Advances on Wright's Conjecture: Counting and discounting periodic orbits in Wright's equation

Jonathan Jaquette
Rutgers University

Jan Bouwe van den Berg
VU Amsterdam

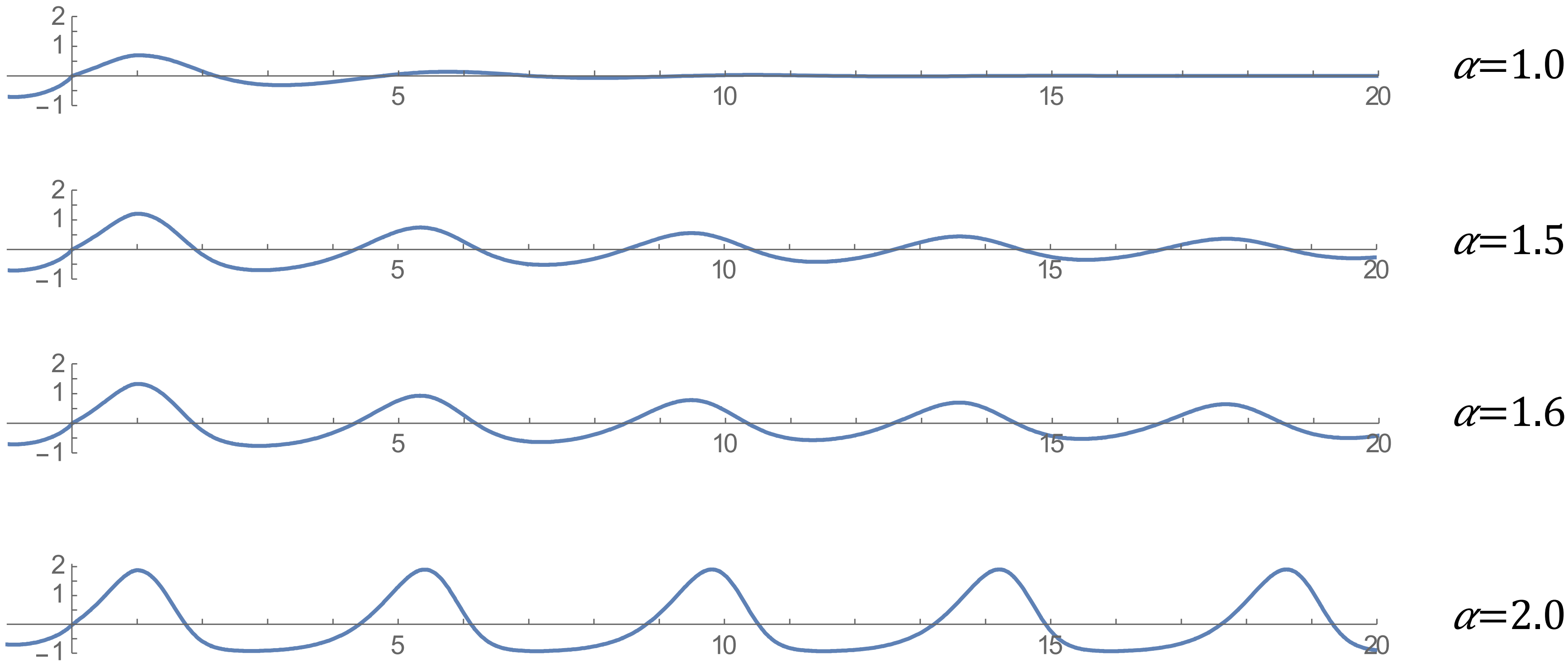
Jean-Philippe Lessard
Université Laval

Konstantin Mischaikow
Rutgers University

May 21st, 2017

Wright's Equation

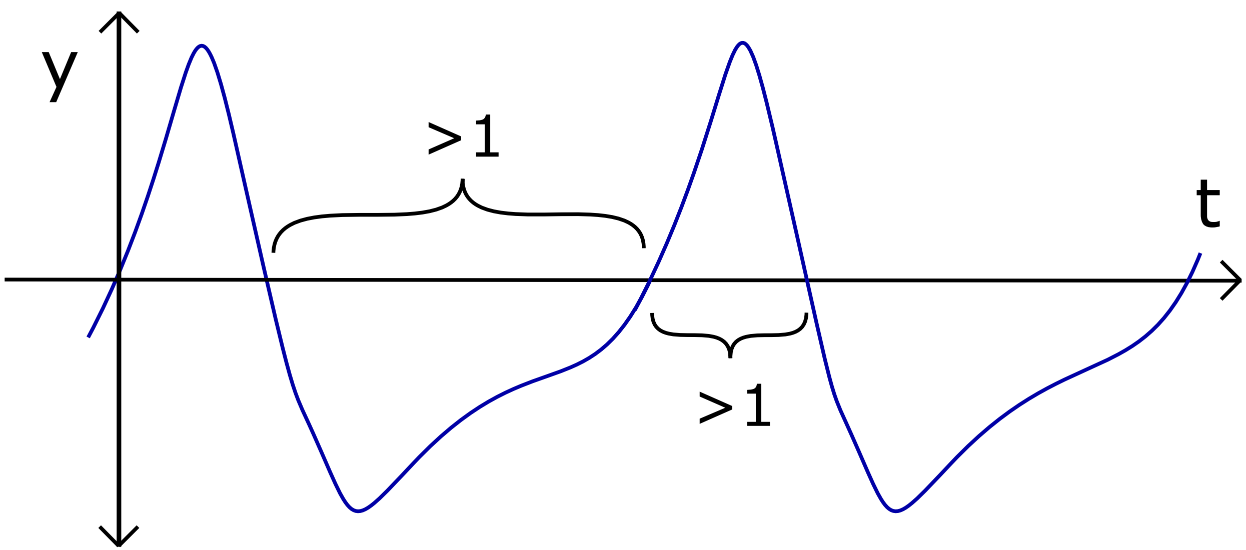
$$y'(t) = -\alpha\, y(t-1)[1+y(t)]$$



Slowly Oscillating Periodic Solutions

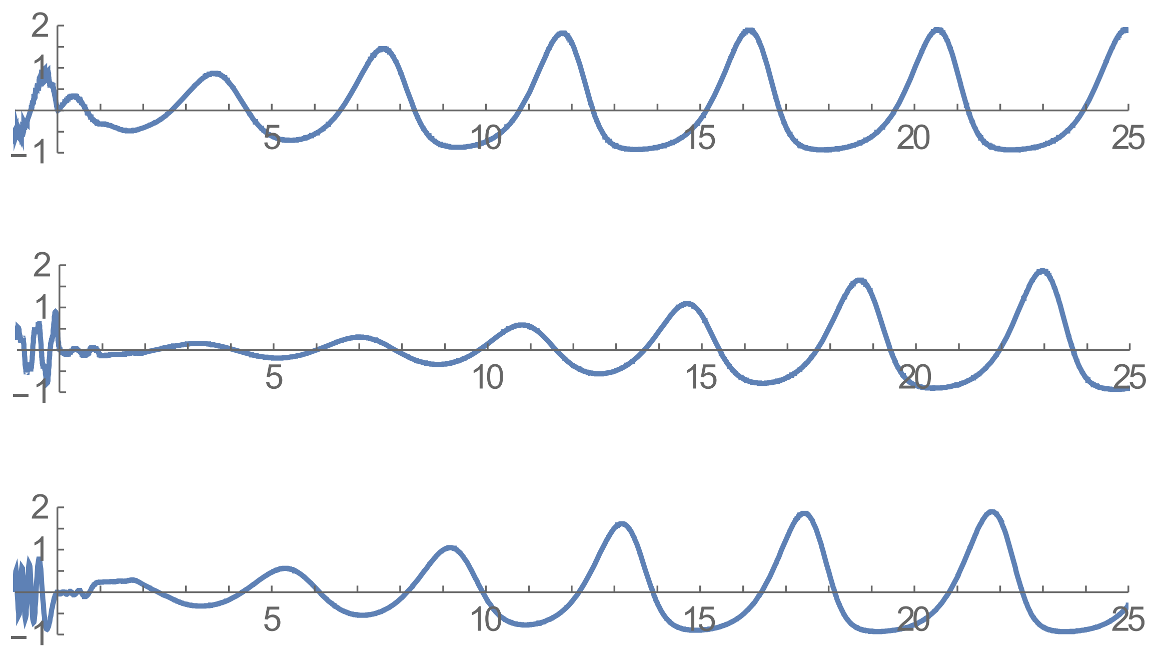
A function is a Slowly Oscillating Periodic Solution (SOPS) if it

- It is a solution to Wright's equation
- It is **positive** for at least one second, and then …
- It is **negative** for at least one second, and then …
- It repeats!



SOPS Exist

Theorem (Jones, 1962): For every $\alpha > \pi/2$ there exists at least one slowly oscillating periodic solution (SOPS) to Wright's equation



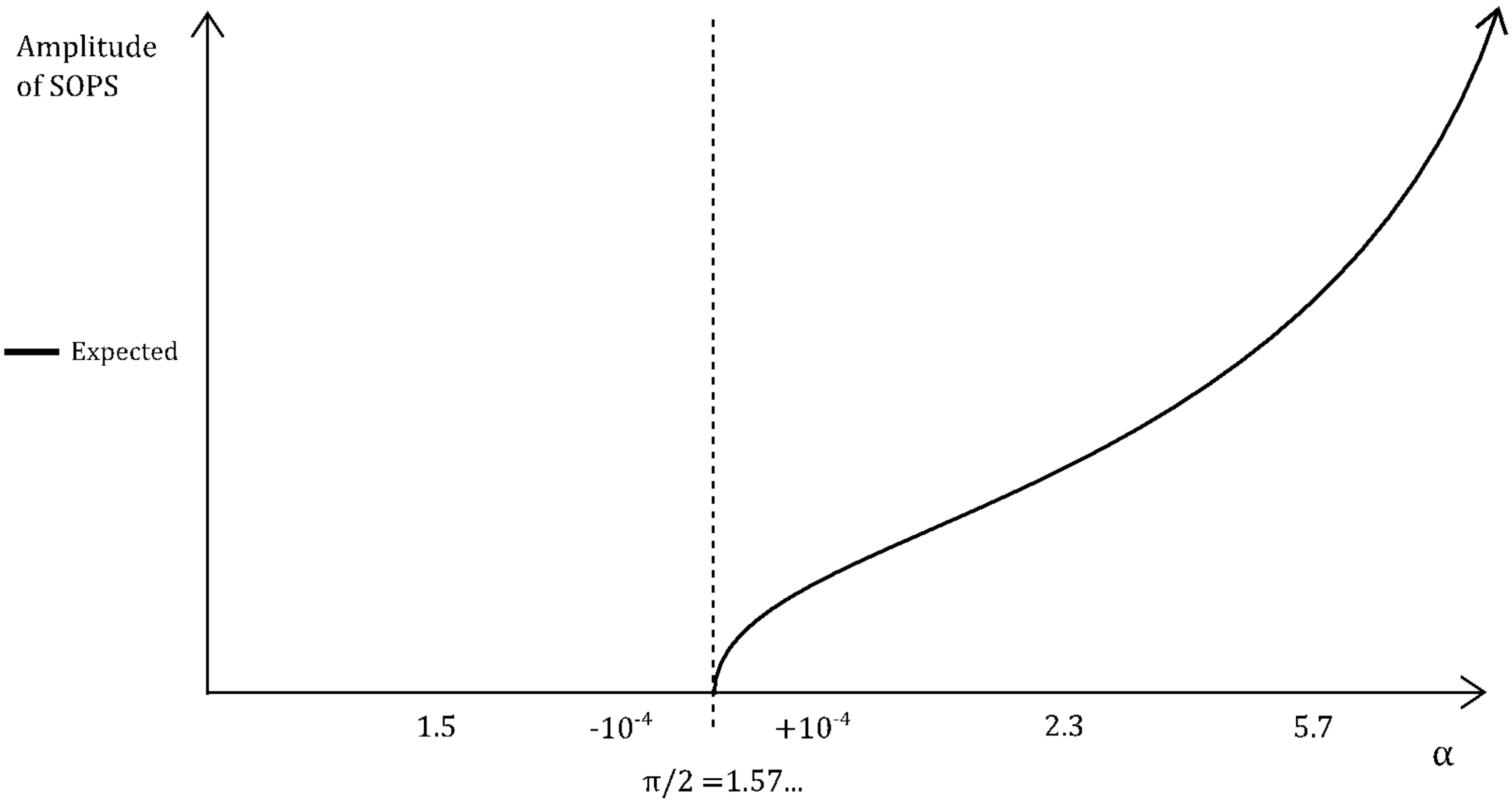
$y'(t) = -\alpha\, y(t-1)[1 + y(t)]$; $\alpha = 2.0$

