Topology of Vortex Trajectories in Wake-Like Flows

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SIAM Conference on Dynamical Systems
Snowbird, Utah
20 May 2015

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Overview

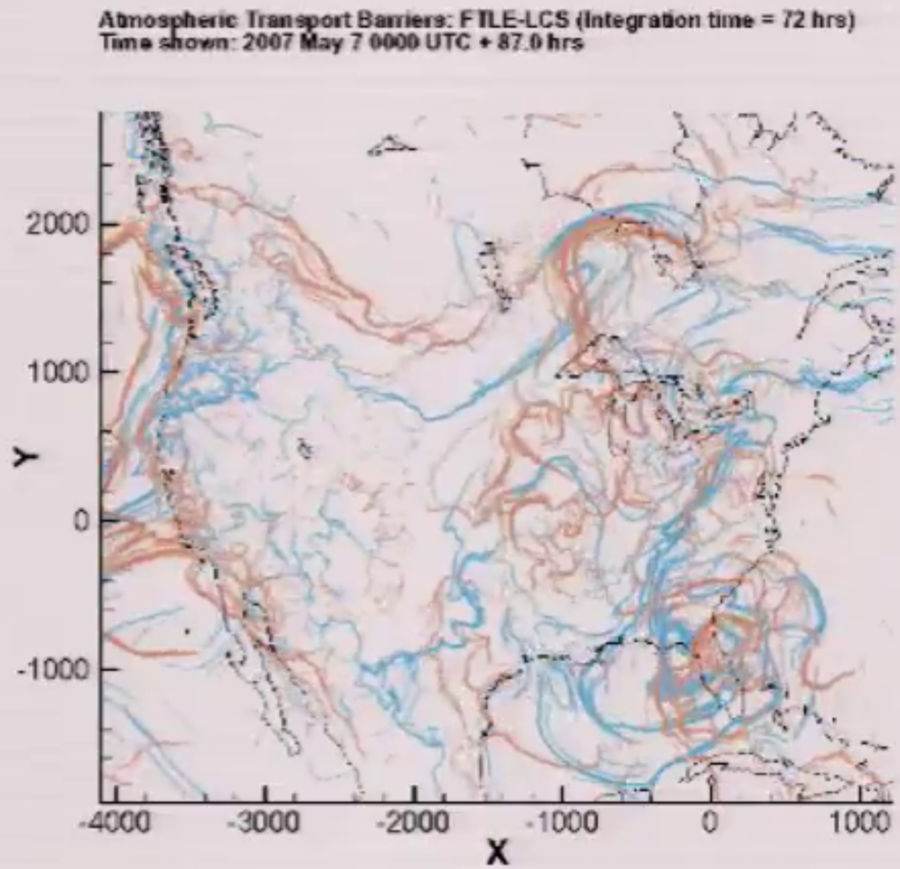
- Background: 'coherent structures', braiding, and topological chaos
- A Hamiltonian point vortex model inspired by '2P mode' bluff body wakes
- Braiding of relative vortex motions in wake-like flows

How do we understand and predict time-dependent transport in flows with very complex dynamics?

Lagrangian Coherent Structures (LCS)

- *hyperbolic* LCS
- ridges of the FTLE field

Haller & Yuan (2000) *Physica D*
Peacock & Haller (2013) *Phy. Today*
Haller (2015) *Ann. Rev. Fluid Mech.*



sequences of LCS over the United States
(S.D. Ross, private communication)
orange = repelling *blue = attracting*

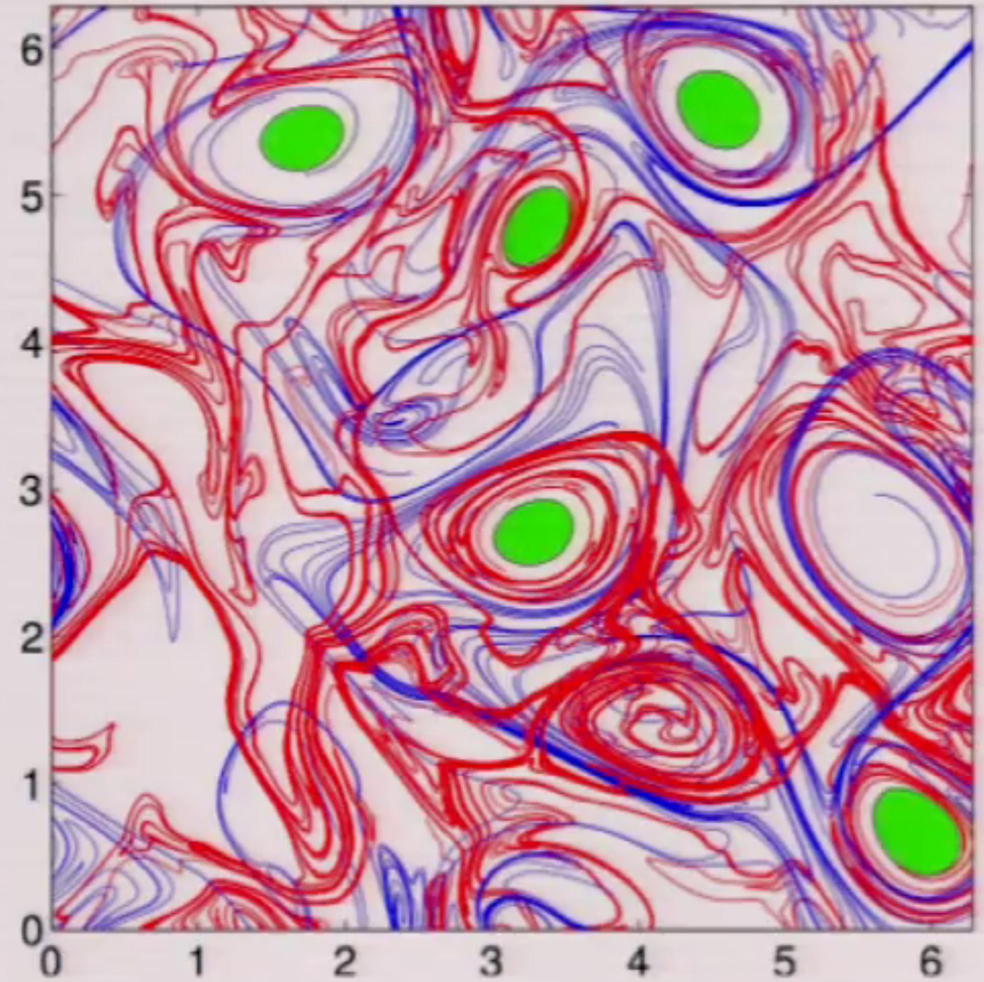
How do we understand and predict time-dependent transport in flows with very complex dynamics?

Lagrangian Coherent Structures (LCS)

- *elliptic* LCS
- rotation-dominated (i.e. vortex-like) regions that move without much stretching or folding

Haller (2005) *JFM*

Farazmand & Haller (2015) *arXiv*



attracting/repelling hyperbolic LCS *(red/blue)*
elliptic LCS *(green)*
(Mohammad Farazmand, Wikipedia)

How do we understand and predict time-dependent transport in flows with very complex dynamics?

