Overview

Introduction

Rank preserving property

Secant varieties

Prolongations and general rank preserving property

Rank decomposition

Let A, B and C be vector spaces over $\mathbb{C}$, and $T \in A \otimes B \otimes C$.

T is said to have *rank one* if $T_{ijk} = a_i b_j c_k$ for some vectors a, b, c.

Definition

The *rank*, denoted by $\mathrm{rk}(T)$, of a tensor T is the minimum integer r such that

$$T = \sum_{i=1}^{r} a_i \otimes b_i \otimes c_i.$$

Such a decomposition is called a *rank-r decomposition*.

Border rank

The set of tensors with rank $\leq r$ is not closed.

Definition

The *border rank*, denoted by $\text{brk}(T)$, of a tensor T is the minimum integer r such that T is a limit of rank-r tensors.

Definition

The *symmetric border rank*, denoted by $\text{brk}_S(T)$, of a symmetric tensor T is the minimum integer r such that T is a limit of symmetric rank-r tensors.

X-rank decomposition

Let $X \subset \mathbb{P}V$ be an irreducible nondegenerate projective variety.

- nondegenerate (X is not contained in a hyperplane) $\implies$ for any $v \in V$, $v = x_1 + \cdots + x_m$ for some $x_1, \ldots, x_m \in \hat{X}$.
- projective $\implies$ $v = x_1 + \cdots + x_m$ instead of $v = c_1 x_1 + \cdots + c_r x_m$ for some coefficients $c_1, \ldots, c_m$.
- irreducible $\implies$ no ambiguity.

Definition (Zak)

For $v \in V$, the *X-rank* of v, denoted by $\mathrm{rk}_X(v)$, is the minimum integer r such that

$$v = x_1 + \cdots + x_r,$$

where $x_1, \ldots, x_r \in \hat{X}$.

Border rank

The set of tensors with rank $\leq r$ is not closed.

Definition

The *border rank*, denoted by $\mathrm{brk}(T)$, of a tensor T is the minimum integer r such that T is a limit of rank-r tensors.

Definition

The *symmetric border rank*, denoted by $\mathrm{brk}_S(T)$, of a symmetric tensor T is the minimum integer r such that T is a limit of symmetric rank-r tensors.

X-rank decomposition

Let $X \subset \mathbb{P}V$ be an irreducible nondegenerate projective variety.

- nondegenerate (X is not contained in a hyperplane) $\implies$ for any $v \in V$, $v = x_1 + \cdots + x_m$ for some $x_1, \ldots, x_m \in \widehat{X}$.
- projective $\implies$ $v = x_1 + \cdots + x_m$ instead of $v = c_1 x_1 + \cdots + c_r x_m$ for some coefficients $c_1, \ldots, c_m$.
- irreducible $\implies$ no ambiguity.

Definition (Zak)

For $v \in V$, the *X-rank* of v, denoted by $\mathrm{rk}_X(v)$, is the minimum integer r such that

$$v = x_1 + \cdots + x_r,$$

where $x_1, \ldots, x_r \in \widehat{X}$.

Border and Generic X-rank

Definition

For $v \in V$, the *X-border-rank* of v, denoted by $\mathrm{brk}_X(v)$, is the minimum integer r such that v is a limit of X-rank-r points.

Definition

An X-rank r is called *generic* if the set of X-rank-r points contains a Zariski open subset of V.

There is only one generic X-rank over $\mathbb{C}$.

Examples

The Segre variety is defined to be the image of

$$\begin{aligned}\mathrm{Seg}: \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n &\to \mathbb{P}(V_1 \otimes \cdots \otimes V_n)\\ ([v_1], \ldots, [v_n]) &\mapsto [v_1 \otimes \cdots \otimes v_n].\end{aligned}$$

The Veronese variety is defined to be the image of

$$\nu_d : \mathbb{P}V \to \mathbb{P}S^d V, \quad [v] \mapsto [v^{\otimes d}].$$

Example

- "The tensor rank in $V_1 \otimes \cdots \otimes V_n$" $= \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$-rank.
- "The symmetric rank in $S^d V$" $= \nu_d(\mathbb{P}V)$-rank.
- the generic rank $r_g(\mathrm{Seg}(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{n-1})) = \lceil \frac{n^3}{3n-2} \rceil$ if $n \neq 3$.
- $\mathrm{rk}_X(v) \geq \mathrm{brk}_X(v)$.

Overview

Introduction

Rank preserving property

Secant varieties

Prolongations and general rank preserving property

Rank preserving property

Let $X \subseteq \mathbb{P}V$ be a nondegenerate irreducible projective variety, and $L \subset \mathbb{P}V$ be a linear subspace. Let $Y := X \cap L$.