Conjectures

- **Wright's Conjecture:**
  For $\alpha \in (0, \pi/2]$ zero is the global attractor

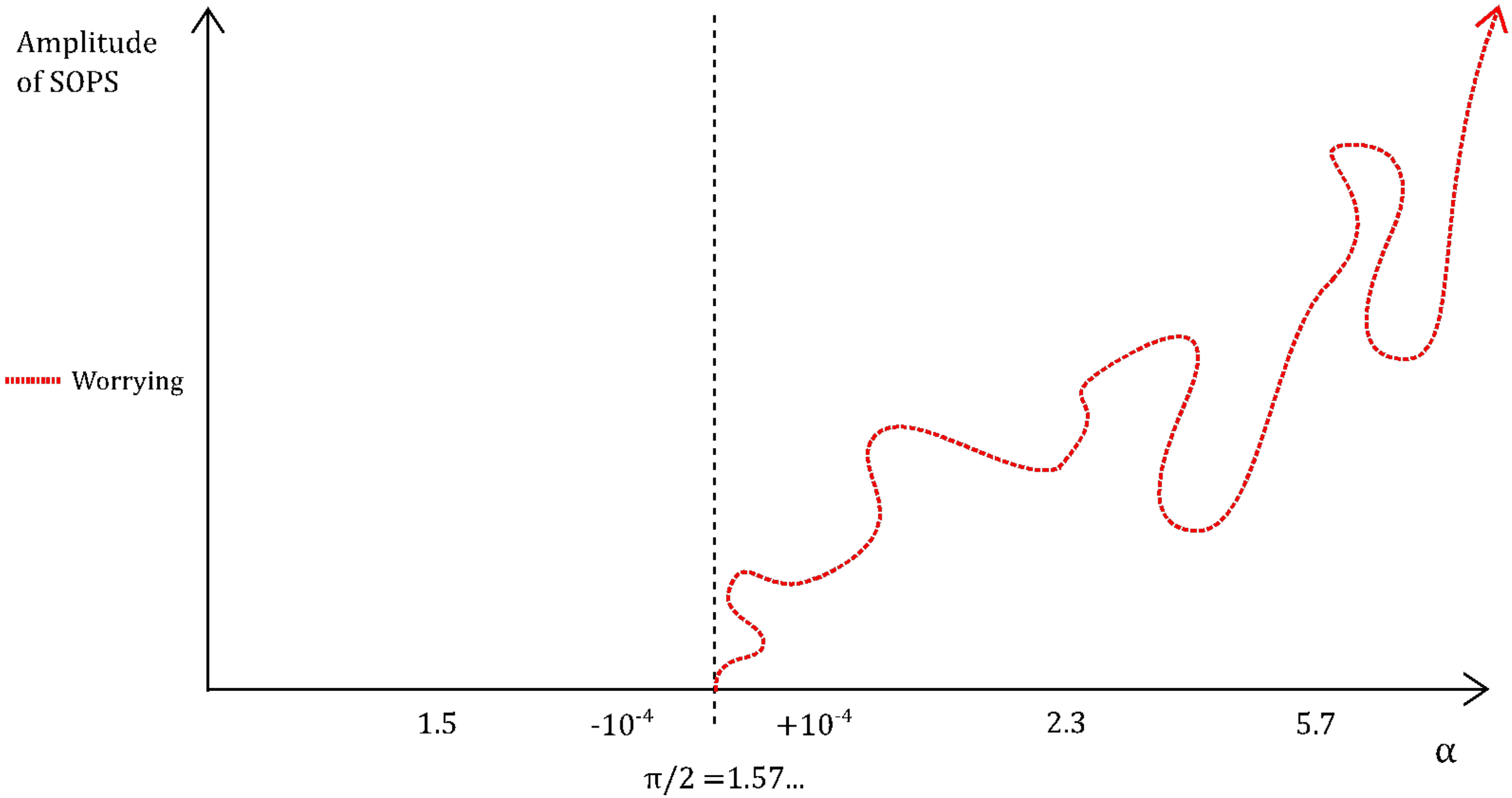
- **Jones' Conjecture:**
  For $\alpha > \pi/2$ there is a unique slowly oscillating periodic orbit (SOPS)

The conjectured bifurcation diagram for Wright's equation

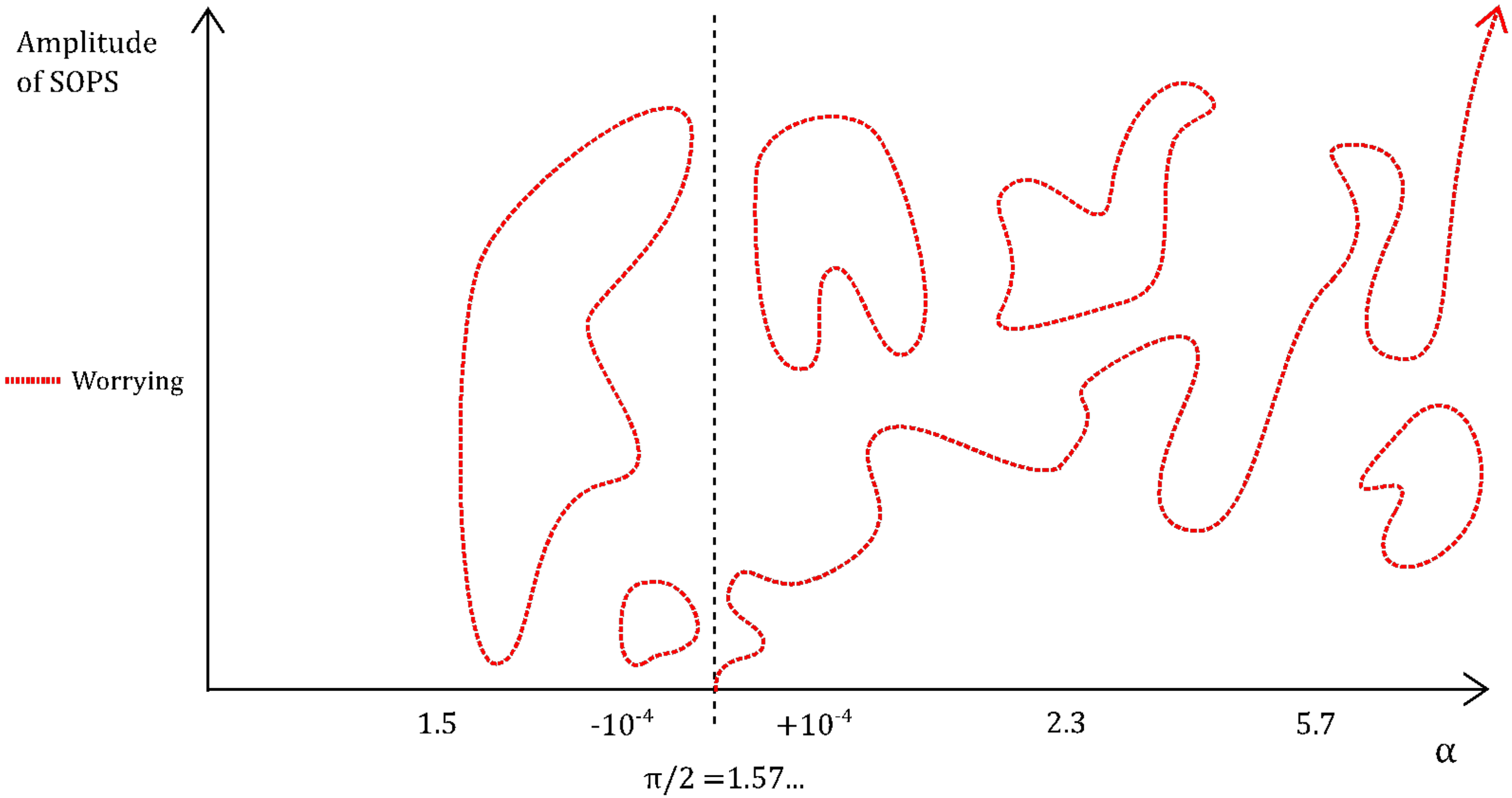


What could go wrong?