Almost Invariant Sets (AIS)

regions of phase space that 'stick together' for a significant length of time

Dellnitz & Junge (1999) *SIAM JNA*

Froyland & Dellnitz (2003) *SIAM JSC*

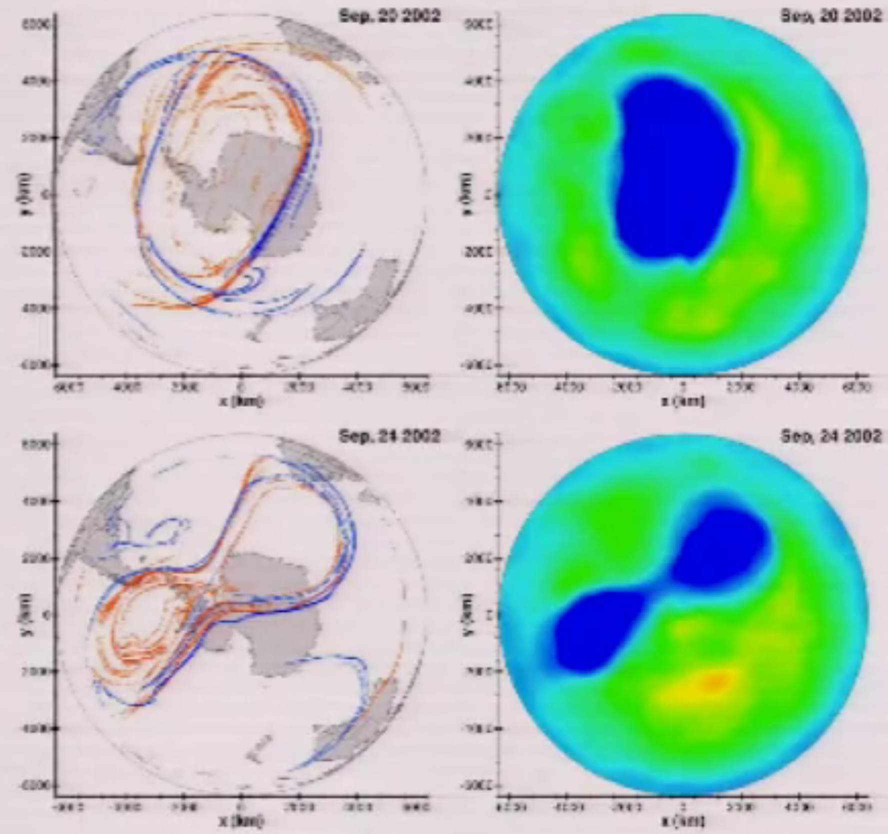
Froyland (2013) *Physica D*

also: the Koopman operator

Mezic & Banaszuk (2004) *Physica D*

Budisic, Mohr & Mezic (2012) *Chaos*

splitting of the ozone hole



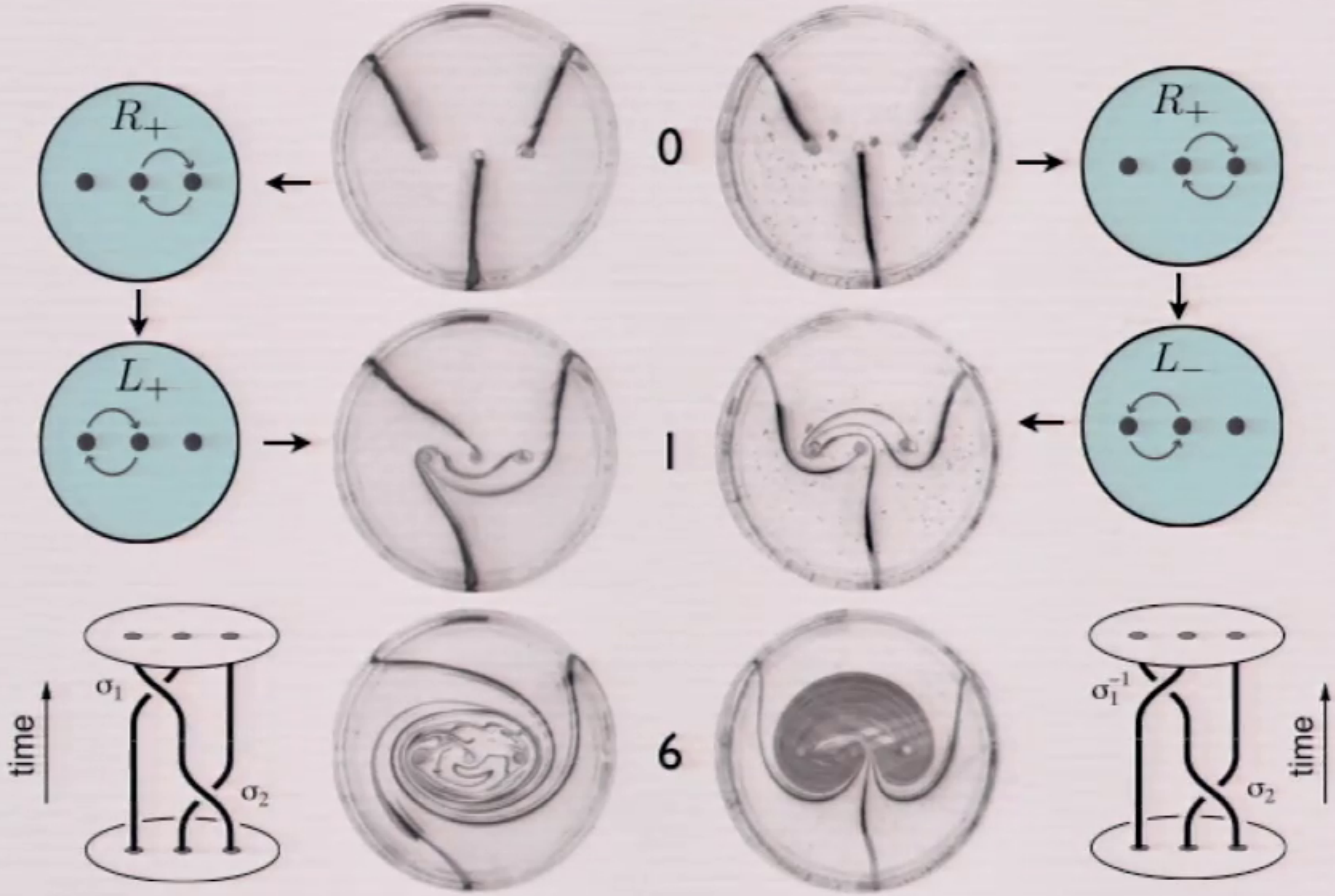
LCS AIS

F. Lekien & S.D. Ross (2010) *Chaos*

Topological fluid mechanics of stirring

J. Fluid Mech. (2000), vol. 403, pp. 277–304.

By PHILIP L. BOYLAND[1], HASSAN AREF[2] AND MARK A. STREMLER[2]



Thurston–Nielsen Classification Theorem

• Thurston (1988) • Casson & Bleiler (1988)

f = a stirrer motion

isotopy ↔ g = the "Thurston-Nielsen representative"

(i) **finite order** (FO): the *n*th iterate of g is the identity

(ii) **pseudoAnosov** (pA):

