text{ for all } x \in F\}.$$

Weighted Fermat-Torricelli problem

Let $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, m$ and let $c_i \neq 0$ for $i = 1, \ldots, m$ be real numbers. In the remainder of this section, we study the following generalized version of the Fermat-Torricelli problem:

$$\text{minimize } f(x) := \sum_{i=1}^{m} c_i \rho_F(x - a_i), \ x \in \mathbb{R}^n.$$

The function f has the following obvious DC decomposition:

$$f(x) = \sum_{c_i > 0} c_i \rho_F(x - a_i) - \sum_{c_i < 0} (-c_i) \rho_F(x - a_i).$$

Let $I := \{i \mid c_i > 0\}$ and $J := \{i \mid c_i < 0\}$ with $\alpha_i = c_i$ if $i \in I$, and $\beta_i = -c_i$ if $i \in J$. Then

$$f(x) = \sum_{i \in I} \alpha_i \rho_F(x - a_i) - \sum_{j \in J} \beta_j \rho_F(x - a_j).$$

Weighted Fermat-Torricelli problem

Let $a_i \in \mathbb{R}^n$ for $i = 1, \ldots, m$ and let $c_i \neq 0$ for $i = 1, \ldots, m$ be real numbers. In the remainder of this section, we study the following generalized version of the Fermat-Torricelli problem:

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Let $I := \{i \mid c_i > 0\}$ and $J := \{i \mid c_i < 0\}$ with $\alpha_i = c_i$ if $i \in I$, and $\beta_i = -c_i$ if $i \in J$. Then

$$f(x) = \sum_{i \in I} \alpha_i \rho_F(x - a_i) - \sum_{j \in J} \beta_j \rho_F(x - a_j).$$

Weighted Fermat-Torricelli problem

Theorem

Let $\gamma_1 := \sup\{r > 0 \mid B(0; r) \subset F\}$ *and* $\gamma_2 := \inf\{r > 0 \mid F \subset B(0; r)\}$. *Suppose that*

$$\gamma_1 \sum_{i \in I} \alpha_i > \gamma_2 \sum_{j \in J} \beta_j.$$

Then the function f *and its approximation* f_μ *have absolute minima.*

Smoothing the Minkowski Gauge

Given any $a \in \mathbb{R}^n$ and $\mu > 0$, a Nesterov smoothing approximation of $\varphi(x) := \rho_F(x-a)$ has the representation

$$\varphi_\mu(x) = \frac{1}{2\mu}\|x-a\|^2 - \frac{\mu}{2}\big[d(\frac{x-a}{\mu};F^\circ)\big]^2.$$

Moreover, $\nabla\varphi_\mu(x) = P(\frac{x-a}{\mu};F^\circ)$ and

$$\varphi_\mu(x) \le \varphi(x) \le \varphi_\mu(x) + \frac{\mu}{2}\|F^\circ\|^2,$$

where $\|F^\circ\| := \sup\{\|u\| \mid u \in F\}$.

Smoothing the Minkowski Gauge

Theorem

Given any $\mu > 0$, an approximation of the function f is the following DC function:

$$f_\mu(x) := g_\mu(x) - h_\mu(x),\ x \in \mathbb{R}^n,$$

where

$$g_\mu(x) := \sum_{i \in I} \frac{\alpha_i}{2\mu}\|x - a_i\|^2,$$

$$h_\mu(x) := \sum_{i \in I} \frac{\mu\alpha_i}{2}\Big[d(\frac{x - a_i}{\mu}; F^\circ)\Big]^2 + \sum_{j \in J} \beta_j \rho_F(x - a^j).$$

Moreover, $f_\mu(x) \leq f(x) \leq f_\mu(x) + \frac{\mu\|F^\circ\|^2}{2}\sum_{i \in I}\alpha_i$ for all $x \in \mathbb{R}^n$.

The DCA for Weighted Fermat-Torricelli Problems

Theorem

Given any $\mu > 0$, an approximation of the function f is the following DC function:

$$f_\mu(x) := g_\mu(x) - h_\mu(x),\ x \in \mathbb{R}^n,$$

where

$$g_\mu(x) := \sum_{i \in I} \frac{\alpha_i}{2\mu} \|x - a_i\|^2,$$

$$h_\mu(x) := \sum_{i \in I} \frac{\mu \alpha_i}{2} \Big[d\Big(\frac{x - a_i}{\mu}; F^\circ\Big)\Big]^2 + \sum_{j \in J} \beta_j \rho_F(x - a^j).$$

Moreover, $f_\mu(x) \le f(x) \le f_\mu(x) + \frac{\mu \|F^\circ\|^2}{2} \sum_{i \in I} \alpha_i$ *for all* $x \in \mathbb{R}^n$*.*

The DCA for Weighted Fermat-Torricelli Problems

Algorithm 3.

