Coagulation dynamics in branching processes

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Modest goals: In the ‘garden of branching processes,’ do some weeding.

- Explain the source of coagulation equations in work of Bertoin-Le Gall
- Improve continuum limit analysis: Galton-Watson $\to$ continuous state (CSBP)
- Simplify continuum limit criteria, via *Bernstein function* theory
- New results on universality (type 2 and type 3)
- Develop analogy: GW $\to$ CSBP $\quad\sim\quad X_1 + \ldots X_n \to Y$ *infinitely divisible*
- Dynamic renormalization $\sim$ *dilation* in Lévy-Khintchine representation

Smoluchowski's coagulation equation (weak form, $K=2$)

The size distribution: $\displaystyle\int_{(0,x]} \nu_t(dz) = \#$ of clusters of size $\le x$

of a system of clustering particles evolves according to $\quad (z_1, z_2 \mapsto z_1 + z_2 = x)$

$$\boxed{\begin{gathered}\partial_t \int_0^\infty a(x)\,\nu_t(dx) = \int_0^\infty \int_0^\infty \tilde{a}(z_1,z_2)\,\nu_t(dz_1)\,\nu_t(dz_2)\,, \\ \tilde{a}(z_1,z_2) = a(z_1+z_2) - a(z_1) - a(z_2).\end{gathered}}$$

- Choosing $\quad a(x) = 1 - e^{-qx} \quad$ yields $\quad \tilde{a}(z_1,z_2) = -(1-e^{-qz_1})(1-e^{-sz_2})$.

Then $\quad \displaystyle\varphi(t,q) := \int_0^\infty (1-e^{-qx})\,\nu_t(dx) \quad \Longrightarrow \quad \boxed{\partial_t \varphi = -\varphi^2}$

Bernstein's theorem and topology of Laplace transforms

Definition $g : (0,\infty) \to (0,\infty)$ is *completely monotone* (CM) if g is C^∞ and

$$(-1)^k g^{(k)}(q) \geq 0 \qquad \forall q > 0.$$

- **Theorem** (Bernstein) g is completely monotone if and only if

$$g(q) = \int_{[0,\infty)} e^{-qx}\, G(dx) \qquad =: \mathcal{L}G(q)$$

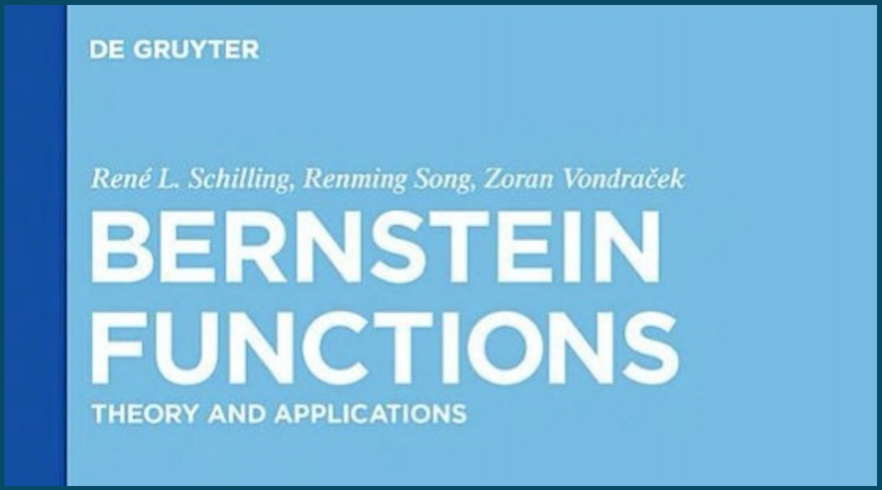
for some measure G on $[0,\infty)$. Notation: $G(x) = \int_{[0,x]} G(dx)$

- **Continuity theorem** for Laplace transforms: As $n \to \infty$,

i) $G_n(x) \to G(x)$ $a.e.$ $\implies$ $\mathcal{L}G_n(q) \to \mathcal{L}G(q)$ $\forall q > 0.$

ii) $\mathcal{L}G_n(q) \to g(q)$ $\forall q > 0$ $\implies$ $g = \mathcal{L}G$ with $G_n(x) \to G(x)$ $a.e.$

Bernstein transforms and the topology of Lévy triples



Definition $\varphi : (0,\infty) \to (0,\infty)$ is Bernstein if φ is C^∞ and φ' is CM.