g has dense orbits, Markov partition with transition matrix A

$\lambda_{\mathrm{TN}} > 1$: *expansion* or *dilation* = PF eigenvalue of A

topological entropy $h_{\mathrm{TN}}(g) = \ln(\lambda_{\mathrm{TN}})$

Handel (1985): complex dynamics of pA map remain under isotopy

(iii) **reducible**: g contains both f.o. and pA regions

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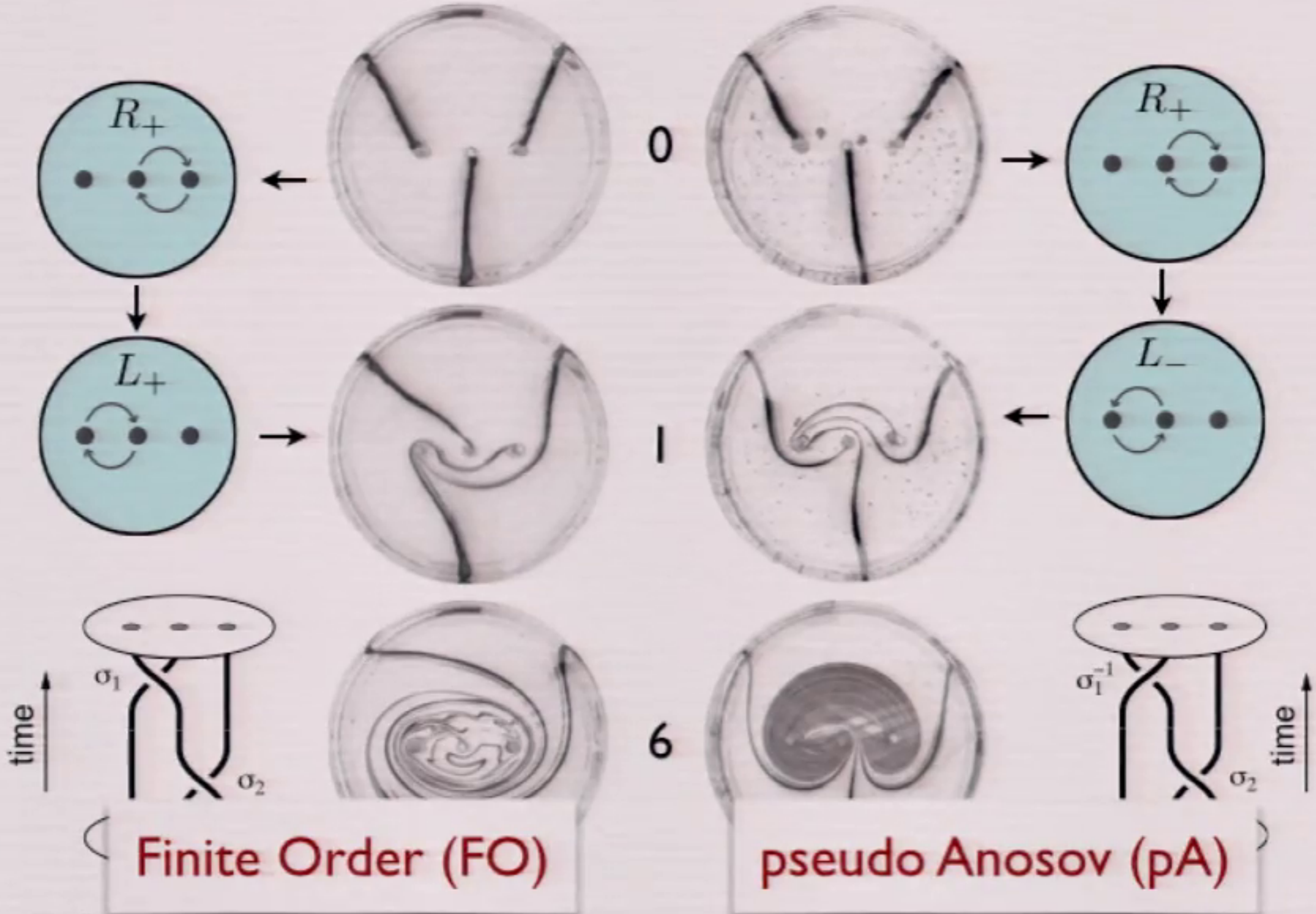
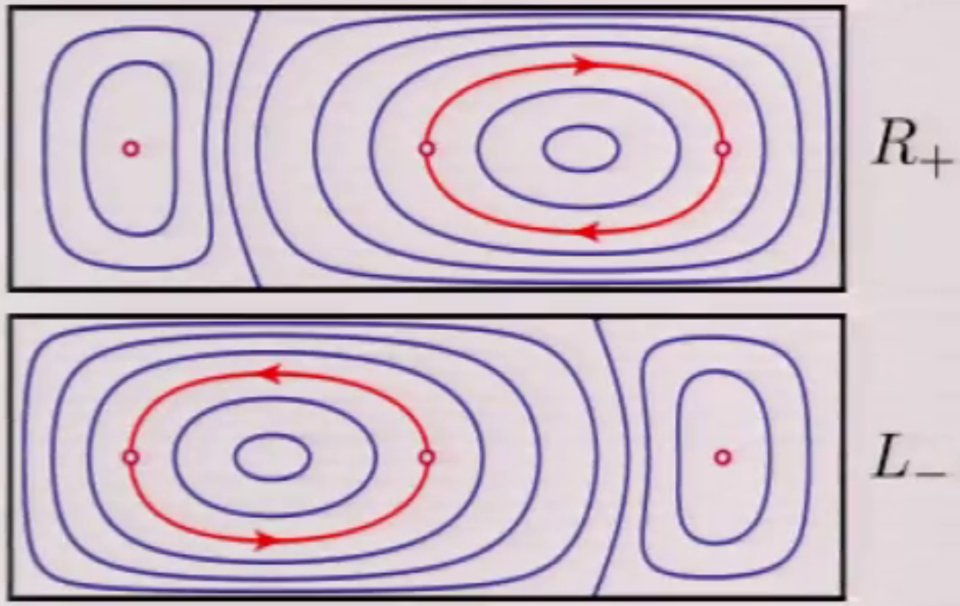
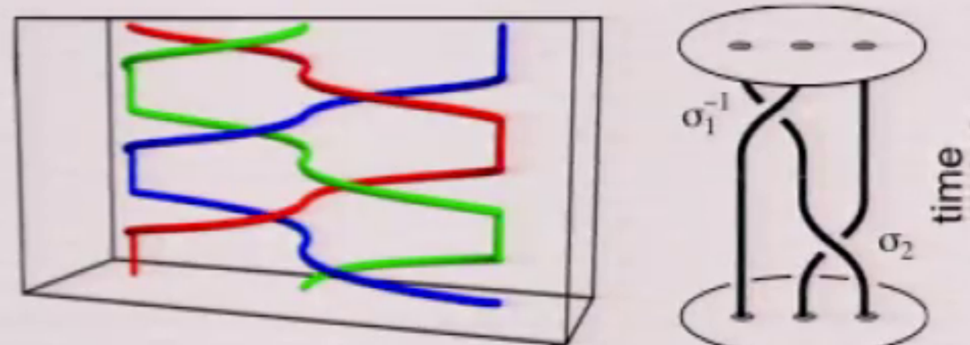


pA braiding of AIS from relatively simple motion: blinking viscous flow in a lid-driven cavity





apply *TN theorem* to mathematical braid

$$h_{\mathrm{TN},3} = \ln \lambda_{\mathrm{TN},3} = 0.96242\ldots$$

Boyland, Aref & Stremler (2000) *JFM*

topological entropy of the flow computed from line stretching

$$h_{\mathrm{flow}} = \ln \lambda \approx 0.968$$

Stremler, Ross, Grover & Kumar (2011) *PRL*; Grover, Ross, Stremler & Kumar (2012) *Chaos*

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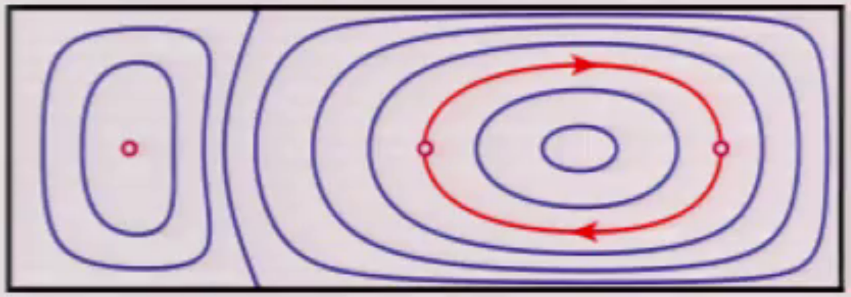
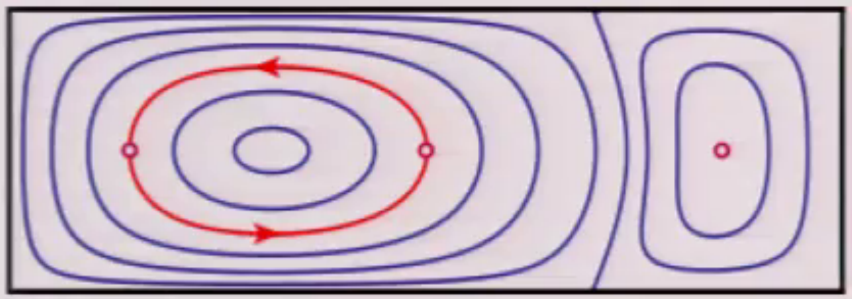
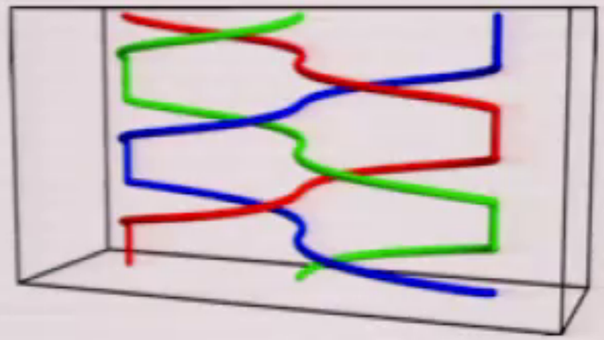
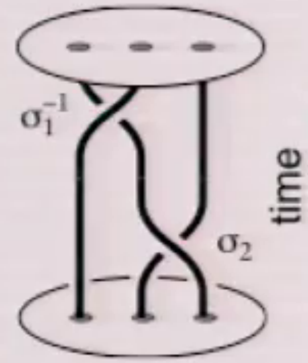