Definition (Buczyński–Ginensky–Landsberg)

Y is said to have the *rank-r preserving property* for a fixed r if

- the linear span $\mathrm{Span}\{Y\}$ is L;
- $\mathrm{rk}_X(v) = r$ for all $v \in L$ with $\mathrm{rk}_Y(v) = r$.

Definition

Y is said to have the *general rank-r preserving property* if

- $\mathrm{Span}\{Y\} = L$;
- $\mathrm{rk}_X(v) = r$ for a general rk_Y-r point $v \in L$.

Similarly we can define the *border rank-r preserving property* by replacing rk with brk.

Example – symmetric version of Strassen's conjecture

Conjecture

Given vector spaces V and W such that $V \cap W = \{0\}$, and tensors $A \in S^d V$ and $B \in S^d W$. Then

$$\mathrm{rk}_S(A \oplus B) = \mathrm{rk}_S(A) + \mathrm{rk}_S(B),$$

where $A \oplus B \in S^d(V \oplus W)$.

$$X = \nu_d(\mathbb{P}(V \oplus W)), \quad L = \mathbb{P}(S^d V \oplus S^d W)$$
$$Y = X \cap L = \nu_d(\mathbb{P}V) \cup \nu_d(\mathbb{P}W)$$

The symmetric version of Strassen's direct sum conjecture asks if Y has the symmetric rank-r preserving property.

Example – Vandermonde rank decompositions

Let V be an $(n+1)$-dimensional vector space. Fix a basis $\{e_1, \dots, e_{n+1}\}$ for V. A symmetric tensor

$$H := \sum_{1 \leq i_1, \dots, i_d \leq n+1} H_{i_1 \dots i_d} e_{i_1} \cdots e_{i_d} \in S^d V$$

is called *Hankel* if there is a vector $h := (h_0, \dots, h_{nd})$ such that

$$H_{i_1 \dots i_d} = h_{i_1 + \dots + i_d - d}.$$

Identify V with $S^n W$ for some 2-dim vector space W. Then H is Hankel if and only if H has the form

$$H = \sum_{i=1}^{r} (w_i^{\otimes n})^{\otimes d}, \tag{1}$$

where $w_1, \dots, w_r \in W$, and r is minimum. r is called the *Vandermonde rank* of H.

Vandermonde rank decompositions continued

Conjecture (Nie – Ye' 16)

For a general Vandermonde rank-r Hankel tensor, its symmetric rank and rank are also r.

$$X_1 = \nu_d(\mathbb{P}V), \quad X_2 = \mathrm{Seg}(\mathbb{P}V^{\times d}), \quad L = \mathbb{P}(S^{dn}W)$$
$$Y = X_1 \cap L = X_2 \cap L = \nu_{dn}(\mathbb{P}W)$$

Then the conjecture by Nie and Ye asks if Y has the general (symmetric) rank-r preserving property.

Overview

Introduction

Rank preserving property

Secant varieties

Prolongations and general rank preserving property

Join Variety

Geometric definition:

For projective varieties $X_1, \dots, X_r \subseteq \mathbb{P}V$ over $\mathbb{C}$, let $\widehat{X}_i$ denote the affine cone of X_i.

Definition

The *join map* is defined by

$$J : \widehat{X}_1 \times \cdots \times \widehat{X}_r \to V, \quad (x_1, \dots, x_r) \mapsto x_1 + \cdots + x_r.$$

The Zariski closure of the image $J(\widehat{X}_1 \times \cdots \times \widehat{X}_r)$ in V is the affine cone of some projective variety, which is denoted by $J(X_1, \dots, X_r)$, and called the *join variety* of $X_1, \dots, X_r$.

Secant varieties

Definition

When $X_1 = \cdots = X_r = X$, we denote $J(X_1, \ldots, X_r)$ by $\sigma_r(X)$, and call it the *rth secant variety* of X.

Equivalently,

Definition

When X is an irreducible projective variety,

$$\sigma_r(X) = \overline{\bigcup_{x_1,\ldots,x_r \text{ general in } X} \operatorname{Span}\{x_1, \ldots, x_r\}}.$$

Connection with tensors

Let $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_n)$ be the Segre variety. The image of the join map $J(\widehat{X}^{\times r})$ is the set of tensors with rank $\leq r$, and $\widehat{\sigma_r(X)}$ is the set of tensors with border rank $\leq r$.

Similarly, let $Y = \nu_d(\mathbb{P}V)$ be the Veronese variety. Then $J(\widehat{Y}^{\times r})$ is the set of symmetric tensors with symmetric rank $\leq r$, and $\widehat{\sigma_r(Y)}$ is the set of symmetric tensors with symmetric border rank $\leq r$.

(Border) Rank preserving property

Theorem (Nie – Ye)

There is a Hankel tensor whose Vandermonde rank is greater than its symmetric rank.