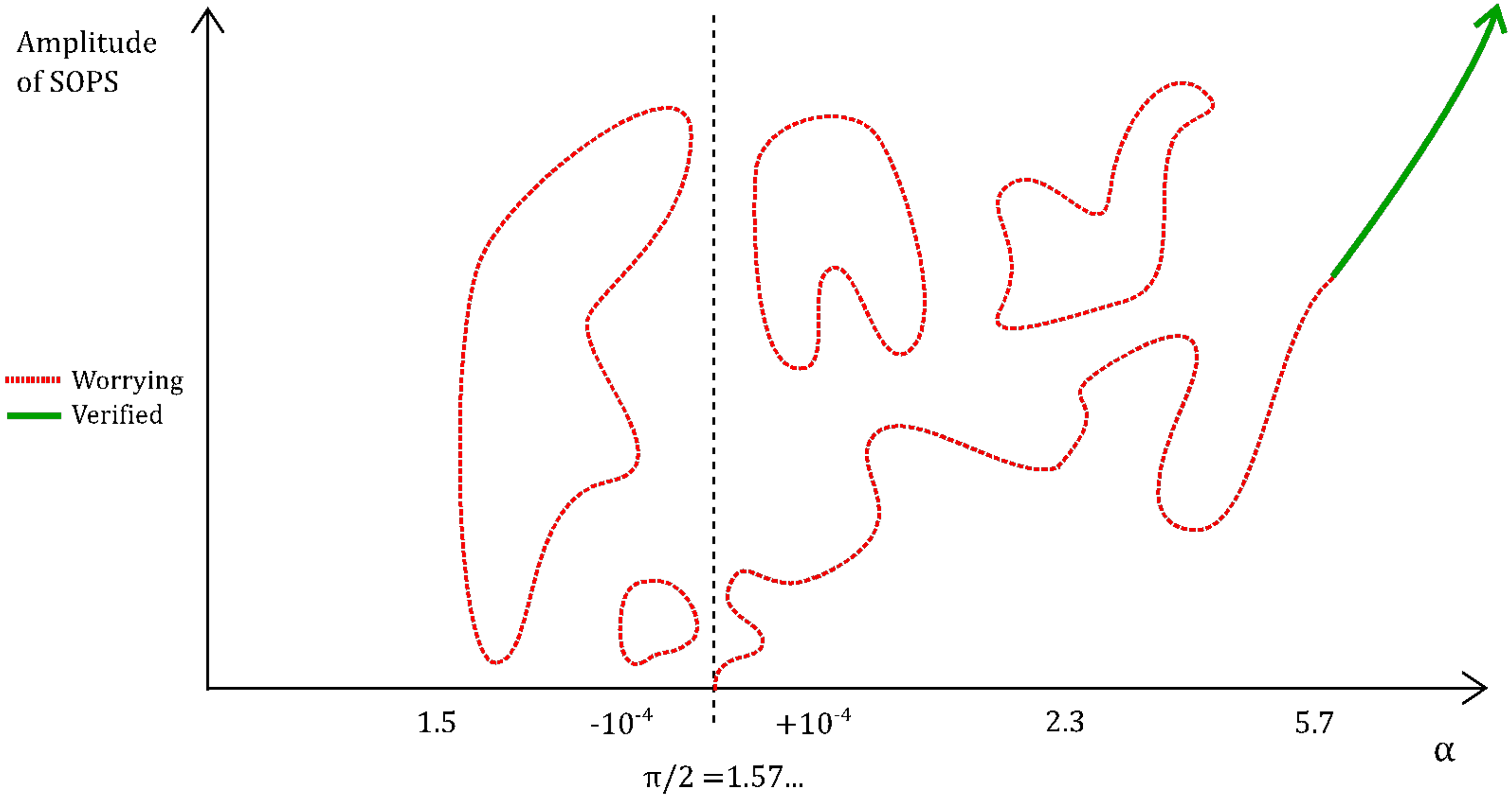
Fold bifurcations!



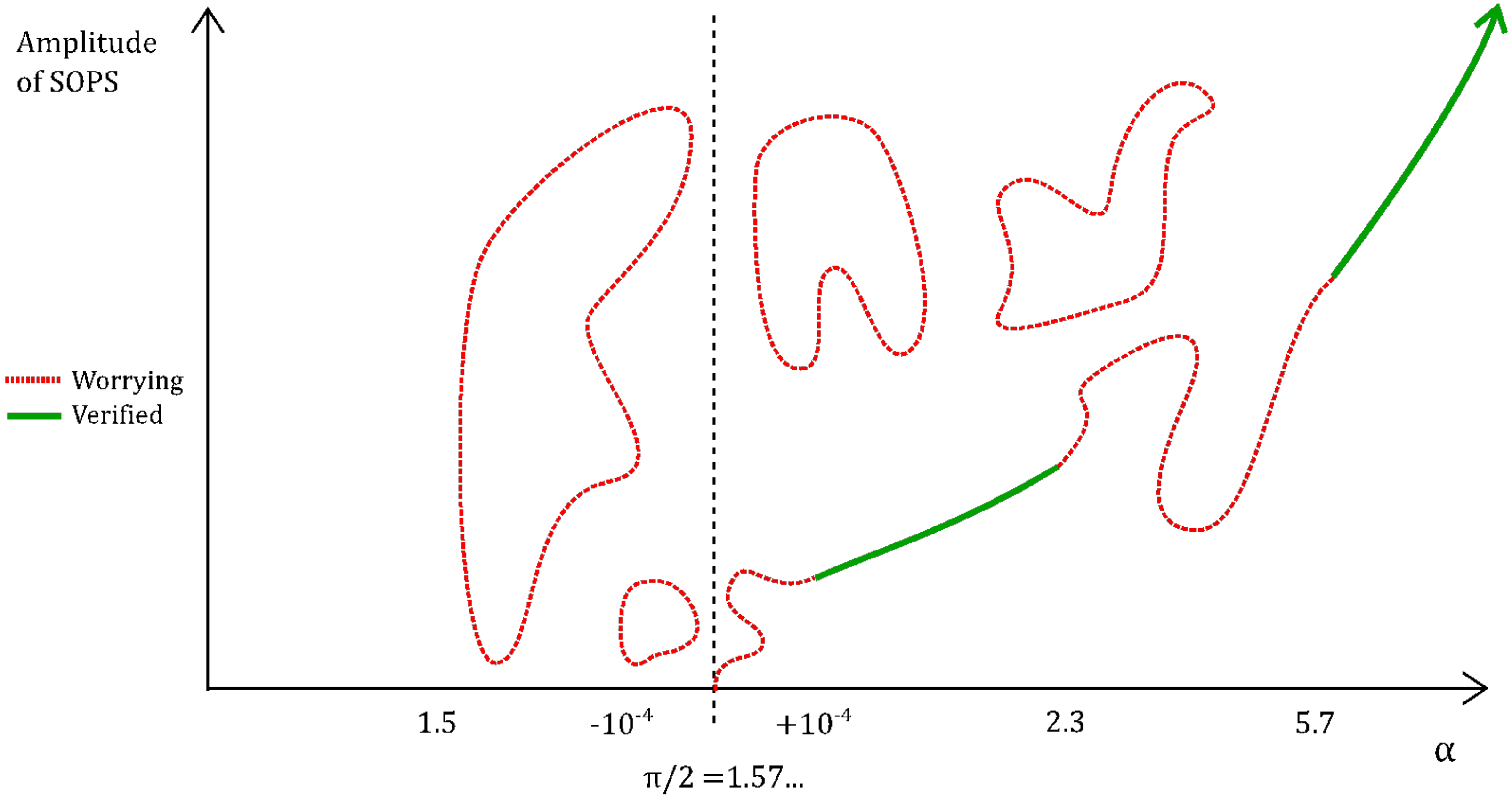
What could go wrong? Isolas of SOPS!