R_+



L_-





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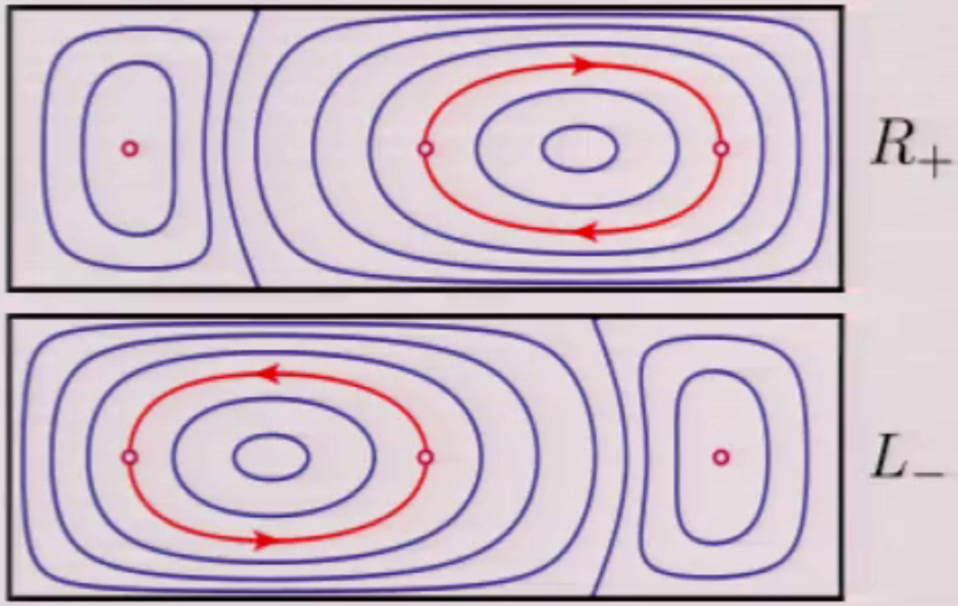

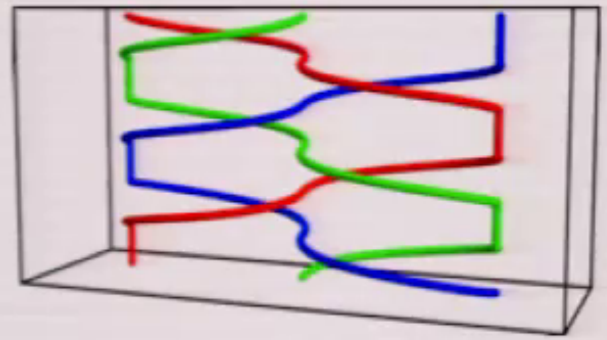

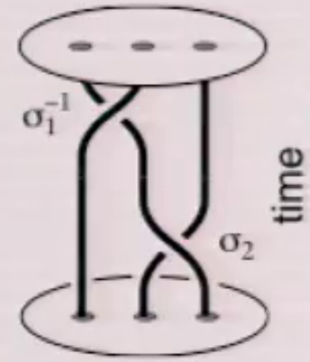

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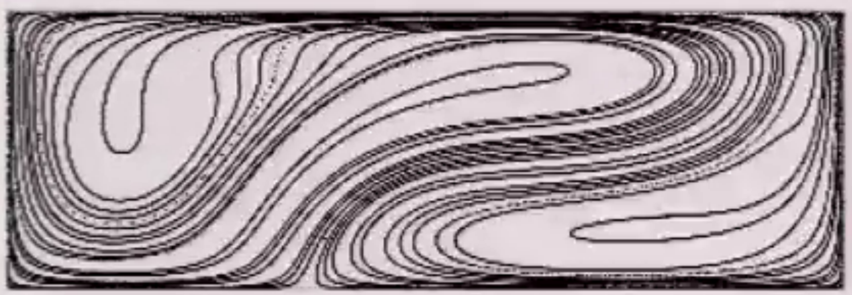

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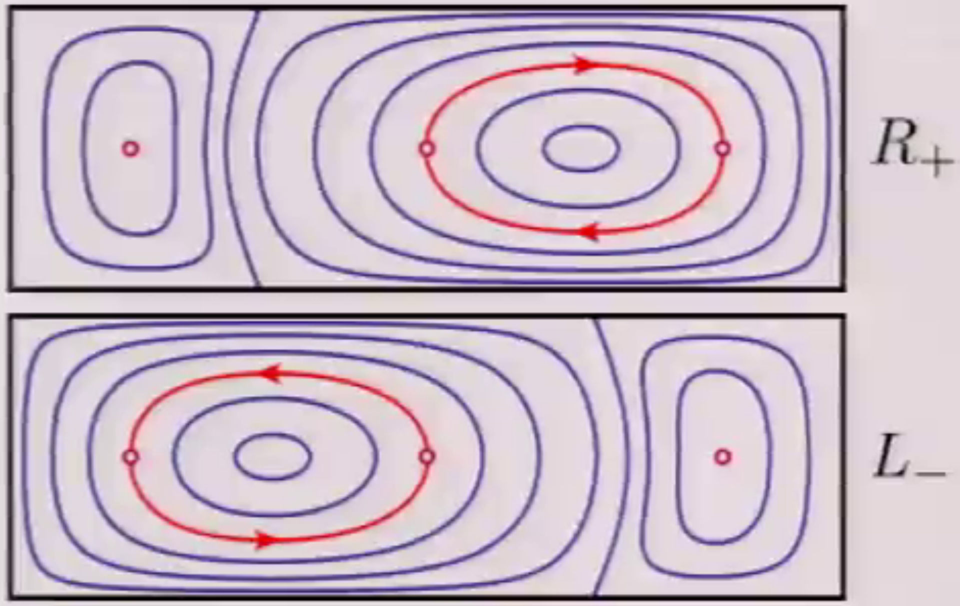

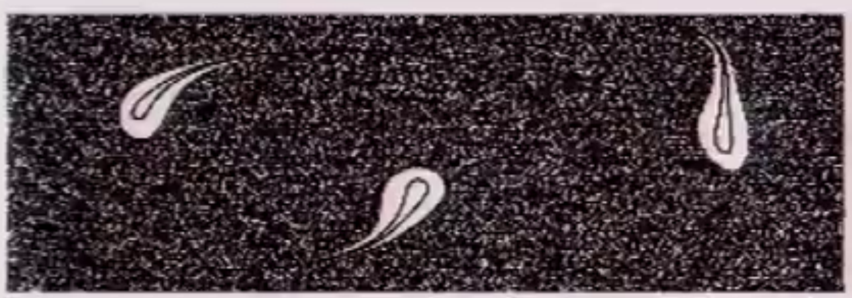

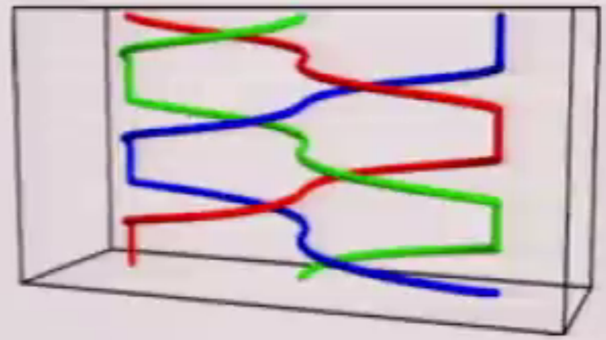

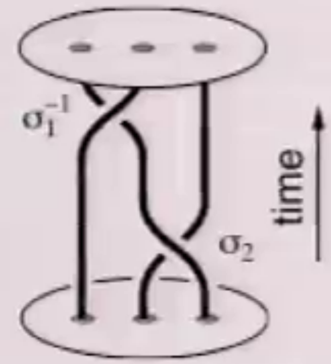

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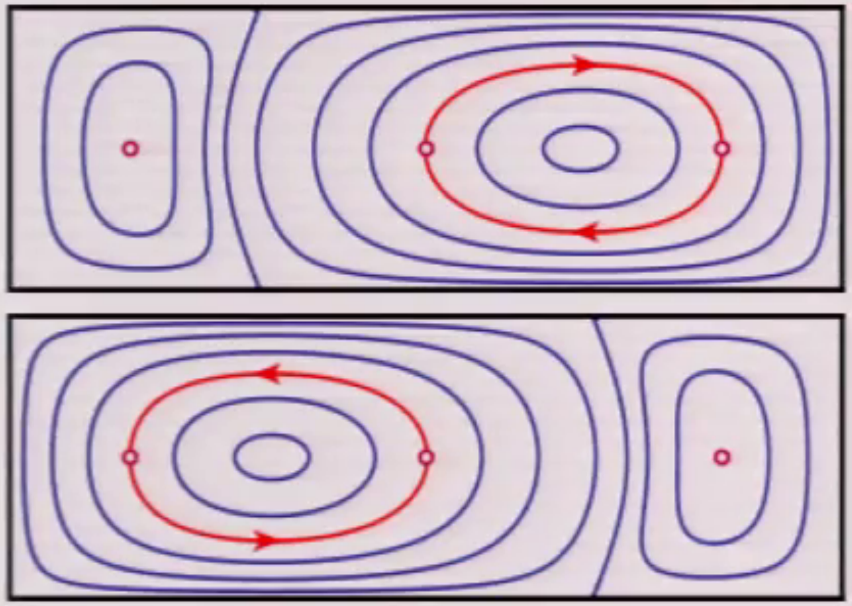