- **Theorem** φ is Bernstein $\Leftrightarrow$ $\boxed{\varphi(q) = a_0 q + a_\infty + \int_E (1 - e^{-qz})\mu(dz)}$

for some Lévy triple (a_0, a_∞, μ): $a_0, a_\infty \geq 0$ and $\int_E (z \wedge 1)\mu(dz) < \infty$.

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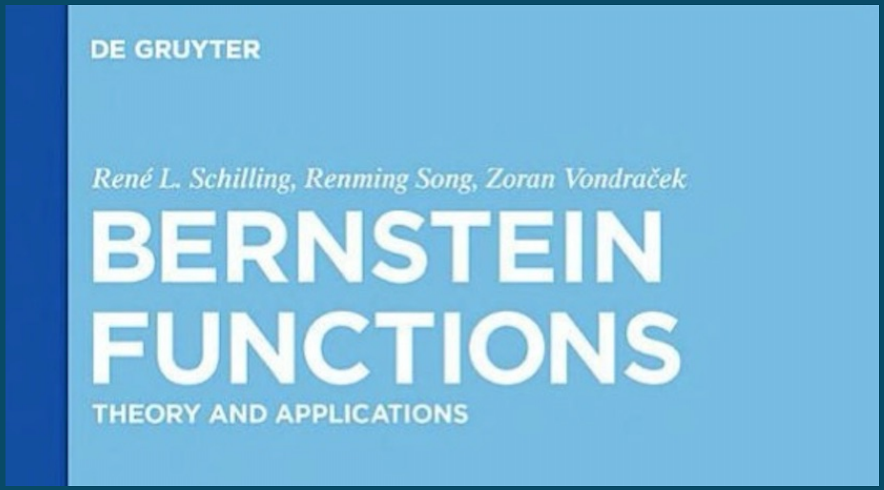

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- We associate the measure $\kappa(dz) = a_0\delta_0 + (z \wedge 1)\mu(dz) + a_\infty\delta_\infty$ on $[0,\infty]$

Continuity theorem for Lévy triples (cf. Menon-P 2008)

Let $(a_0^{(n)}, a_\infty^{(n)}, \mu^{(n)})$ be a sequence of Lévy triples,

$$\varphi^{(n)}(q) = a_0^{(n)} q + a_\infty^{(n)} + \int_E (1 - e^{-qz}) \mu^{(n)}(dz)$$

$$\kappa^{(n)}(dz) = a_0^{(n)} \delta_0 + (z \wedge 1) \mu^{(n)}(dz) + a_\infty^{(n)} \delta_\infty$$

Then TFAE:

(i) $\varphi(q) := \lim_{n\to\infty} \varphi^{(n)}(q)$ exists for each $q > 0$.

(iii) $\kappa^{(n)}$ converges weakly to some measure κ on $[0, \infty]$, meaning

$$\langle f, \kappa^{(n)} \rangle \to \langle f, \kappa \rangle \quad \text{for all } f \in C([0, \infty]).$$

If these conditions hold, the limit quantity φ, κ is that associated as above with a unique Lévy triple (a_0, a_∞, μ).

Generalized Smoluchowski dynamics in branching processes

Bertoin-Le Gall 2006: Critical continuous-state branching processes (CSBPs) that are extinct a.s. are associated with the generalized Smoluchowski equation

$$\partial_t \langle a, \nu_t \rangle = \sum_{k=2}^{\infty} R_k(t)\, I_k(a, \nu_t) \tag{GS}$$

$I_k(a, \nu_t)$ is the expected change in the 'moment' $\langle a, \nu_t \rangle = \int_0^\infty a(z)\, \nu_t(dz)$

$$I_k(a, \nu_t) = \int_{(0,\infty)^k} \left(a(z_1 + \cdots + z_k) - \sum_{j=1}^{k} a(z_j) \right) \prod_{j=1}^{k} \frac{\nu_t(dz_j)}{\langle 1, \nu_t \rangle}.$$