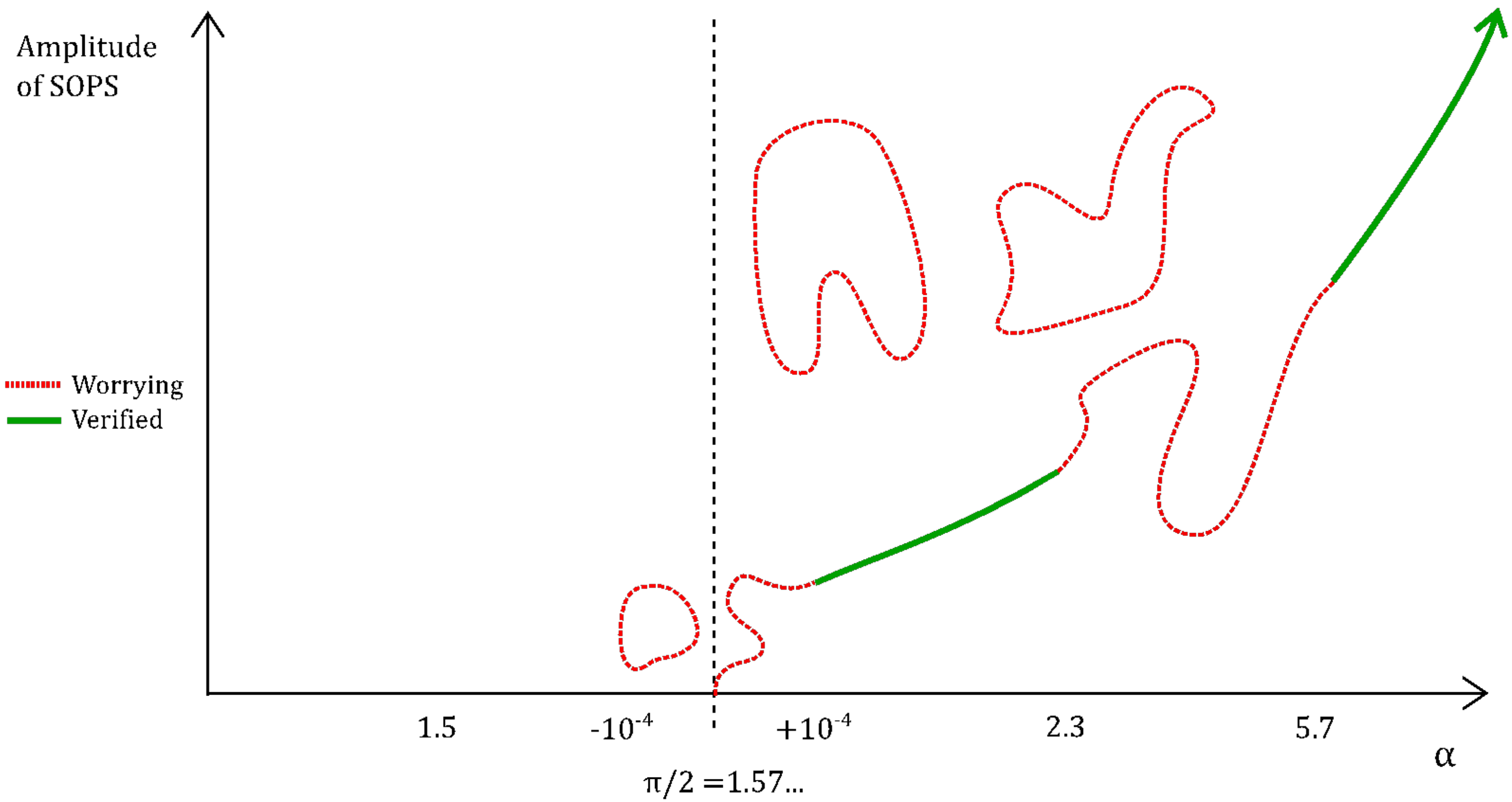
(1991) Xie



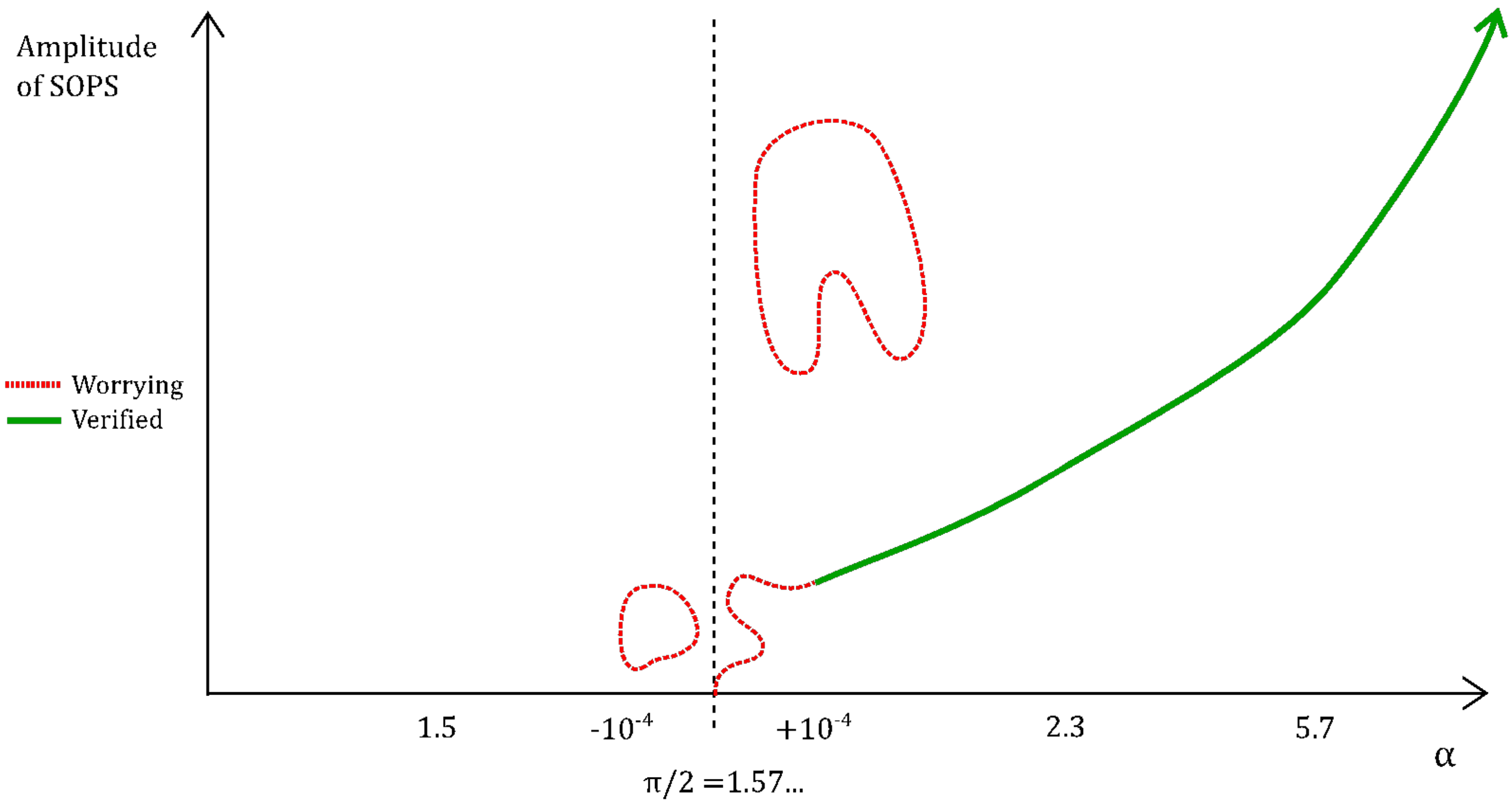
(2010) Lessard



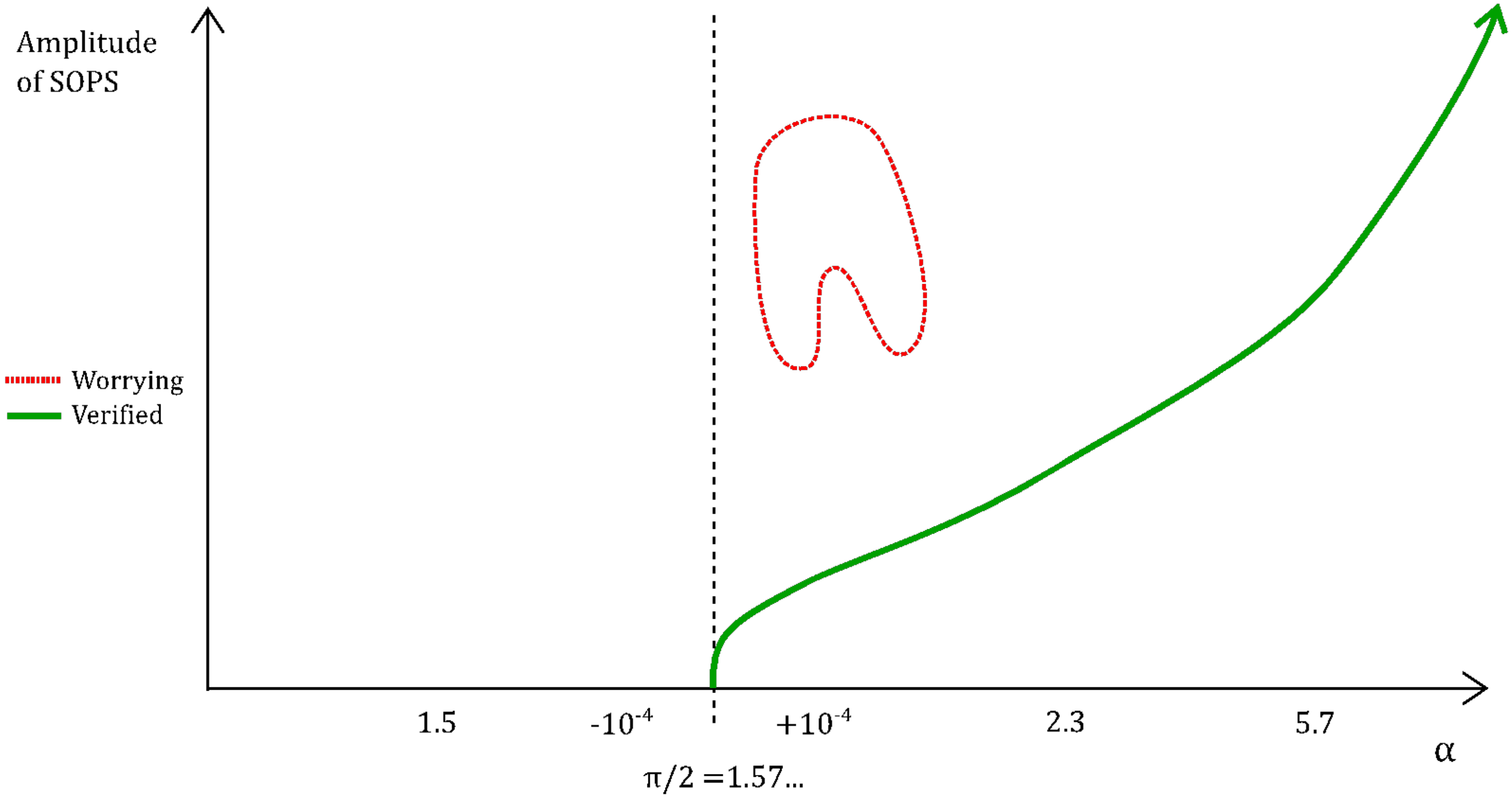
(2014) Banhelyi, Csendes, Krisztin, Neumaier



(2017)* Jaquette, Lessard, Mischaikow



(2017)* van den Berg, Jaquette



Our Results

- For $\alpha \in (0, \pi/2]$ zero is the global attractor
- For $\alpha \in [1.9, 6.0]$ there is a unique SOPS to Wright's equation
- There are no subsequent bifurcations in the branch of SOPS originating at $\alpha = \pi/2$

Proof of Wright's Conjecture

- **Wright's Conjecture:** For $\alpha \in (0, \pi/2]$ zero is the global attractor
  - Zero is the global attractor ⇔ no SOPS
  - It suffices to show that there are no SOPS for $\alpha \in [1.5706, \pi/2\ ]$

Outline of the Rest of Talk

- Interpret Wright's Equation as a Functional Equation
  - $y(t)$ is periodic $\Leftrightarrow$ $F(x)=0$
- Fixed Point Problem
  - $F(x)=0$ $\Leftrightarrow$ $T:B\to B$ has a fixed point
- Radii Polynomials
  - Technique for proving Banach fixed point theorem
- Tie together local and global results

Fourier Series

- For frequency $\omega > 0$ we can write a periodic solution as

$$y(t) = \sum_{k \in \mathbb{Z}} a_k e^{i\omega k t}$$
$$a_k \in \mathbb{C}$$

- Since $y(t) \in \mathbb{R}$ then $a_{-k} = a_k^*$
- Solutions to Wright's equation satisfy $a_0 = 0$
- Define the space

$$\ell^1 := \{\{a_k\}_{k \geq 1} : \sum_{k \geq 1} |a_k| < \infty\}$$

Wright's Equation in Fourier Space

- For frequency ω we can write a periodic solution as

$$y(t)=\sum_{k\in\mathbb{Z}} a_k e^{i\omega k t}$$
$$a_k \in \mathbb{C}$$

- We can rewrite Wright's equation …

$$y'(t)=-\alpha\, y(t-1)[1+y(t)]$$

- … in each mode using the function $G(\alpha,\omega,a)$

$$[G(\alpha,\omega,a)]_k := i\omega k\, a_k + \alpha e^{-i\omega k} a_k + \alpha \sum_{k_1+k_2=k} e^{-i\omega k_1} a_{k_1} a_{k_2}$$

Equivalence Theorem (1)

- Let $a \in \ell^1$, $\alpha > 0, \omega > 0$
- Define $y: \mathbb{R} \to \mathbb{R}$ as

$$y(t) = \sum_{k=1}^{\infty} a_k e^{i\omega k t} + a_k^* e^{-i\omega k t}$$

- Then $y(t)$ is a periodic solution to Wright's equation if and only if $G(\alpha, \omega, a) = 0$

Banach Algebra

- Define basis vectors $e_k \in \ell^1$ as

$$[e_k]_j = \begin{cases} 1 & \text{if } k=j \\ 0 & \text{if } k \neq j \end{cases}$$

- We define the norm on ℓ^1 as follows

$$\|a\| = \|a\|_{\ell^1} := 2 \sum_{k=1}^{\infty} |a_k|$$

- For $a, a \in \ell^1$ we define the discrete convolution

$$[a * a]_k = \sum_{k_1, k_2 \in \mathbb{Z},\ k_1 + k_2 = k} a_{k_1} a_{k_2}$$

$$a_{-k} = a_k^*$$

- Then we have $\{a * a\}_{k \geq 1} \in \ell^1$ and

$$\|a * a\| \leq \|a\| \cdot \|a\|$$

Defining Some Operators

- We define a compact operator K
  - $[Ka]_k := a_k / k$
  - $Ka = \{a_1/1, a_2/2, a_3/3, a_4/4, a_5/5, \ldots\}$
- … and a unitary operator U_ω
  - $[U_\omega a]_k := e^{-ik\omega} a_k$
  - $U_\omega a = \{e^{-i\omega} a_1, e^{-2i\omega} a_2, \ldots\}$
- … so that we can write our function

$$[G(\alpha,\omega,a)]_k = i\omega k\, a_k + \alpha e^{-i\omega k} a_k + \alpha \sum_{k_1+k_2=k} e^{-i\omega k_1} a_{k_1} a_{k_2}$$

- all in one condensed equation

$$G(\alpha,\omega,a) = (i\omega K^{-1} + \alpha U_\omega) a + \alpha (U_\omega a) * a$$

