LDDMM Models of a Heartbeat

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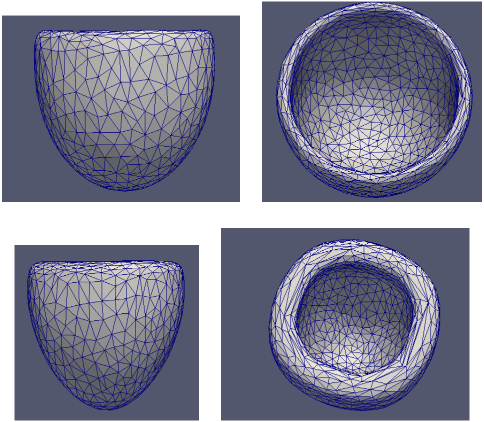
A toy model of a mouse's heart

General problem: Given the contours of a relaxed (diastole) and contracted (systole) heart, recover the whole motion of a contraction.



We want to deform a shape onto another while preventing non-realistic motions like self-intersection → LDDMM.

A Brief Summary of LDDMM

Purpose of LDDMM: Compare two "shapes" q_0 and q_1 in $\mathbb{R}^d$ in a way that takes into account their **geometric properties** (smoothness, self-intersection...). The initial shape q_0 is the **template**, q_1 is the **target**.

Examples of shapes:

- Parametrized embedded curves, surfaces and submanifolds: $q \in \mathrm{Emb}^k(M, \mathbb{R}^d)$
- Unparametrized embedded curves, surfaces and submanifolds: $q \in \mathrm{Emb}^k(M, \mathbb{R}^d)/\mathcal{D}^k(M)$ **(Bauer, Bruveris, Michor)**
- Landmarks: $q = (x_1, \ldots x_n)$, $x_i \in \mathbb{R}^d$

A Brief Summary of LDDMM

Here, $q_0 : S \to \mathbb{R}^d$ parametrized embedded manifold.

Method: Let $(V, \langle\cdot,\cdot\rangle)$ be a Hilbert space of vector fields with continuous inclusion in $\mathcal{C}_0^1(\mathbb{R}^d, \mathbb{R}^d)$. A time-dependent vector field $v \in L^2(0,1;V)$ admits a unique flow $\varphi(\cdot)$ such that

$$\partial_t \varphi(t,x) = v(t, \varphi(t,x)), \quad t \in [0,1], \ x \in \mathbb{R}^d.$$

Action of the flow on the template q_0 (composition on the left), deforming it into $q(t,s) = \varphi(t, q_0(s))$, or, for short,

$$q(t) = \varphi(t) \cdot q_0.$$

Infinitesimal action: $\partial_t q(t,s) = v(t, q(t,s))$, i.e.,

$$\dot{q}(t) = v(t) \cdot q(t).$$

A Brief Summary of LDDMM

Minimize

$$J(v) = \frac{1}{2}\int_0^1 \|v(t)\|_V^2 dt + g(q(1), q_1).$$

Data attachment term: g gets smaller the closer $q(1)$ gets to q_1, and

$$q(0) = q_0, \quad \dot{q}(t) = v(t) \cdot q(t).$$

Plan

1. The LDDMM setting
   - Reproducing Kernel Hilbert Spaces
   - Minimizers and reduction of the problem
2. Fibered model of the heart
   - Fibers
   - Contraction and LDDMM, first model
   - Second model: tracking
   - Third model: elastic energy

RKHS

Reproducing Kernel Hilbert Space of vector field: space V of $\mathcal{C}^1$ vector fields endowed with a Hilbert product $\langle \cdot, \cdot \rangle_V$, such that the inclusion $V \hookrightarrow \mathcal{C}^1_0(\mathbb{R}^d, \mathbb{R}^d)$ is continuous.

Remark: v and all its first order partial derivatives are dominated by $\|v\|_V = \sqrt{\langle v, v \rangle_V}$.