$R_k(t)$ is the rate at which k clusters simultaneously coalesce: with $\rho = \langle 1, \nu_t \rangle$,

$$R_k = \int_0^\infty \frac{e^{-\rho z} (\rho z)^k}{k!} \pi(dz) \quad \text{where} \quad \int_0^\infty (z^2 \wedge z)\, \pi(dz) < \infty. \quad \text{Why??}$$

Bernstein transform of (GS)

The Bernstein transform $\varphi(t,q) := \int_0^\infty (1 - e^{-qs})\, \nu_t(ds)$ satisfies

$$\boxed{\partial_t \varphi = -\Psi(\varphi)} \quad , \quad \Psi(u) = \hat{\beta}\,\frac{u^2}{2} + \int_{(0,\infty)} \left(e^{-uz} - 1 + uz\right) \pi(dz).$$

This equation is well-known in the CSBP literature: A CSBP $X(t,x)$ is a Lévy process in x, with Laplace exponent φ, and Lévy jump measure $\nu_t(ds)$:

$$\mathbb{E}(e^{-qX(t,x)}) = e^{-x\varphi(t,q)}$$

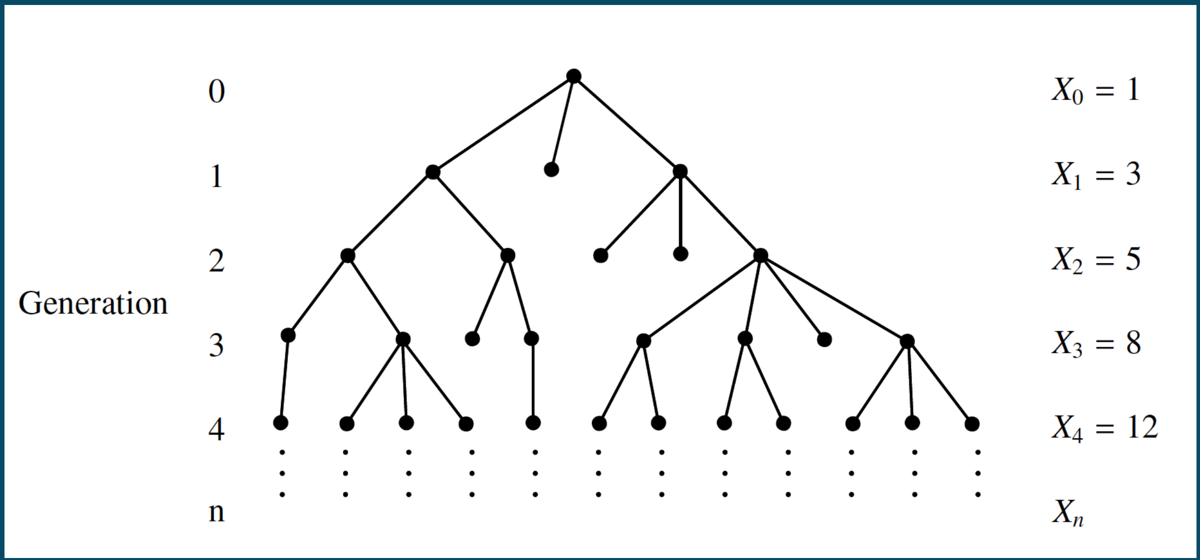
The rates $R_k = \int_0^\infty \frac{e^{-\rho z}(\rho z)^k}{k!} \pi(dz) \quad = \frac{(-\rho)^k \Psi^{(k)}(\rho)}{k!}$

Qs: Where do ν_t, π come from? Can one do scaling limit analysis?