Hopf Bifurcation

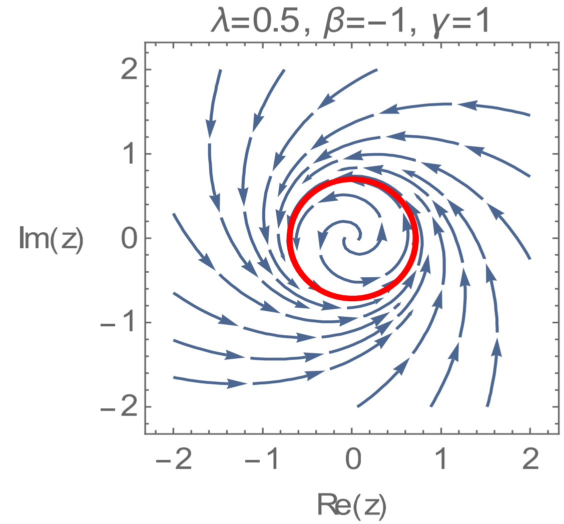
- The Hopf normal form is
  $z'(t)=$
  $z(\lambda+i)+z|z|^{2}(\beta+i\gamma)$
- If $\beta<0$ then the bifurcation is **supercritical**
- There is a stable limit cycle for $\lambda>0$ given by…

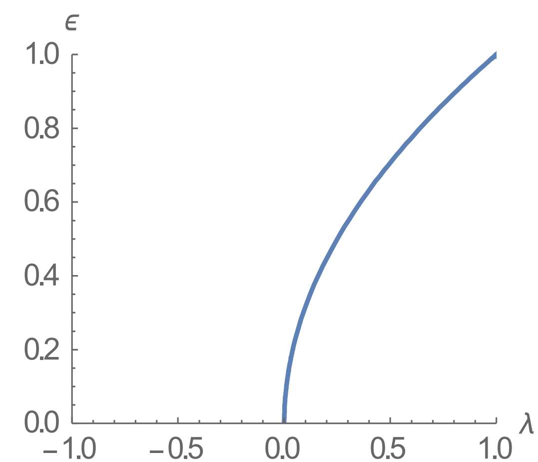
$z(t)=\epsilon e^{i\omega t}$

where

$\epsilon=\sqrt{-\lambda/\beta}$

$\omega=1+\gamma\epsilon^{2}$





Phase Condition

- If $y(t)$ is a periodic solution, then so is $y(t+\tau)$
- Write $a_1 = \epsilon\, e^{i\theta}$ with $\epsilon \geq 0$
  and choose $\tau := -\theta/\omega$,

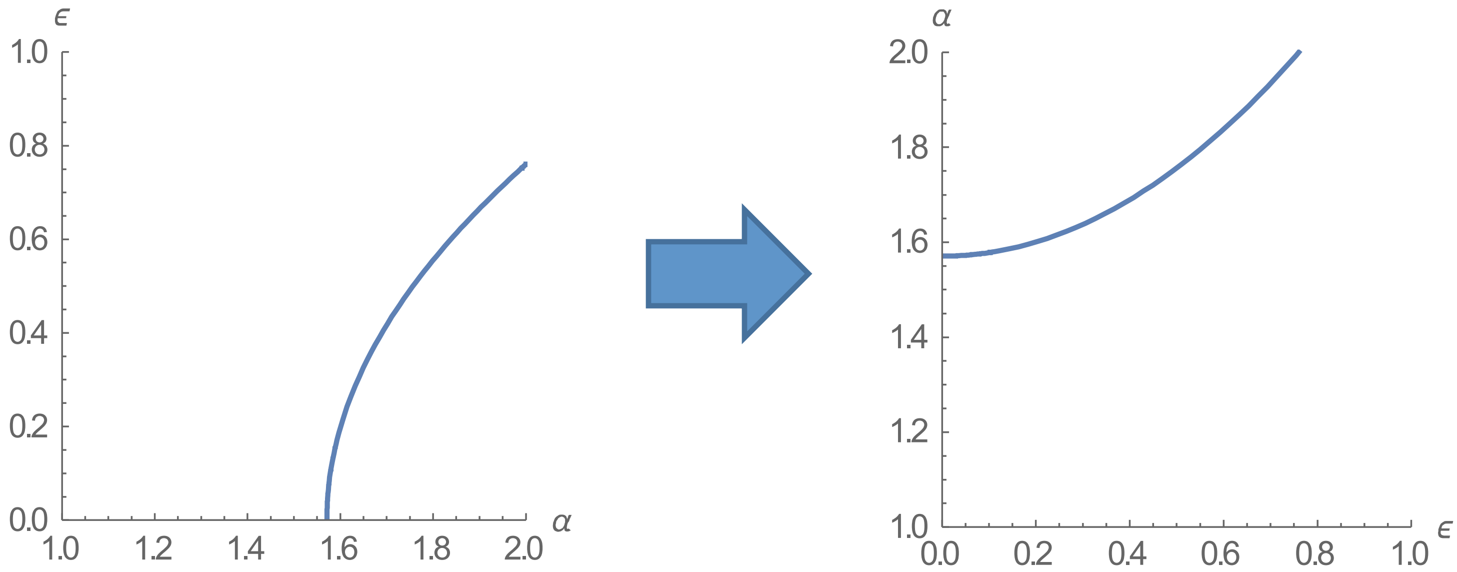
$$a_1\, e^{i\omega(t+\tau)} = \epsilon\, e^{i\omega t}$$

- Without loss of generality, write $a \in \ell^1$ as

$$a = \epsilon e_1 + c, \qquad \text{with } c \in \ell_0^1$$

$$\ell_0^1 := \{c \in \ell^1 : c_1 = 0\}$$

Turning ϵ into a Parameter



Turning ϵ into a Parameter

- Rewrite the function $G(\alpha,\omega,a)$

$$G(\alpha,\omega,a) := (i\omega K^{-1} + \alpha U_{\omega})a + \alpha(U_{\omega} a) * a$$

using the change of variables, $a = \epsilon e_1 + c$

$$G(\alpha,\omega,\epsilon e_1 + c) = F_{\epsilon}(\alpha,\omega,c)$$

- $F_{\epsilon}(\alpha,\omega,c) :=$
$(i\omega + \alpha e^{-i\omega})\epsilon e_1 + (i\omega K^{-1} + \alpha U_{\omega})c$
$+ \epsilon^2 \alpha e^{-i\omega} e_2 + \alpha\epsilon L_{\omega} c + \alpha(U_{\omega} c) * c$
where we define
  - $L_{\omega} := \sigma^{+}(e^{-i\omega} I + U_{\omega}) + \sigma^{-}(e^{i\omega} I + U_{\omega})$
  - σ^{+} is the right shift operator
  - σ^{-} is the left shift operator

Equivalence Theorem (2)

- Let $\epsilon \geq 0, c \in \ell_0^1, \alpha > 0, \omega > 0$

- Define $y : \mathbb{R} \to \mathbb{R}$ as

$$y(t) = \epsilon(e^{i\omega t} + e^{-i\omega t}) + \sum_{k=2}^{\infty} c_k e^{i\omega k t} + c_k^* e^{-i\omega k t}$$

- Then $y(t)$ is a periodic solution to Wright's equation

  if and only if $F_\epsilon(\alpha, \omega, c) = 0$

Epsilon Rescaling

- We want to use a Newton-like method to solve $F_\epsilon(\alpha,\omega,c)=0$ for small values of ϵ
- At the bifurcation point $DF_0(\pi/2,\pi/2,0)$ is not invertible
- Make the change of variables $c=\epsilon c$ and define

$$F_\epsilon(\alpha,\omega,\epsilon c)=\epsilon F_\epsilon(\alpha,\omega,c)$$

$$F_\epsilon(\alpha,\omega,c):=(i\omega+\alpha e^{-i\omega})e_1+(i\omega K^{-1}+\alpha U_\omega)c+\epsilon\alpha(e^{-i\omega}e_2+L_\omega c+(U_\omega c)*c)$$

Equivalence Theorem (3)

- Let $\epsilon > 0$, $c \in \ell_0^1$, $\alpha > 0, \omega > 0$

- Define $y : \mathbb{R} \to \mathbb{R}$ as

$$y(t) = \epsilon\left(e^{i\omega t} + e^{-i\omega t}\right) + \epsilon \sum_{k=2}^{\infty} c_k e^{i\omega k t} + c_k^{*} e^{-i\omega k t}$$

- Then $y(t)$ is a periodic solution to Wright's equation

  if and only if $F_\epsilon(\alpha, \omega, c) = 0$

Newton's Method

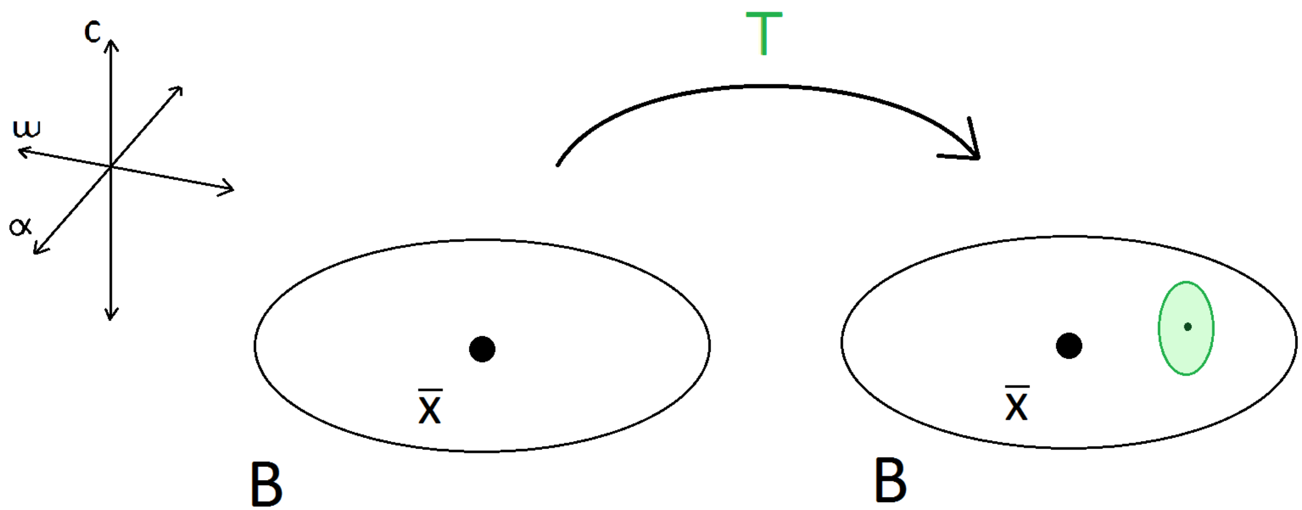
- Newton's Method produces a sequence by

$$x_{n+1} := x_n - f(x_n)/f'(x_n)$$

- The same principal works in infinite dimensions
  - Need approximate solution
  - Need approximate inverse-derivative