INPUTS: $\mu > 0$, $x_1 \in \mathbb{R}^n$, $N \in N$, F, $a^1, \ldots, a^m \in \mathbb{R}^n$, $c_1, \ldots, c_m \in \mathbb{R}$.

for $k = 1, \ldots, N$ **do**

Find $y_k = u_k + v_k$, where

$$u_k := \sum_{i \in I} \alpha_i \left[\frac{x_k - a_i}{\mu} - P\left(\frac{x_k - a_i}{\mu}; F^\circ\right)\right],$$

$$v_k \in \sum_{j \in J} \beta_j \partial \rho_F(x_k - a^j).$$

Find $x_{k+1} = \frac{y_k + \sum_{i \in I} \alpha_i a_i / \mu}{\sum_{i \in I} \alpha_i / \mu}$.

OUTPUT: x_{N+1}.

Multifacility Location

We now consider the multifacility location problem: given m points $a_1, \ldots, a_m \in \mathbb{R}^n$,

$$\text{minimize} \qquad f(x_1, \ldots, x_k) = \sum_{i=1}^{m} \min_{\ell=1,\ldots,k} \rho_F(x_\ell - a_i).$$

where F is a nonempty, closed and bound convex set containing the origin, and $\rho_F(x) = \inf_{x \in tF} \{t > 0\}$ is the Minkowski gauge.

When F is the closed unit ball B, the problem becomes

$$\text{minimize} \qquad f(x_1, \ldots, x_k) = \sum_{i=1}^{m} \min_{\ell=1,\ldots,k} \|x_\ell - a_i\|.$$

It can be shown that a globally optimal solution exists.

DC Decomposition

$$f(x_1, \ldots, x_k) = \sum_{i=1}^{m} \min_{\ell=1,\ldots,k} \|x_\ell - a_i\|$$

We will utilize the fact that

$$f(x_1, \ldots, x_k) = \sum_{i=1}^{m} \left[\sum_{\ell=1}^{k} \|x_\ell - a_i\| - \max_{r=1,\ldots,k} \sum_{\substack{\ell=1 \\ \ell \neq r}}^{k} \|x_\ell - a_i\| \right]$$

$$= \sum_{i=1}^{m} \left(\sum_{\ell=1}^{k} \|x_\ell - a_i\| \right) - \sum_{i=1}^{m} \left(\max_{r=1,\ldots,k} \sum_{\substack{\ell=1 \\ \ell \neq r}}^{k} \|x_\ell - a_i\| \right).$$

DC Decomposition

We obtain the μ-smoothing approximation $f_\mu = g_\mu - h_\mu$, where

$$g_\mu(x_1, \ldots, x_k) = \frac{1}{2\mu} \sum_{i=1}^{m} \sum_{\ell=1}^{k} \|x_\ell - a_i\|^2$$

$$h_\mu(x_1, \ldots, x_k) = \frac{\mu}{2} \sum_{i=1}^{m} \sum_{\ell=1}^{k} \left[d\left(\frac{x_\ell - a_i}{\mu}; B \right) \right]^2 + \sum_{i=1}^{m} \max_{r=1,\ldots,k} \sum_{\substack{\ell=1 \\ \ell \neq r}}^{k} \left(\frac{1}{2\mu} \|x_\ell - a_i\|^2 - \frac{\mu}{2} \left[d\left(\frac{x_\ell - a_i}{\mu}; B \right) \right]^2 \right)$$

To implement DCA, we need ∂g_μ^* and $\partial h_\mu \ldots$

∂g_μ

Using the Frobenius norm in a space of matrices, we express g_μ as

$$G_\mu(X) = \frac{m}{2\mu}\|X\|^2 - \frac{1}{\mu}\langle X, B\rangle + \frac{k}{2\mu}\|A\|^2,$$

with the inner product $\langle A, B\rangle = \sum_{\ell}^{k}\sum_{j}^{n} a_{\ell j}b_{\ell j}$,
X is the $k \times n$ matrix with rows $x_1, \ldots, x_k$,
A is $m \times n$ with rows $a_1, \ldots, a_m$, and
B is $k \times n$ whose every row is the sum $a_1 + \cdots + a_m$.