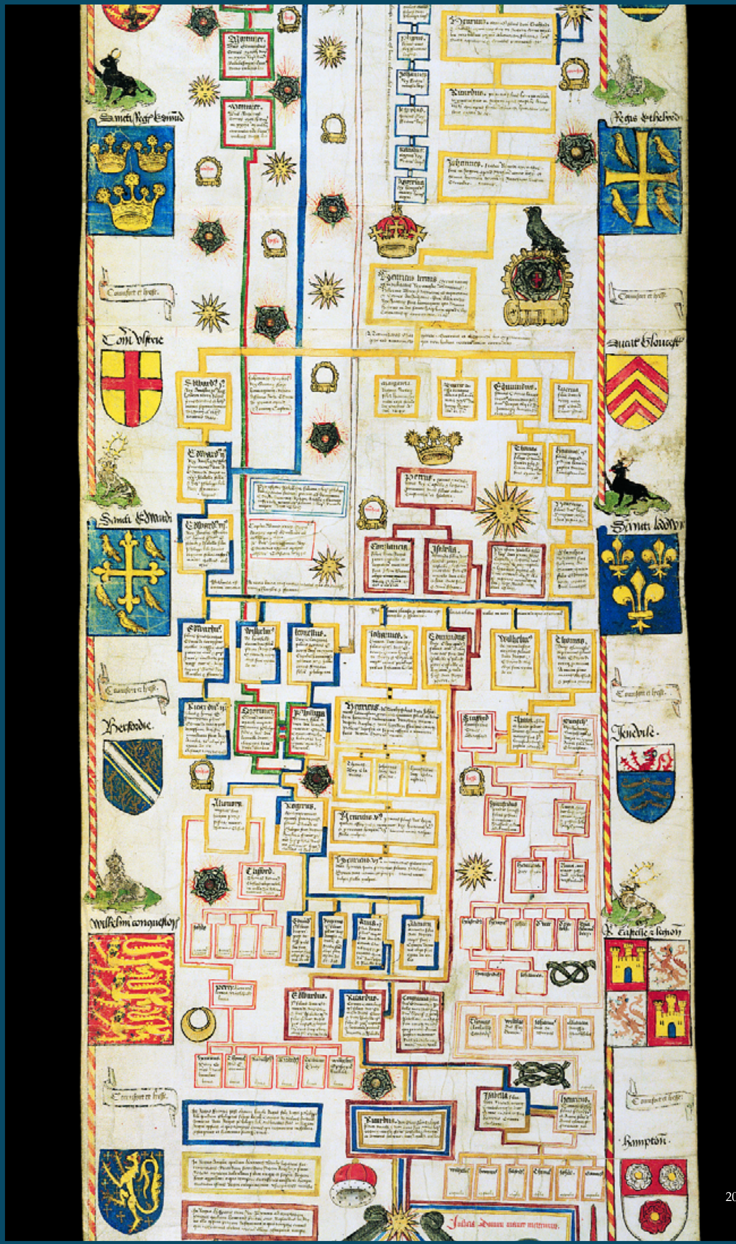
A **Galton-Watson branching process**

is a Markov chain $n \mapsto X_n \quad \in \mathbb{N} \cup \{0\}$

$$X_n = \text{total population at generation } n$$
$$= \# \text{ of nodes at level } n \text{ in a random tree}$$



Each *parent* has k *children* iid with law $\hat{\pi}(k)$



Galton-Watson dynamics

$P_n(j,k) := \Pr\{X_n = k \mid X_0 = j\}$ is the n-step transition probability of $j \to k$

$\boxed{\nu_n(k) := P_n(1,k)}$ is the *clan size distribution* after n generations.

$\boxed{\hat{\pi}(k) := P_1(1,k)}$ is the *family size distribution*.

- The branching property implies $X_{n+1} = \sum_{i=1}^{X_n} \xi_{n,i}$ with iid $\xi_{n,i} \sim \hat{\pi}$, thus

$$\nu_{n+1}(k) = \sum_{j\geq 1} \nu_n(j)\hat{\pi}^{*j}(k) \quad = \sum_{j\geq 1} P_n(1,j)P_1(j,k).$$

Markov $\Longrightarrow$ $\boxed{\nu_{n+1}(x) = \sum_{j\geq 1} \hat{\pi}(j)\, \nu_n^{*j}(x)} \quad = \sum_{j\geq 1} P_1(1,j)P_n(j,x)$

Generating function and Bernstein transform

$$G_n(z) = \sum_{j\geq 1} \nu_n(j) z^j \quad \Longrightarrow \quad G_{n+1} = G_1 \circ G_n$$