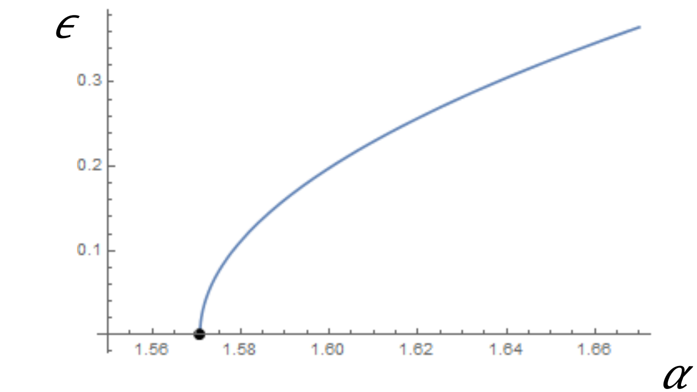
Newton-Like Operator

- Apply contraction mapping principle to a Newton-like operator $T(x):=x-A^{\dagger} F(x)$
- While $A^{\dagger}$, F, T all depend on $\epsilon\geq 0$, we suppress this in the notation



Approximate Solution

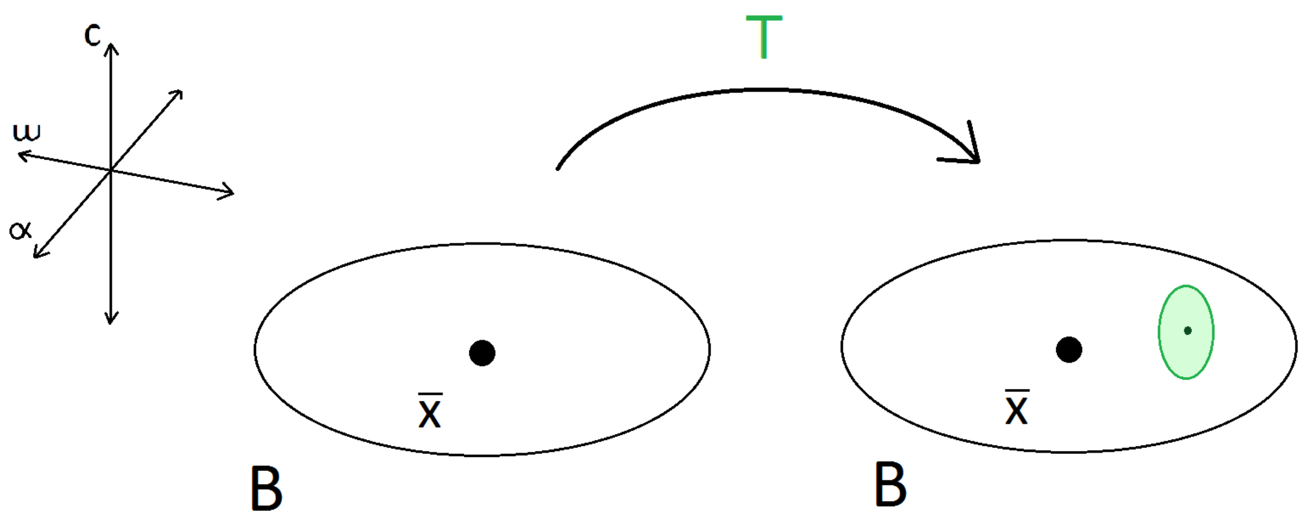
- Using normal forms theory, we define the approximate solution $x_\epsilon : \mathbb{R}_+ \to \mathbb{R}^2 \times \ell_0^1$
- $x_\epsilon = \{\alpha_\epsilon, \omega_\epsilon, c_\epsilon\}$
  - $\alpha_\epsilon := \pi/2 + \epsilon^2/5\,(3\pi/2 - 1)$
  - $\omega_\epsilon := \pi/2 - \epsilon^2/5$
  - $c_\epsilon := (2 - i/5)\,\epsilon\, e_2$



Approximate Derivative

- Next we define $A^{\dagger}$ for our Newton-like operator

$$T(x) := x - A^{\dagger} F(x)$$



Approximate Derivative

- Define the map $A^{\dagger} = DF(x_{\epsilon})^{-1} + \mathcal{O}(\epsilon^{2})$ by

$$\boldsymbol{A^{\dagger} \coloneqq A_{0}^{-1} - \epsilon A_{0}^{-1} A_{1} A_{0}^{-1}}$$

- Define the maps
  - $i_{\mathbb{C}}(s,t) \coloneqq s + i\,t$
  - $A_{0}\,x = A_{0}(\alpha,\omega,c) \coloneqq i_{\mathbb{C}}\, A_{0,1} \begin{bmatrix} \alpha \\ \omega \end{bmatrix} e_{1} + A_{0,*}\, c$
  - $A_{1}\,x = A_{1}(\alpha,\omega,c) \coloneqq i_{\mathbb{C}}\, A_{1,2} \begin{bmatrix} \alpha \\ \omega \end{bmatrix} e_{2} + A_{1,*}\, c$
- Define $\omega_{0} = \pi/2$, and the maps…

$$A_{0,1} \coloneqq \begin{bmatrix} 0 & -\pi/2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2-3\pi/2 \\ -4 & 2(2+\pi) \end{bmatrix} \qquad A_{1,2} \coloneqq 1/5$$

$$A_{0,*} \coloneqq \pi/2\,\left(iK^{-1} + U_{\omega_{0}}\right) L_{\omega_{0}} \qquad A_{1,*} \coloneqq \pi/2$$

Equivalence Theorem (4)

- Define the Newton-like operator

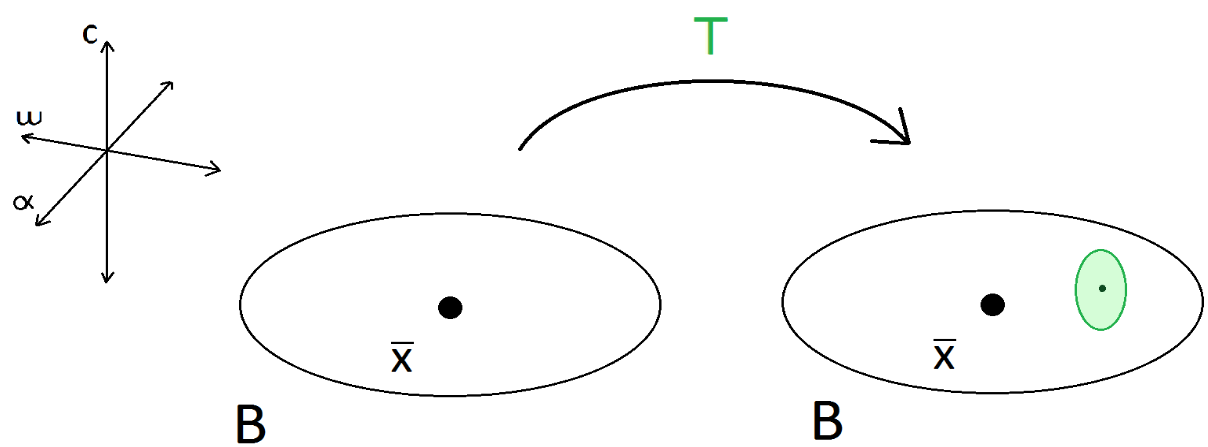
$$T(x) := x - A^{\dagger} F(x)$$

- For $0 \leq \epsilon < 0.79$ the operator $A^{\dagger}$ is injective and the following are equivalent:
  - Fixed points of $T(x)$
  - Zeros of $F(x)$
  - SOPS to Wright's equation

Newton-Like Operator

- To have a contraction mapping, we need

$$T(B) \subseteq B$$



A Ball about our Approximation

- Let $\epsilon \geq 0$, $r = \{r_\alpha, r_\omega, r_c\} \in \mathbb{R}^3$, $\rho > 0$
- Define $\boldsymbol{B_\epsilon(r, \rho)}$ to be the collection of points $\{\alpha, \omega, c\} \in \mathbb{R}^2 \times \ell_0^1$ satisfying …
    - $|\alpha - \alpha_\epsilon| \leq r_\alpha$
    - $|\omega - \omega_\epsilon| \leq r_\omega$
    - $||c - c_\epsilon|| \leq r_c$
    - $||K^{-1} c|| \leq \rho$
        - ρ makes the ball compact!

$\{\alpha_\epsilon, \omega_\epsilon, c_\epsilon\}$ is the approximate solution

Radii Polynomials

- For $T(x_\epsilon) - x_\epsilon \in \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1$
  we define $Y(\epsilon) \in \mathbb{R}^3$
  which provides a component-wise bound
- For $DT(x) \in \mathcal{L}(\mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1, \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1)$
  we define $Z(\epsilon, r, \rho) \in \mathrm{Mat}(\mathbb{R}^3, \mathbb{R}^3)$
  which provides a component-wise bound for all $x \in B_\epsilon(r, \rho)$
- Define the radii polynomials:

$$P(\epsilon, r, \rho) := Y(\epsilon) - [I - Z(\epsilon, r, \rho)] \cdot r$$

Radii Polynomials: Uniform in ϵ

- $P(\epsilon, r, \rho)$ is increasing in ϵ
- If $0 \le \epsilon \le \epsilon_0$ then

$$P(\epsilon_0, r, \rho) < 0 \Rightarrow P(\epsilon, r, \rho) < 0$$

- If each component of $P(\epsilon_0, r, \rho)$ is negative, then **for all** $\mathbf{0 \le \epsilon \le \epsilon_0}$ there is a unique $x_\epsilon \in B_\epsilon(r, \rho)$ such that $T(x_\epsilon) = x_\epsilon$

Radii Polynomials: Uniform in ϵ

- For $T(x_\epsilon) - x_\epsilon \in \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1$
  we define $Y(\epsilon) \in \mathbb{R}^3$
  which provides a component-wise bound
- For $DT(x) \in \mathcal{L}(\mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1, \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1)$
  we define $Z(\epsilon, r, \rho) \in \mathrm{Mat}(\mathbb{R}^3, \mathbb{R}^3)$
  which provides a component-wise bound for all $x \in B_\epsilon(r, \rho)$
- Define the radii polynomials:

$$P(\epsilon, r, \rho) := Y(\epsilon) - [I - Z(\epsilon, r, \rho)] \cdot r$$

Radii Polynomials: Uniform in ϵ^2

- $P(\epsilon,\epsilon^2 r,\rho)$ is increasing in ϵ
- If $0\leq\epsilon\leq\epsilon_0$ then

$$P(\epsilon_0,\boldsymbol{\epsilon_0^2}\, r,\rho)<0\Rightarrow P(\epsilon,\boldsymbol{\epsilon^2}\, r,\rho)<0$$

- If each component of $P(\epsilon_0,\epsilon_0^2 r,\rho)$ is negative,
  then for all $0\leq\epsilon\leq\epsilon_0$ there is a unique $x_\epsilon\in B_\epsilon(\epsilon^2 r,\rho)$ such that $T(x_\epsilon)=x_\epsilon$