$\nabla G_\mu(X) = \frac{m}{\mu}X - \frac{1}{\mu}B$
$\nabla G^*_\mu(Y) = \frac{1}{m}(B + \mu Y)$

$\boxed{X \in \partial G^*(Y) \text{ iff } Y \in \partial G(X)}$

∂h_μ

$$h_\mu(x_1,\ldots,x_k) = \frac{\mu}{2}\sum_{i=1}^{m}\sum_{\ell=1}^{k}\left[d\left(\frac{x_\ell - a_i}{\mu}; B\right)\right]^2$$

$$+\sum_{i=1}^{m}\max_{r=1,\ldots,k}\sum_{\substack{\ell=1\\ \ell\neq r}}^{k}\left(\frac{1}{2\mu}\|x_\ell - a_i\|^2 - \frac{\mu}{2}\left[d\left(\frac{x_\ell - a_i}{\mu}; B\right)\right]^2\right)$$

∂h_μ

$$\nabla H_1(X) = \begin{bmatrix} \sum_i \frac{x_1 - a_i}{\mu} - P_B\left(\frac{x_1 - a_i}{\mu}\right) \\ \vdots \\ \sum_i \frac{x_k - a_i}{\mu} - P_B\left(\frac{x_k - a_i}{\mu}\right) \end{bmatrix}$$

For each $i = 1, \ldots, m$, there is some R_i such that the R-excluded sum is maximal. If we call this sum F_{R_i}, then ∇F_{R_i} is a $k \times n$ matrix whose $\ell^{\text{th}} \neq R$ row is $P_B\left(\frac{x_\ell - a_i}{\mu}\right)$, and R^{th} row is $\mathbf{0}$.

$$V \in \partial H_2(X) \text{ iff } V = \sum_{i=1}^{m} F_{R_i}$$

Multifacility Location Algorithm

INPUT: $X_1 \in \operatorname{dom} g$, $N \in \mathbb{N}$

for $k = 1, \ldots, N$ **do**

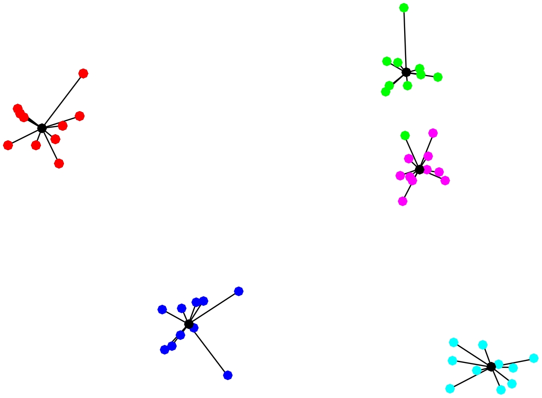
Compute $Y_k = \nabla H_1(X_k) + V_k$

Compute $X_{k+1} = \frac{1}{m}(B + \mu Y_k)$

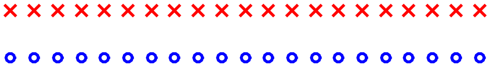
end for

OUTPUT: x_{N+1}

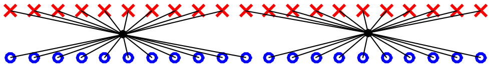
Clustering



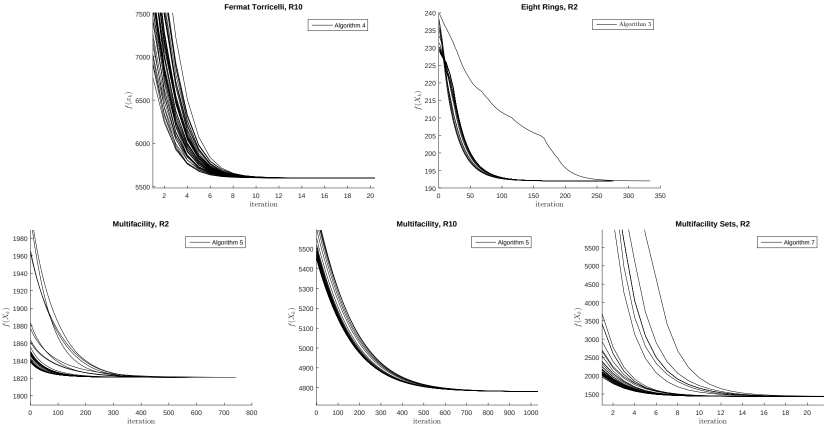
Clustering



Clustering



Results


Fermat Torricelli, R10
Algorithm 4
$f(x_k)$
iteration
Eight Rings, R2
Algorithm 5
$f(X_k)$
iteration
Multifacility, R2
Algorithm 5
$f(X_k)$
iteration
Multifacility, R10
Algorithm 5
$f(X_k)$
iteration
Multifacility Sets, R2
Algorithm 7
$f(X_k)$
iteration

References

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