The Bernstein transform $\quad \hat{\varphi}_n(q) := \sum_{j\geq 1} \nu_n(j)(1-e^{-qj}) \quad = 1 - G_n(e^{-q})$

satisfies $\quad \boxed{\hat{\varphi}_{n+1}(q) - \hat{\varphi}_n(q) = -\hat{\Psi}(\hat{\varphi}_n(q))} \quad , \qquad \hat{\Psi}(s) = G(1-s) - 1 + s\,.$

If X is critical ($\sum_{j\geq 0} \hat{\pi}(j) = 1$) then

$$\hat{\Psi}(s) = \sum_{j\geq 2} \Big((1-s)^j - 1 + js \Big) \hat{\pi}(j).$$

Continuous-size, continuous-time limits (CSBP)

Let $h =$ grid size, $\tau =$ time step. Rescale size via $j \mapsto jh$ and let

$$\tilde{\nu}_n(dx) = \frac{1}{h}\sum_{j>0} \nu_n(j)\delta_{jh}(dx), \qquad \tilde{\pi}(dx) = \frac{1}{\tau h}\sum_{j>0} \hat{\pi}(j)\delta_{jh}(dx).$$

One finds $\tilde{\varphi}_n(q) := \langle 1 - e^{-qx}, \tilde{\nu}_n\rangle$ satisfies $\boxed{\dfrac{\tilde{\varphi}_{n+1}(q) - \tilde{\varphi}_n(q)}{\tau} = -\tilde{\Psi}(\tilde{\varphi}_n(q)),}$

$$\tilde{\Psi}(u) = \int_{(0,\infty)} \Big((1-hu)^{x/h} - 1 + xu\Big)\, \tilde{\pi}(dx).$$

- **Theorem**

a) Let $h_k, \tau_k \to 0$ and $\boxed{(x \wedge 1)(x - h_k)\tilde{\pi}_k(dx) \to \kappa(dz)}$ weak-* on $[0,\infty]$.

Then $\tilde{\Psi}(u) \to \quad \Psi(u) = a_0\dfrac{u^2}{2} + a_\infty + \displaystyle\int_{(0,\infty)} \left(e^{-uz} - 1 + uz\right) z^{-1}\mu(dz)\,.$

Continuous-time limits: coalescence with multiple clustering

b) Assume further $(x \wedge 1)\nu_0(dx) \to \delta_0$ weak-* on $[0,\infty]$. (E.g., $p_0(1)=1$)

Then $n\tau \to t \implies \tilde{\varphi}_n(q) \to \varphi(t,q)$ where

$$\boxed{\partial_t \varphi = -\Psi(\varphi)} \quad \forall t>0, \qquad \varphi(0,q)=q.$$

This entails $(x\wedge 1)\tilde{\nu}_n \to (x \wedge 1)\nu_t$, with $\varphi(t,q) = \langle 1 - e^{-qx}, \nu_t\rangle$.

Ala Bertoin-Le Gall, we infer ν_t solves (GS) provided $\rho = \langle 1, \nu_t\rangle < \infty$ for $t>0$. This is known to be equivalent to

$$\int_1^\infty \frac{du}{\Psi(u)} < \infty \qquad \text{(E)}$$

We call this ν_t the *fundamental solution* of (GS): $(x\wedge 1)\nu_t(dx) \to \delta_0$ as $t \to 0$.

Universality 1: *typical* limits

Suppose the family-size distribution $\hat{\pi}$ has a finite second moment:

$$m_2 = \sum_{j=1}^{\infty} j^2 \hat{\pi}(j) < \infty.$$

Then with $\tau = h\,(m_2 - 1)$ we have $(x \wedge 1)(x - h)\tilde{\pi}(dx) \to \kappa = \delta_0$ as $h \to 0$, whence

$$\Psi(u) = \frac{1}{2}u^2$$

and ν_t is the fundamental solution of Smoluchowski's equation with constant kernel $K = 1$.

Universality 2: *arbitrary* limits (a la Doeblin)

Theorem There exists *some* (far from unique) family-size distribution $\hat{\pi}$ and sequences h_n, $\tau_n \to 0$, such that: For *every* finite measure

$$\kappa(dz) = a_0\delta_0 + (z \wedge 1)z\pi(dz) + a_\infty\delta_\infty \quad \text{on } [0, \infty],$$

there is some subsequence h_{n_k}, τ_{n_k} along which the hypothesis of a) holds.