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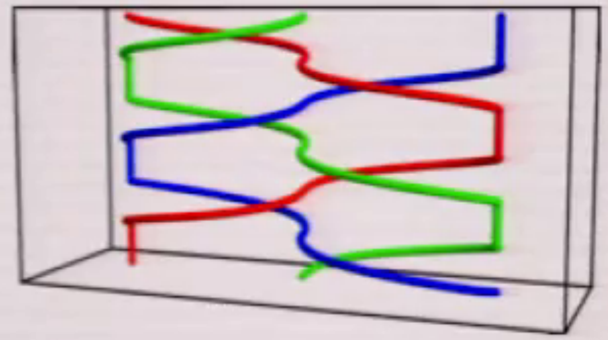

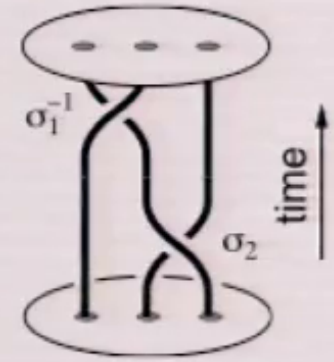

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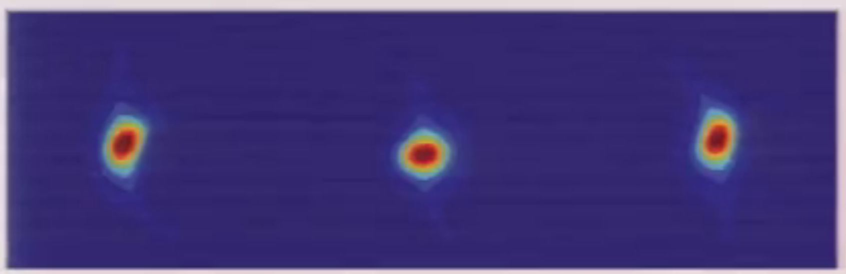

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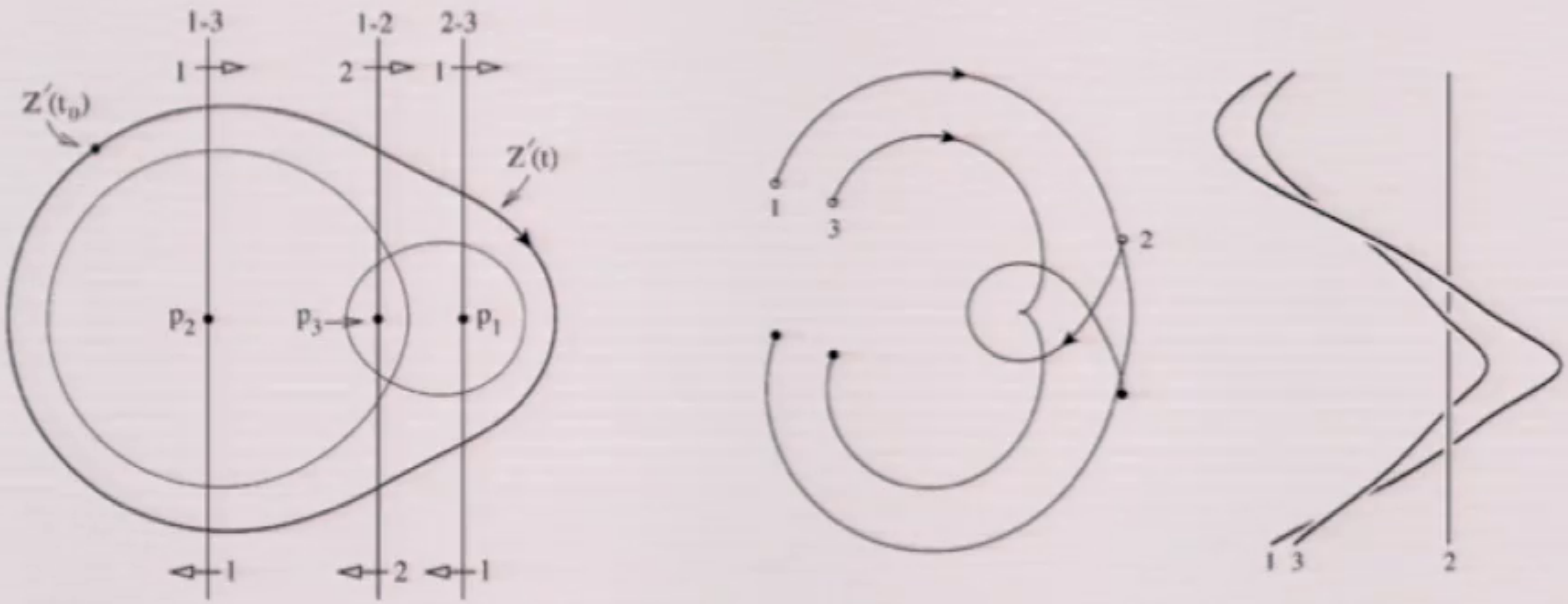
Topological fluid mechanics of point vortex motions   Physica D 175 (2003) 69–95

Philip Boyland[a,*], Mark Stremler[b], Hassan Aref[c]

- 3 vortices in the plane, zero net circulation $\sum_{\alpha} \Gamma_\alpha = 0$

Example: $\Gamma_1 : \Gamma_2 : \Gamma_3 = 2 : 1 : (-3)$



phase space   physical space   braid representation

every braid representation is *reducible* with all parts *finite order*

Topological fluid mechanics of point vortex motions

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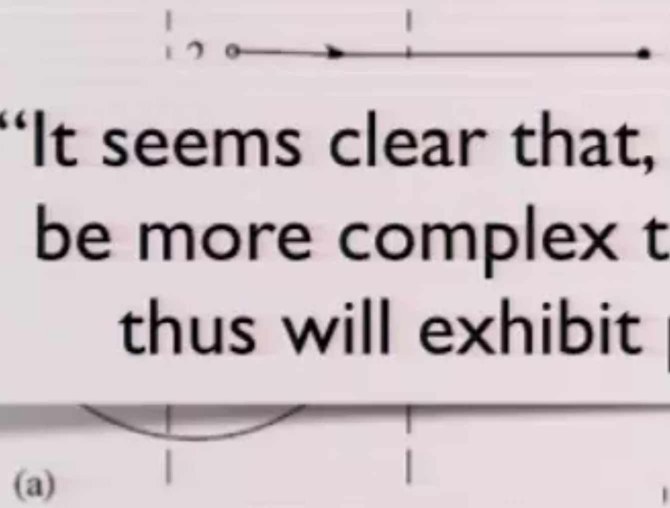
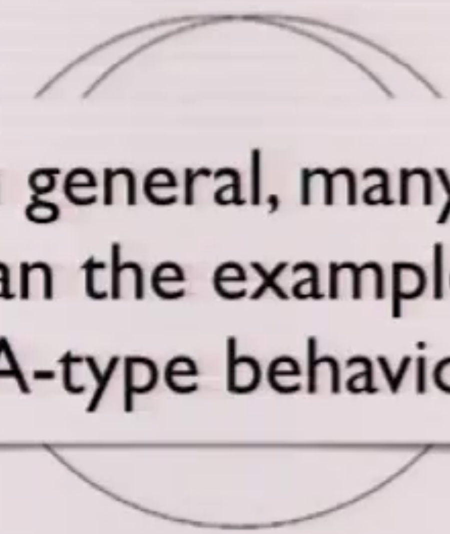
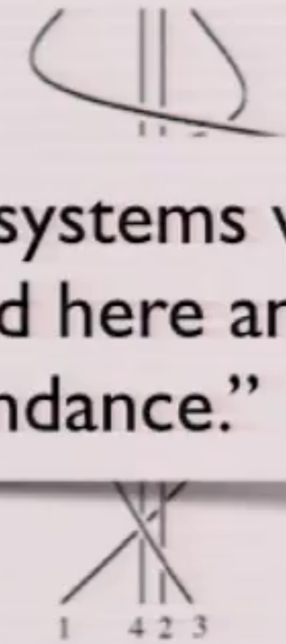