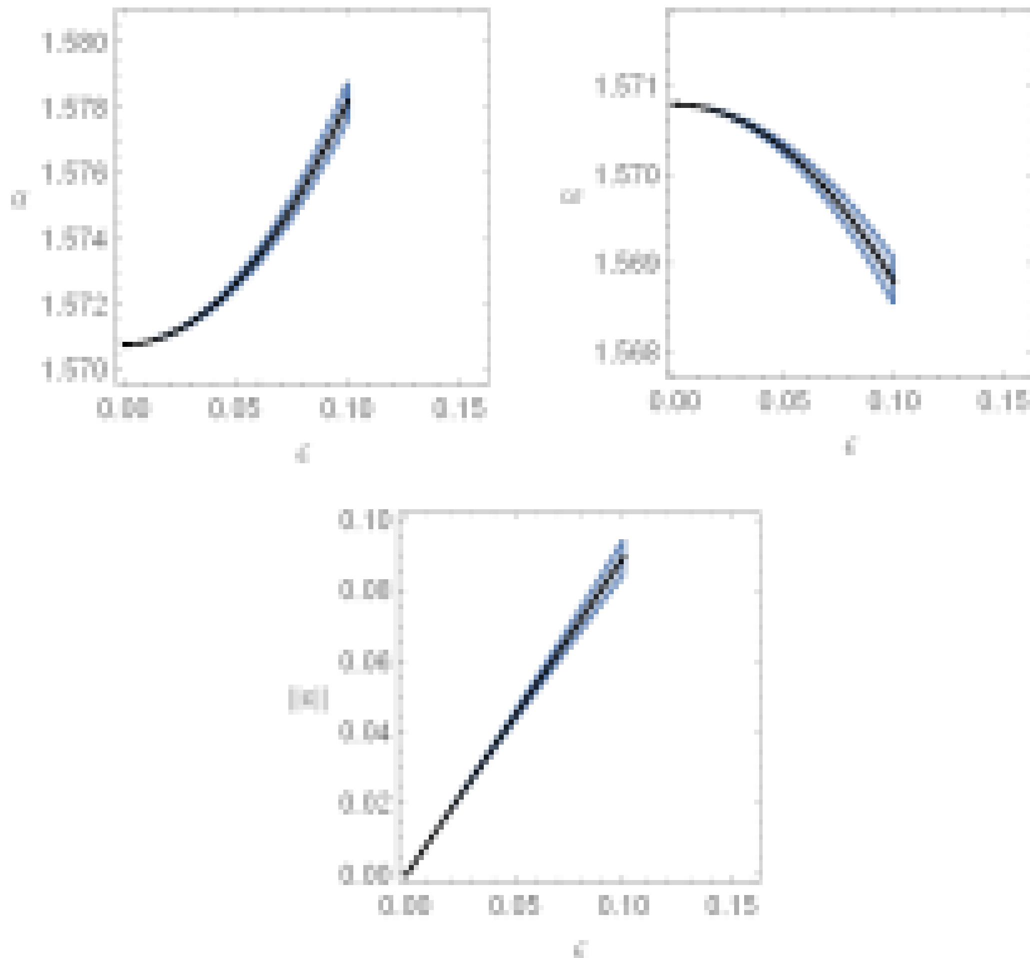
Radii Polynomials: Uniform in ϵ^2

- For $T(x_\epsilon) - x_\epsilon \in \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1$
  we define $Y(\epsilon) \in \mathbb{R}^3$
  which provides a component-wise bound
- For $DT(x) \in \mathcal{L}(\mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1, \mathbb{R}^1 \times \mathbb{R}^1 \times \ell_0^1)$
  we define $Z(\epsilon, r, \rho) \in \mathrm{Mat}(\mathbb{R}^3, \mathbb{R}^3)$
  which provides a component-wise bound for all $x \in B_\epsilon(r, \rho)$
- Define the radii polynomials:

$$P(\epsilon, r, \rho) := Y(\epsilon) - [I - Z(\epsilon, r, \rho)] \cdot r$$

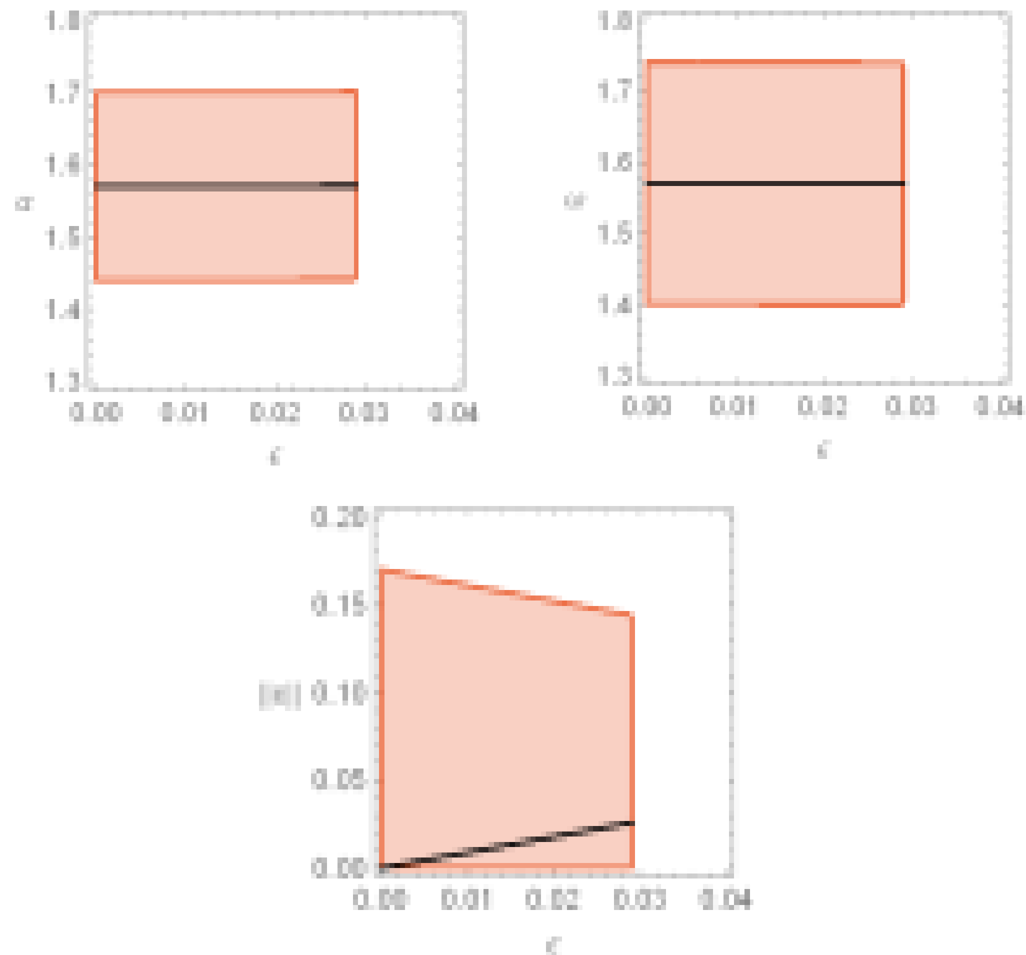
Applications (1)

- Fix the constants:
  - $\epsilon_0 = 0.10$
  - $r_\alpha = 0.0594$
  - $r_\omega = 0.0260$
  - $r_c = 0.4929$
  - $\rho = 0.3191$
- The black line is $\boldsymbol{x_\epsilon}$
  The **blue region** is $B_\epsilon(\epsilon^2 r, \rho)$
- For all $0 \le \epsilon \le \epsilon_0$
  there is a unique
  $x_\epsilon \in B_\epsilon(\epsilon^2 r, \rho)$
  such that $T(x_\epsilon) = x_\epsilon$
- For $\epsilon > 0$ these solutions $F(\alpha_\epsilon, \omega_\epsilon, c_\epsilon) = 0$
  satisfy $\alpha_\epsilon > \pi/2$



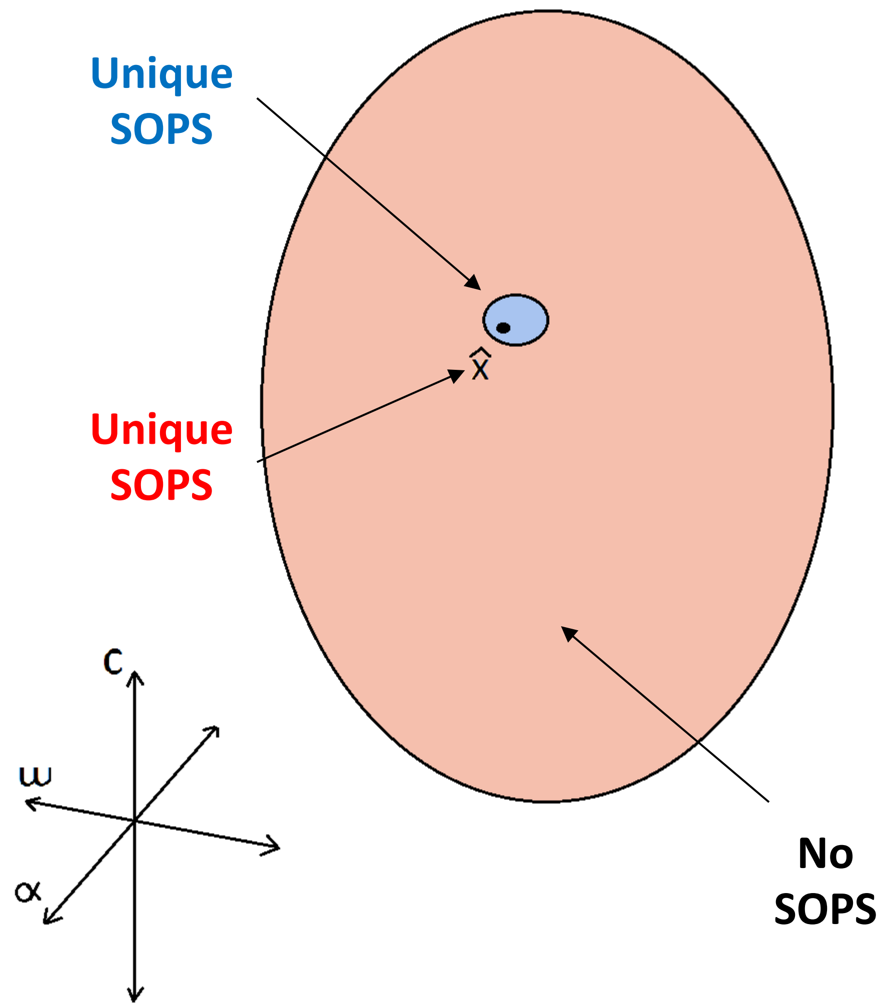
Applications (2)

- Fix the constants:
  - $\epsilon_0 = 0.029$
  - $r_\alpha = 0.13$
  - $r_\omega = 0.17$
  - $r_c = 0.17$
  - $\rho = 1.78$
- The black line is $\boldsymbol{x_\epsilon}$
  The **red region** is $B_\epsilon(r,\rho)$
- For all $0 \le \epsilon \le \epsilon_0$ there is a unique $x_\epsilon \in B_\epsilon(r,\rho)$ such that $T(x_\epsilon) = x_\epsilon$