The conclusion implies that *every possible* critical CSBP is a limit of rescalings of one particular "universal" Galton-Watson process, along some subsequence.

The proof exploits a resemblance between *Bernoulli shifts* and the rescalings induced on κ-measures (Lévy triples) by

$$\kappa(dz) = (z \wedge 1)\, z\, \pi(dz) \qquad \mapsto \qquad \tilde{\kappa}(dz) = (z \wedge 1)\, \frac{1}{\tau}\, \frac{z}{h}\, \pi\left(\frac{dz}{h}\right)$$

The same technique of "packing the tail" of the starting distribution is described by Feller to constuct *Doeblin's universal laws* in probability.

Universality 3: long-time scaling limits for (GS): necessary and sufficient conditions

Theorem Assume (E). Let ν_t be the fundamental solution of (GS). Then TFAE:

(i) there exist a probability measure $\hat{\mu}$ and $\lambda(t) > 0$ such that the rescalings

$$\tilde{\nu}_t(dx) := \frac{\nu_t(\lambda(t)^{-1}dx)}{\langle 1, \nu_t \rangle} \quad \underset{t\to\infty}{\longrightarrow} \hat{\mu}(dx)$$

weakly on $(0, \infty)$.

(ii) Ψ is regularly varying at 0 with index $1 + r \in (1, 2]$, and $\hat{\mu} = \hat{\mu}_1$ where $\hat{\mu}_t$ is self-similar, with generalized Mittag-Leffler profile

$$\int_0^x \hat{\mu}_1(dy) = F_{r,1}(\beta x) = -\sum_{k=1}^{\infty} \frac{(r)_k}{k!} \frac{(-(\beta x)^r)^k}{\Gamma(1 + rk)},$$

where $\beta = \langle x, \hat{\mu}_1 \rangle^{-1}$. Furthermore, $\lambda(t) \sim \beta \langle 1, \hat{\mu}_t \rangle^{-1}$.

Idea of the proof (well, not exactly)

Rescale and dilate time via $(\rho(s) = \langle 1, \nu_s \rangle)$

$$t = s\hat{t}, \qquad \nu^s_{\hat{t}}(dx) = \frac{\nu_{s\hat{t}}(\lambda(s)^{-1}\, dx)}{\rho(s)}, \qquad \varphi_s(\hat{t}, q) = \frac{\varphi(s\hat{t}, \lambda(s) q)}{\rho(s)},$$

getting a *renormalized* equation

$$\partial_{\hat{t}} \varphi_s = -\Psi_s(\varphi_s), \qquad \Psi_s(u) = \frac{s\Psi(\rho(s)u)}{\rho(s)} = \int_{(0,\infty)} (e^{-uz} - 1 + uz) \pi_s(dx).$$

- With $\hat{\varphi}(q) = \langle 1 - e^{-qx}, \hat{\mu} \rangle$ the hypothesis means

$$\varphi_s(1, q) \underset{s\to\infty}{\longrightarrow} \hat{\varphi}(q) \qquad \forall q \in [0, \infty].$$

By study of the renormalized solution formula for the ODE and the assumed uniqueness of the limit, prove that necessarily

$\lim_{s\to\infty} \Psi_s(u)$ *exists*, hence (by scaling rigidity) $= cu^{1+r}$ and Ψ is r.v.

Conclusions for critical CSBPs that go extinct a.s.

Below we write $\delta_{\lambda,\alpha}X(x) := \lambda X(\alpha x)$.

Theorem Let $X(t,x)$ be a critical CSBP with branching mechanism Ψ satisfying

$$\int_1^\infty \frac{du}{\Psi(u)} < \infty \tag{E}$$

Then TFAE:

(i) There exists a nondegenerate Lévy process $\hat{X} = \hat{X}(x)$ and functions $\alpha, \lambda > 0$ such that

$$\delta_{\lambda(t),\alpha(t)}X(t,\cdot) \xrightarrow[t\to\infty]{\mathcal{L}} \hat{X}(\cdot). \tag{2}$$

(ii) Ψ is regularly varying at $u = 0$ with index $1 + r \in (1,2]$.

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Thank you!