physical space



relative motion in projected plane



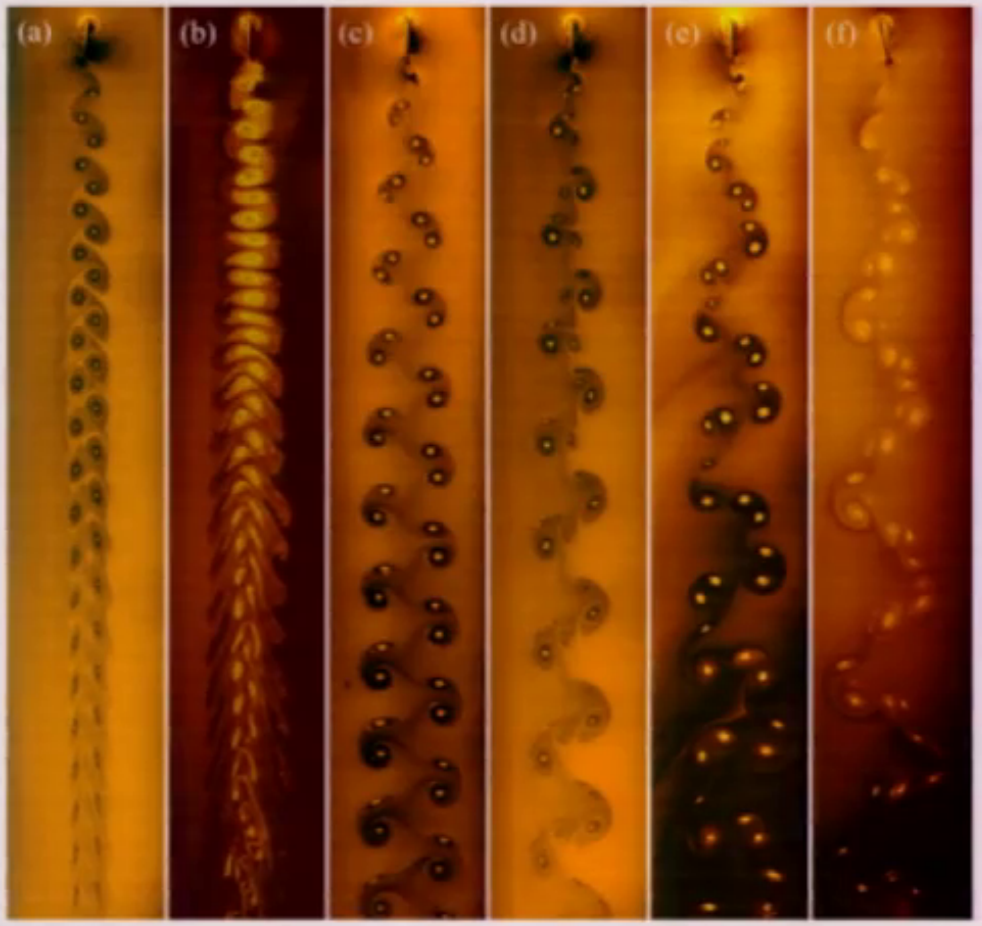
braid representation

"It seems clear that, in general, many vortex systems will be more complex than the examples studied here and thus will exhibit pA-type behavior in abundance."

a subset of the vortex motions generate a braid that is *pseudo Anosov*

Exotic wakes generated by a flapping foil in a flowing soap film

Schnipper, Andersen, and Bohr (2009) *J. Fluid Mechanics* 633, 411–423



2S 2S 2P 2P+2S 4P 4P+2S

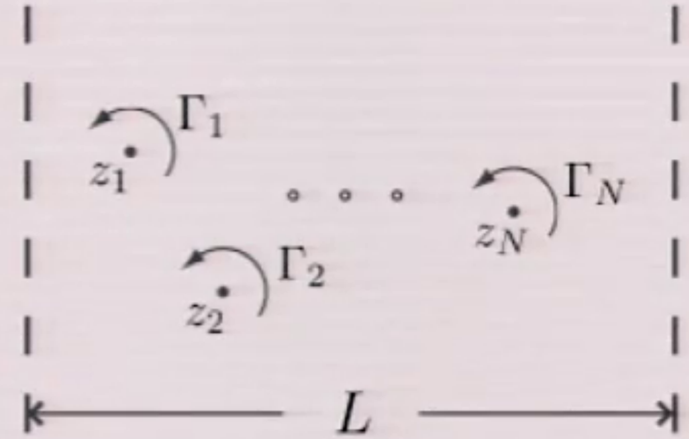
N point vortices in a *periodic strip*

spatially periodic array of point vortices in 2D potential flow

equations of motion:

Friedmann & Poloubarinova (1928)

$$\frac{dz_m^*}{dt} = \frac{1}{2L\,\mathrm{i}} \sum_{n=1}^{N}{}' \, \Gamma_n \cot\left[\frac{\pi}{L}(z_m - z_n)\right]$$



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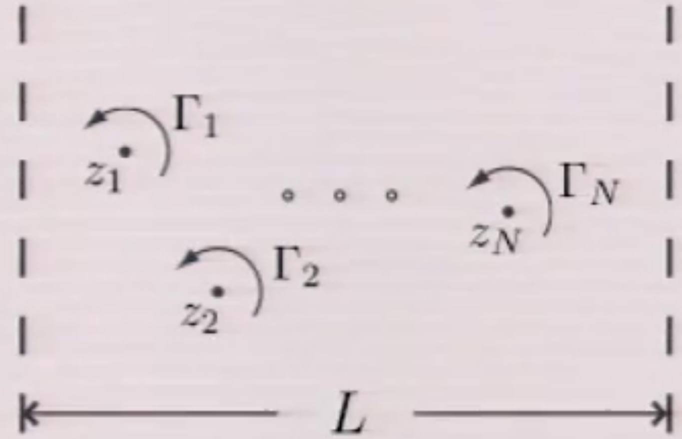


system is *Hamiltonian*: $\Gamma_n \frac{\mathrm{d}x_n}{\mathrm{d}t} = \frac{\partial H}{\partial y_n}$ $\quad \Gamma_n \frac{\mathrm{d}y_n}{\mathrm{d}t} = -\frac{\partial H}{\partial x_n}$

$$H = -\frac{1}{4\pi} \sum_{m,n=1}^{N}{}' \, \Gamma_m \Gamma_n \ln\left|\sin\left[\frac{\pi}{L}(z_m - z_n)\right]\right|$$

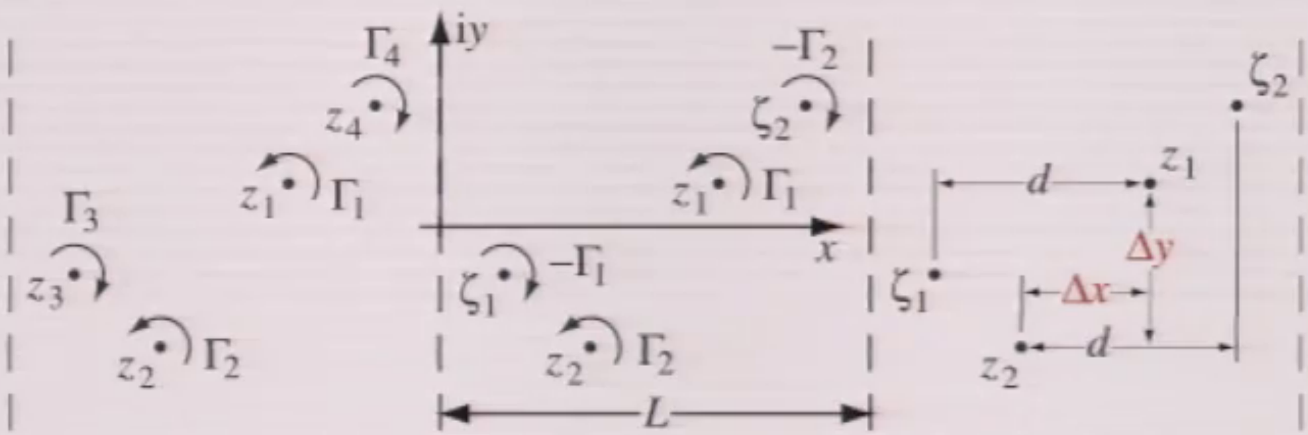
and preserves the *linear impulse*: $\Xi = Q + \mathrm{i}\,P = \sum_{n=1}^{N} \Gamma_n z_n$