Riesz Theorem: Any continuous linear function $P : V \to \mathbb{R}$ (we denote $P \in V^*$), is represented by product with a unique vector field $K_V(P) \in V$:

$$\forall v \in V, \quad P(v) = \langle K_V(P), v \rangle_V .$$

In particular $\|K_V(P)\|_V^2 = P(K_V(P))$

Reproducing Kernel

Reproducing Kernel: Fix a vector $u \in \mathbb{R}^d$ and a point $y \in \mathbb{R}^d$; $u \otimes \delta_y : v \mapsto u \cdot v(y)$ belongs to V^*, and is represented by a vector field $K_V(u \otimes \delta_y) \in V$.

Then, $u \mapsto K_V(u \otimes \delta_y)(x)$ is linear, so we can write

$$K_V(u\delta_y)(x) = K(x, y)u$$

for some mapping $K : \mathbb{R}^d \times \mathbb{R}^d \to M_d(\mathbb{R})$.

Moreover,

$$\|K_V(u\delta_y)\|_V^2 = (u\delta_y, K_V u\delta_y) = u \cdot K(y, y)u.$$

Reproducing Kernel

More generally, take any compactly-supported measure ν on $\mathbb{R}^d$ and any ν-integrable mapping $p : \mathbb{R}^d \to \mathbb{R}^d$. Define $p \otimes \nu \in V^*$ by

$$(p \otimes \nu, v) = \int_{\mathbb{R}^d} p(x) \cdot v(x) d\nu(x).$$

Then we have $K_V(p\nu)(x) = \int_{\mathbb{R}^d} K(x,y)p(y)d\nu(y)$. Moreover,

$$\|K_V(p\nu)\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} p(x) \cdot K(x,y)p(y)d\nu(y)d\nu(x).$$

Important because, usually, $\partial_1 g(q, q_1)$ is of this form.

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Form of the minimizers

For a template $q_0 : S \mapsto \mathbb{R}^d$ that is given by an immersed submanifold of $\mathbb{R}^d$, recall that we want to minimize $J(v) = \frac{1}{2}\int_0^1 \|v(t)\|^2 dt + g(q(1), q_1)$, with $\dot{q}(t) = v(t)\cdot q(t)$ and $q(0) = q_0$.

Proposition 1

Let v^* be a minimizer. If $\partial_1 g(q, q_1)$ can be written as a vector-valued density along S, then there exists $t \mapsto u(t)$ a vector-valued density on S along the trajectory, such that

$$v^*(t,x) = \int_S K(x, q(t,s))u(s)ds = K_V(u(t) \otimes q_* \mathrm{vol}_S).$$

Then

$$\dot{q}(t) = K_V(u(t) \otimes q_* \mathrm{vol}_S)\cdot q(t) =: K_{q(t)}u(t)$$

Reduction

The original problem is then equivalent to minimizing

$$J(u) = \frac{1}{2}\left(u(t), K_{q(t)}u(t)\right) dt + g(q(1), q_1),$$

with $q(0) = q_0$ and $\dot{q}(t) = K_{q(t)}u(t)$, where

$$K_q u(s) = \int_S K(q(s), q(s'))u(s')ds',$$

and

$$(u, K_q u) = \int_S u(s) \cdot (K_q u)(s)ds.$$

Application to our problem

Application to the heart

The motion is not realistic.

$\longrightarrow$ Need to account for the action of the muscle fibers.

Plan

Fibers

Accepted model for direction of fibers on the surface:
At a point $q_0(s)$ of the embedded surface of the relaxed heart, the direction $L_0(q_0(s))$ is obtained by the following process.

- Find a unit vector $l_0(q_0(s))$ that is horizontal and tangent to $q_0(S)$ at $q_0(s)$, oriented counterclockwise around the vertical axis.
- Rotate it by 45 (mouse) or 60 (human) degrees counterclockwise around the outer normal direction to $q_0(S)$ at $q_0(s)$: $L_0(q_0(s)) = R_{n(q_0(s)),\pi/4} l(q_0(s))$.

$L : q_0(S) \mapsto \mathbb{R}^3$ is a unit vector field along the immersion q_0.
The muscle fibers are the integral curves of L_0 without boundary, parametrized by arclength.