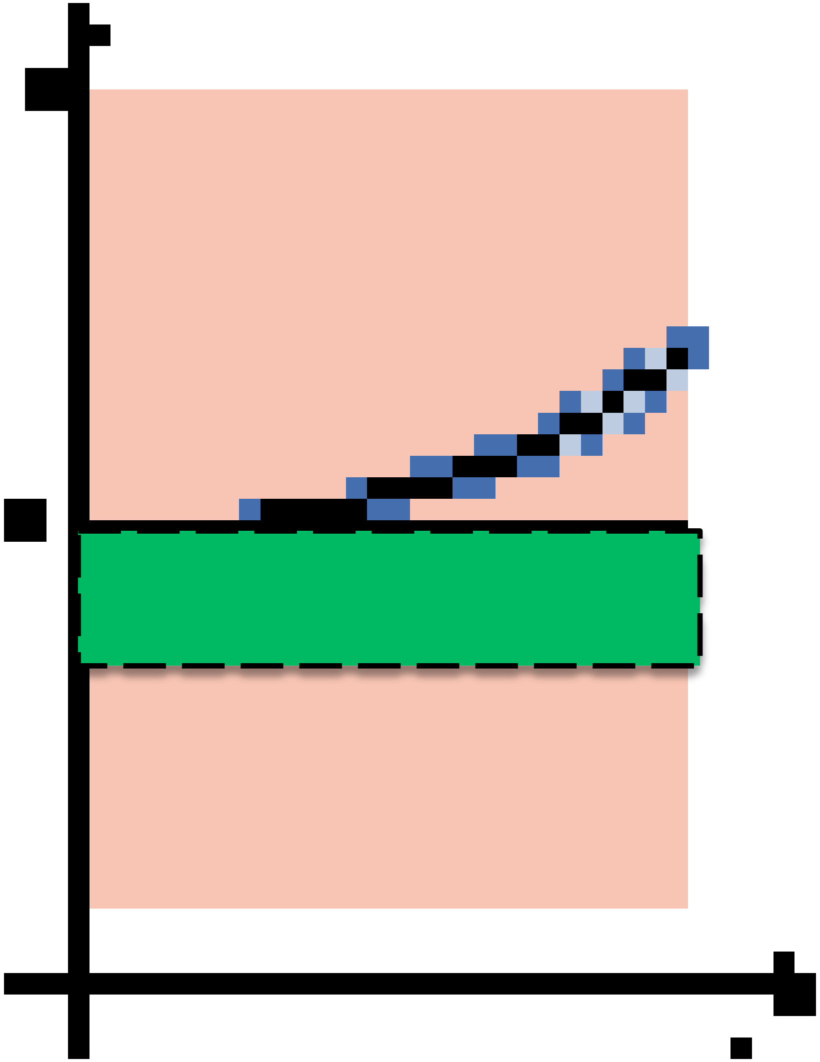


Large and Small Radii

- Fix $0<\epsilon\leq 0.029$
- There is a unique SOPS $x_1 \in B_\epsilon(r_1,\rho_1)$
- There is a unique SOPS $x_2 \in B_\epsilon(r_2,\rho_2)$
- $B_\epsilon(r_1,\rho_1) \subset B_\epsilon(r_2,\rho_2)$
  - $x_1 = x_2$
  - No SOPS in $B_\epsilon(r_2,\rho_2) \backslash B_\epsilon(r_1,\rho_1)$

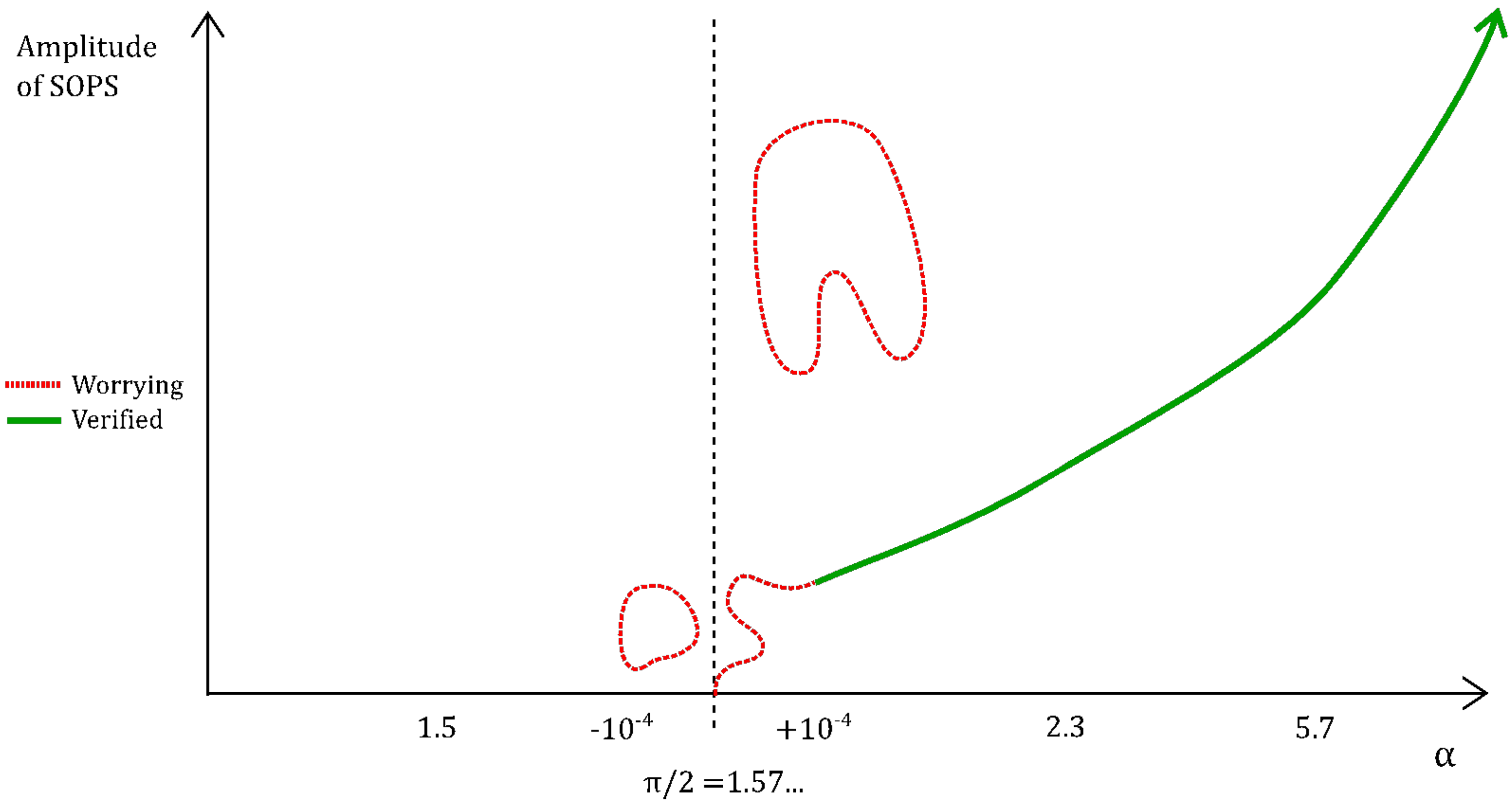


Sketch of Proof

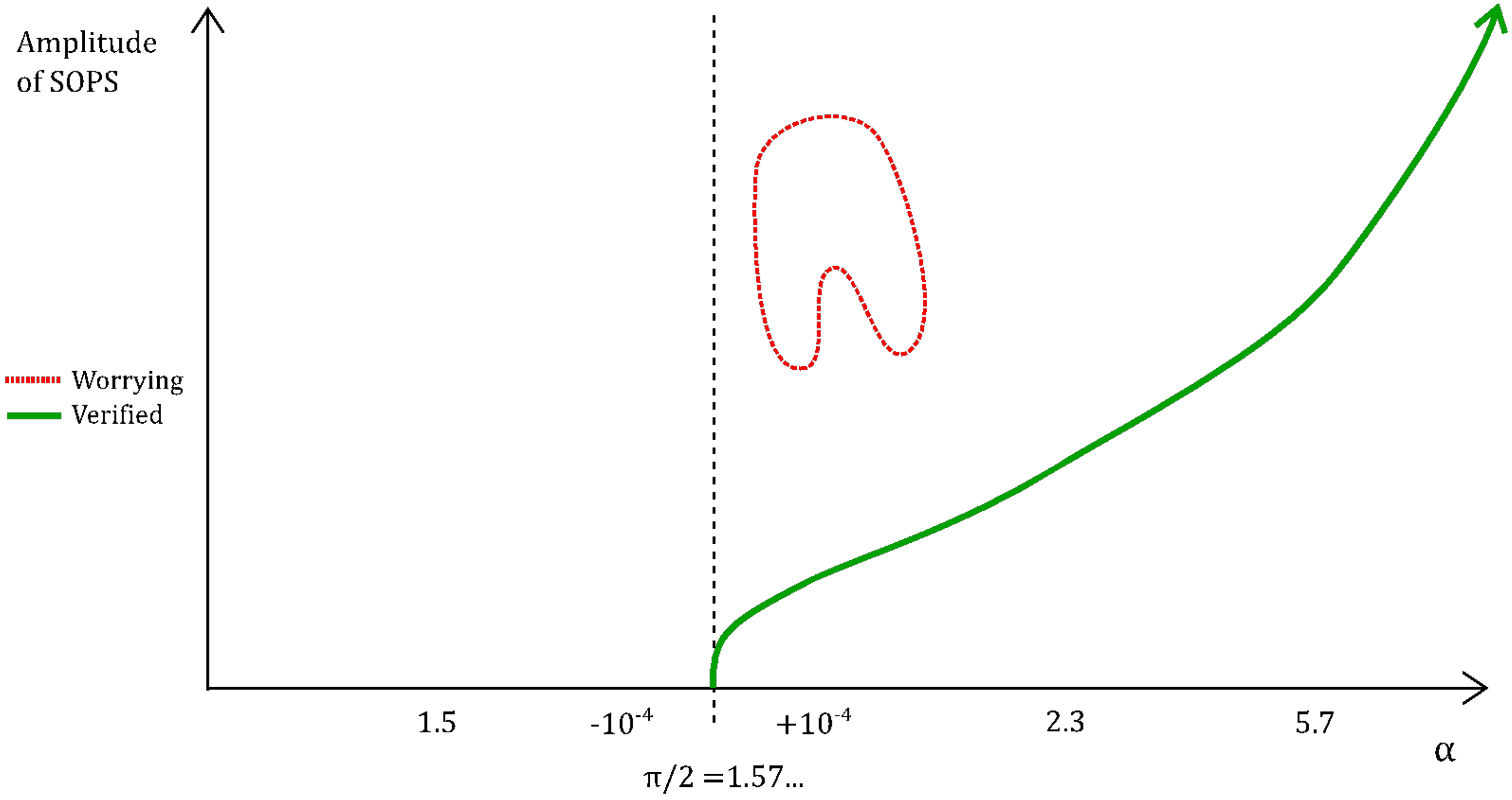


- The **blue region** satisfies $\alpha > \pi/2$
- There cannot be any SOPS in the **red region**
- The **green region** is the only place SOPS could be if $\alpha \in [1.5706, \pi/2\ ]$
- The **green region** is contained inside the **red region**
- Hence, there cannot be any SOPS for $\alpha \in [1.5706, \pi/2\ ]$

Summary



Summary



Thank You!