N=4 point vortices in a periodic strip of width L

general 4-vortex problem in a periodic strip is not integrable

assume glide-reflective symmetry with a shift

$$\Gamma_3 = -\Gamma_1 \qquad z_3 = z_1^* - d = \zeta_1 \qquad z_4 = z_2^* + d = \zeta_2 \qquad \Gamma_4 = -\Gamma_2$$



spatial symmetry must be maintained by the dynamics

$$\frac{dz_1^*}{dt} = \frac{d\zeta_1}{dt} \qquad \frac{dz_2^*}{dt} = \frac{d\zeta_2}{dt} \qquad \Rightarrow \qquad d = nL/2 \quad n \in I$$

Constraining the 4-vortex system with $n = 1$

the *linear impulse*: $$Q + \mathrm{i}P = \Xi = \Gamma_1\,(z_1 - \zeta_1) + \Gamma_2\,(z_2 - \zeta_2)$$

$$\mathbb{Q} = Q/L\mathbb{S} = \gamma - 1/2 \qquad \mathbb{P} = P/L\mathbb{S} = 2\left[\gamma y_1 + (1-\gamma) y_2\right]/L$$

the *Hamiltonian*: $$\mathbb{S} = \Gamma_1 + \Gamma_2 \qquad \gamma = \Gamma_1/\mathbb{S}$$

$$\mathbb{H}(\Delta x, \Delta y; \gamma, \mathbb{P}) = -\frac{1}{2\pi}\left\{\ln\left[\frac{\sin^2(\pi\,\Delta x/L) + \sinh^2(\pi\,\Delta y/L)}{\cos^2(\pi\,\Delta x/L) + \sinh^2\left[\pi\mathbb{P} + \pi(1-2\gamma)\Delta y/L\right]}\right] - \frac{\gamma}{1-\gamma}\ln\Big[\cosh\left[\pi\mathbb{P} + 2\pi(1-\gamma)\Delta y/L\right]\Big] - \frac{1-\gamma}{\gamma}\ln\Big[\cosh\left[\pi\mathbb{P} - 2\pi\gamma\Delta y/L\right]\Big]\right\}$$

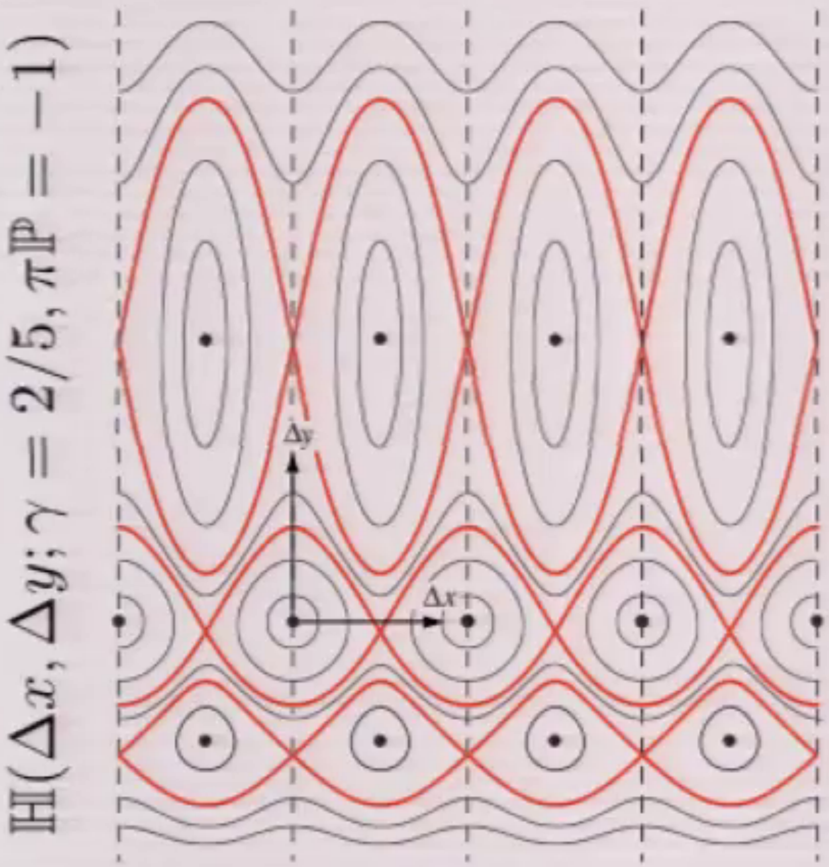
$$z_1 - z_2 = \Delta x + \mathrm{i}\,\Delta y$$

Stremler & Basu (2014) *Fluid Dynamics Research*

Results for the 4-vortex model problem

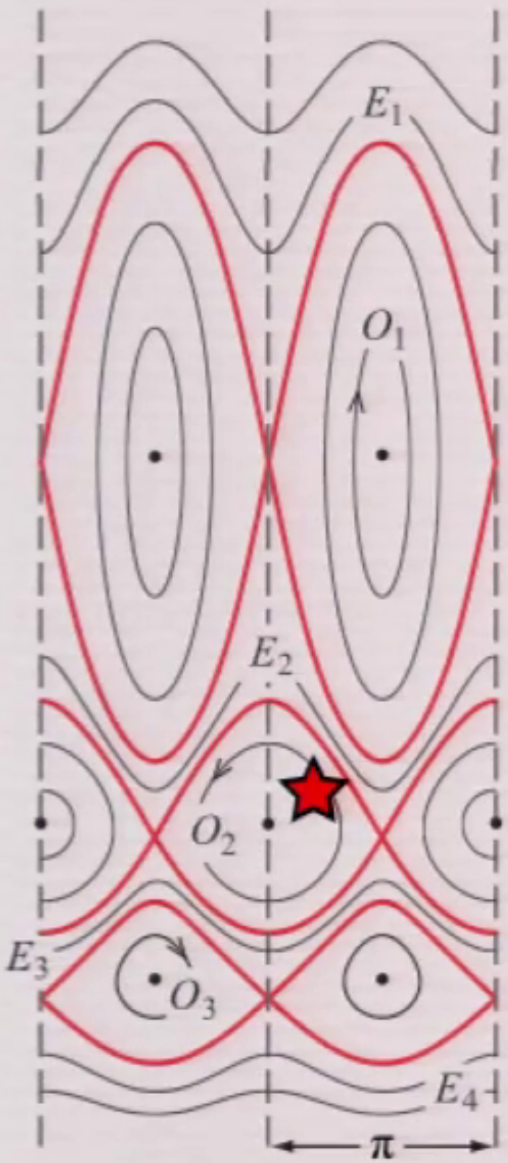
problem reduces to one in the $(\Delta x, \Delta y)-$ plane

$$2\mathbb{S}\,\frac{\mathrm{d}(\Delta x)}{\mathrm{d}t} = \frac{\partial \mathbb{H}}{\partial(\Delta y)} \qquad 2\mathbb{S}\,\frac{\mathrm{d}(\Delta y)}{\mathrm{d}t} = -\frac{\partial \mathbb{H}}{\partial(\Delta x)}$$