Keeping track of fibers

In the LDDMM framework, the shape is deformed through the flow of a vector field $t \mapsto v(t) \in V$: $\dot{q}(t) = v(t) \cdot q(t)$. Since L_0 reprensents the tangent to a curve, it is transported by the differential of this flow, so that

$$\partial_t L(t, q(t,s)) = P^{\perp}_{L(t,q(t,s))} dv(t, q(t,s)).L(t, q(t,s)),$$

which we write abbreviate $\dot{L}(t) = P^{\perp}_{L(t)} dv(t).L(t)$.

Here, $P^{\perp}_a = I_3 - aa^T$ is the projection on the plane perpendicular to a, a unit vector in $\mathbb{R}^3$.

Keeping track of fibers

For v of the form $K_q u$, we get the reduced control system

$$\begin{cases} \dot{q}(t,s) = \displaystyle\int_S K(q(s),q(s'))u(s')ds', \\ \dot{L}(t,q(t,s)) = P^{\perp}_{L(t)} \displaystyle\int_S [\partial_1 K(q(s),q(s')).L(q(s))]u(s')ds', \end{cases}$$

for almost every t in $[0,1]$ and every s in S.

Not finished: we just added additional (and useless) data. Need to constrain the control u so that it somewhat models the fibers.

Plan

Deformation induced by a contraction

Main Idea: The heart is foliated by curves (the heart fibers), and the contraction of the heart is caused by each fiber getting shorter.

Kinematic motion of a curve contracting: $t \mapsto c_t$ of a constant speed curve $\tau \mapsto c(\tau)$ **without boundary** induces a motion along the osculating plane of that curve, i.e., along a combination of the tangent $T(\tau) = \partial_s c(\tau)$ and the normal $N(\tau) = \partial_\tau T(\tau)$ to the curve.

Then, if we are not interested in the parametrization of the curve, but only in its shape, we can restrict ourselves to deformation along the normal N, that is, so that

$$\partial_\tau c_\tau(\tau) = u(t, \tau) N(t, \tau),$$

for some real-valued control field u along c.

Deformation induced by a contraction through LDDMM

We use the LDDMM framework to regularize this motion:

$$v(t, c_t(\tau)) = \int K(c_t(\tau), c_t(\tau'))u(\tau')N_t(\tau')d\tau'.$$

Control

Let $c(\cdot) : \mathbb{R} \to q(S)$ be an integral curve of L:

$$\partial_\tau c(\tau) = L(c(\tau))$$

Then, c is parametrized by arclength, so the normal is given by

$$\partial_\tau L(c(\tau)) = dL(c(\tau)).L(c(\tau)) =: N_L(c(\tau)).$$

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Hence, we choose a control of the form $u(s)N_{L_0}(q_0(s))$, regularized through the kernel:

$$\dot{q}(t) = K_{q(t)}u(t)N_{L(t)} = \int_S K(q(s), q(s'))u(t, s')N_{L(t)}(q'(s))ds',$$

In a way, this represents the interactions between the various points of the heart.

Control system

We finally get our optimal control problem: minimize over every $u \in L^2(0,1;\mathcal{C}^0(S))$ the functional

$$J(u) = \frac{1}{2}\int_0^1 (u(t)N_{L(t)}, K_q(t)u(t)N_{L(t)})dt + g(q(1), q_1),$$

where $q(0) = q_0$, $L(0) = L_0$, and, for almost every t in $[0,1]$,

$$\begin{cases} N_{L(t)} = dL(t).L(t), \\ \dot{q}(t) = K_{q(t)}u(t)N_{L(t)}, \\ \dot{L}(t) = P^{\perp}_{L(t)}d(K_{q(t)}u(t)N_{L(t)}).L(t). \end{cases}$$

Application to our problem

Plan

Adding tracking

In order to counteract LDDMM's tendency to "bounce", we proceed as follows:

1. We do a regular LDDMM matching, obtaining a deformation $\tilde{q}(\cdot)$ with no torsion.
2. we add a term

$$\int_0^1 g(q(t), \tilde{q}(t))dt$$

in the cost of the fibered model.

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Adding tracking

To counteract LDDMM's tendency to "bounce", we add a term with elastic energy of the deformation in the cost function.

Thank you for your attention!