Theorem (Schönhage)

There are $T_1 \in V_1 \otimes V_2 \otimes V_3$ and $T_2 \in W_1 \otimes W_2 \otimes W_3$ such that

$$\mathrm{brk}(T_1 \oplus T_2) < \mathrm{brk}(T_1) + \mathrm{brk}(T_2).$$

Theorem (Shitov)

There are $T_1 \in V_1 \otimes V_2 \otimes V_3$ and $T_2 \in W_1 \otimes W_2 \otimes W_3$ such that

$$\mathrm{rk}(T_1 \oplus T_2) < \mathrm{rk}(T_1) + \mathrm{rk}(T_2).$$

Reasonable to consider the general rank preserving property of Y, i.e.,

$$\sigma_r(Y) \not\subseteq \sigma_{r-1}(X).$$

Overview

Introduction

Rank preserving property

Secant varieties

Prolongations and general rank preserving property

Prolongation

Definition

Let A be a vector subspace of $S^d V$. The k-th *prolongation* of A, denoted by $A^{(k)}$, is defined by

$$A^{(k)} = \{f \in S^{d+k} V \mid \frac{\partial^k f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}} \in A, |\boldsymbol{\alpha}| = k\}.$$

Equivalently,

Definition

For a subspace $A \subset S^d V$, $A^{(k)} = (A \otimes S^k V) \cap S^{d+k} V$.

General Vandermonde rank preserving property

For a homogeneous ideal I, the *initial degree* of I, denoted by $\alpha(I)$, is defined by $\alpha(I) = \min\{k \mid I_k \neq 0\}$.

Let $X \subset \mathbb{P}V$ be a nondegenerate irreducible projective variety, and $L \subseteq V$ be a linear subspace.

Proposition

Let $\alpha(I(X)) = k$, and $Y = X \cap \mathbb{P}L$. Assume (i) $\text{Span}\{Y\} = \mathbb{P}L$, (ii) Y is irreducible. If $I_k^{((k-1)(r-2))}$ is not generated by the linear forms defining $\mathbb{P}L$, then Y has the general rank-r preserving property.

Corollary (Q. – Lim)

For a general Vandermonde rank-r Hankel tensor, its symmetric rank and rank are also r, where $r \leq \lceil \frac{dn+1}{2} \rceil$.

General rank-r preserving property

Assume $V \cap W = \{0\}$. Let $X \subseteq \mathbb{P}V$ and $Y \subseteq \mathbb{P}W$ be nondegenerate subvarieties.

Lemma

$I_\ell(J(X,Y)) \subseteq I_k(Y)^{(\ell-k)} \cap I_{\ell-k}(X)^{(k)}$ *for* $0 \leq k \leq \ell$.

Let $\dim V = n$, $\dim W = m$, and $k = \lfloor d/2 \rfloor$.

Corollary (Q. – Lim)

When $r \leq \binom{n+k-1}{k}$ and $s \leq \binom{m+k-1}{k}$, for a general rk_S-r tensor $T \in S^d V$ and a general rk_S-s tensor $T' \in S^d W$,

$$\mathrm{rk}_S(T \oplus T') = r + s.$$

General Vandermonde rank preserving property

For a homogeneous ideal I, the *initial degree* of I, denoted by $\alpha(I)$, is defined by $\alpha(I) = \min\{k \mid I_k \neq 0\}$.

Let $X \subset \mathbb{P}V$ be a nondegenerate irreducible projective variety, and $L \subseteq V$ be a linear subspace.

Proposition

Let $\alpha(I(X)) = k$, and $Y = X \cap \mathbb{P}L$. Assume (i) $\mathrm{Span}\{Y\} = \mathbb{P}L$, (ii) Y is irreducible. If $I_k^{((k-1)(r-2))}$ is not generated by the linear forms defining $\mathbb{P}L$, then Y has the general rank-r preserving property.

Corollary (Q. – Lim)

For a general Vandermonde rank-r Hankel tensor, its symmetric rank and rank are also r, where $r \leq \lceil \frac{dn+1}{2} \rceil$.

General rank-r preserving property

Assume $V \cap W = \{0\}$. Let $X \subseteq \mathbb{P}V$ and $Y \subseteq \mathbb{P}W$ be nondegenerate subvarieties.

Lemma

$I_\ell(J(X,Y)) \subseteq I_k(Y)^{(\ell-k)} \cap I_{\ell-k}(X)^{(k)}$ *for* $0 \leq k \leq \ell$.

Let $\dim V = n$, $\dim W = m$, and $k = \lfloor d/2 \rfloor$.

Corollary (Q. – Lim)

When $r \leq \binom{n+k-1}{k}$ and $s \leq \binom{m+k-1}{k}$, for a general rk_S-r tensor $T \in S^d V$ and a general rk_S-s tensor $T' \in S^d W$,

$$\mathrm{rk}_S(T \oplus T') = r + s.$$