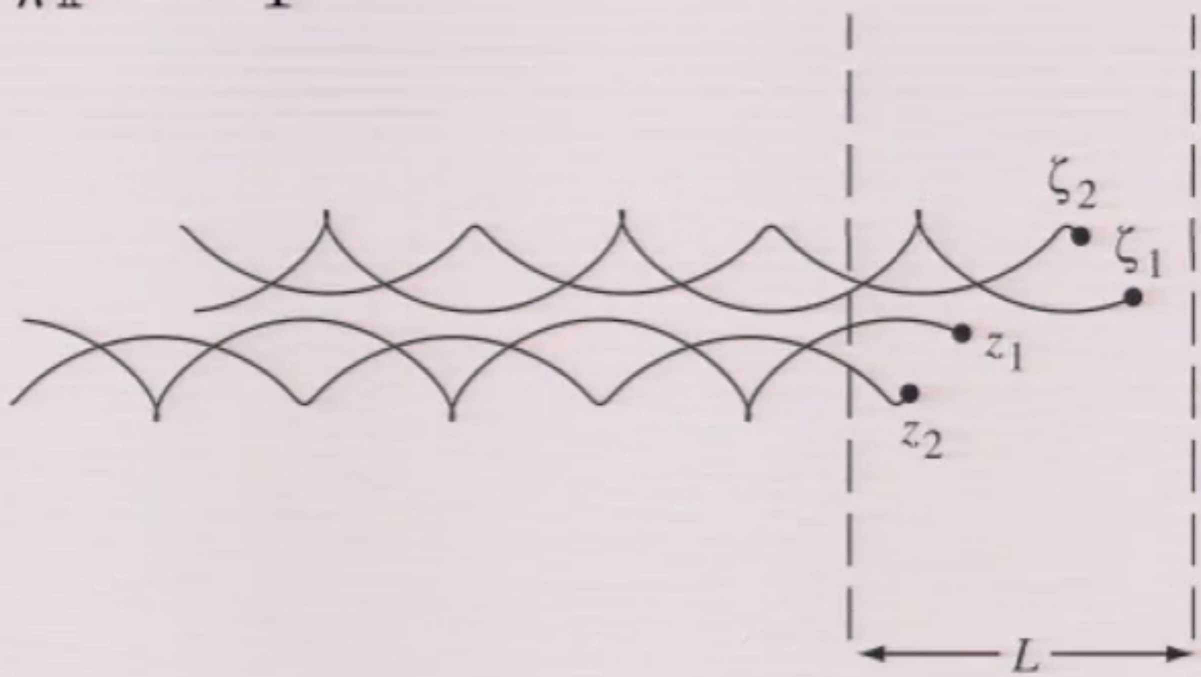


- phase space trajectories given by level curves of Hamiltonian
- phase space is separated into distinct *regimes of motion*
- vortex trajectories found by integrating along the phase space curves
- initial conditions from a single regime generate qualitatively similar vortex trajectories

results for $\gamma = 2/5 \quad \pi\mathbb{P} = -1$



phase space



physical space

From vortex trajectories to braids

1. Define vortex motions relative to vortex 2

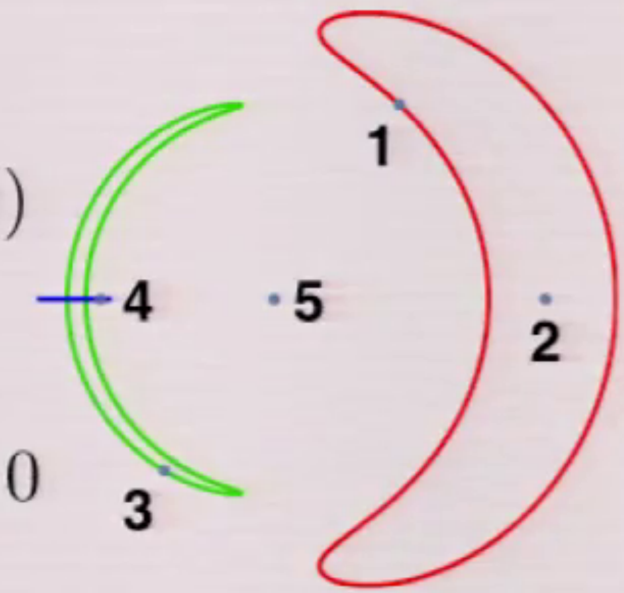
$$c_\alpha(t) = 2\pi L\left[x_\alpha(t) - x_2(t)\right]/L + \left[y_\alpha(t) - y_2(t)\right]/L$$

2. Conformal mapping of cylinder to plane

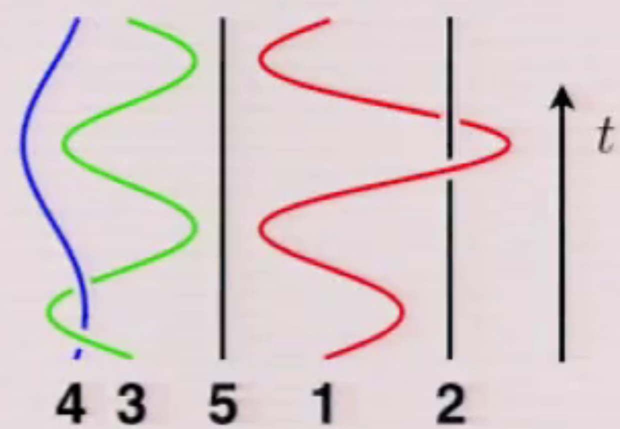
$$T = \exp(iz)$$

$$s_\alpha(t) = T\left(c_\alpha(t)\right)$$

add point from infinity: $s_5(t) = 0$



3. Observe space-time trajectories and crossings from -i axis



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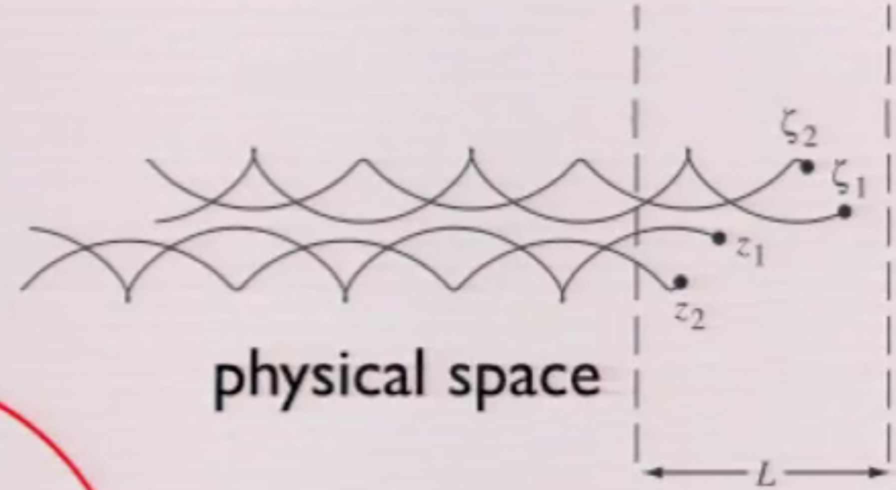

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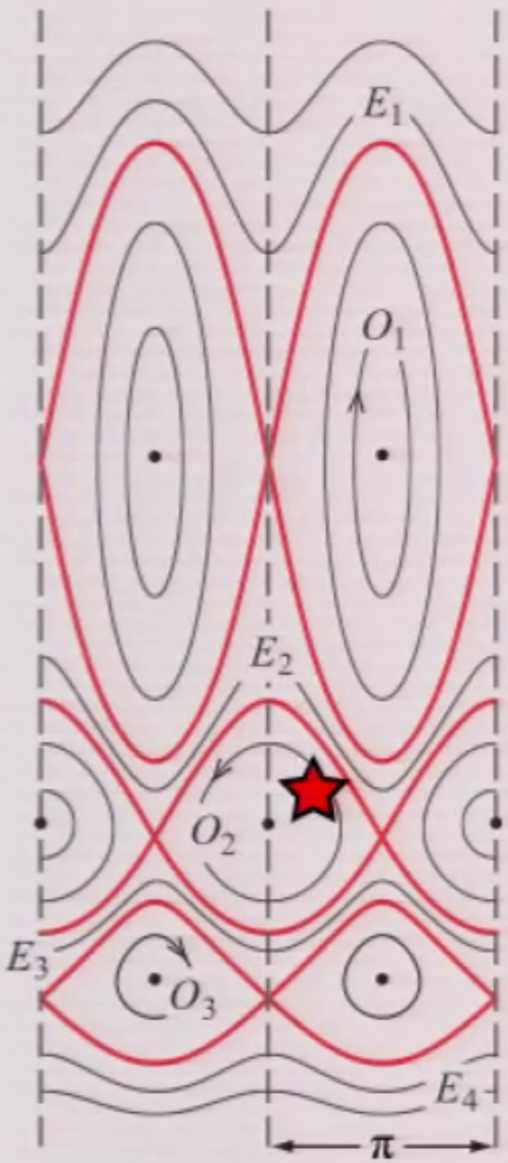

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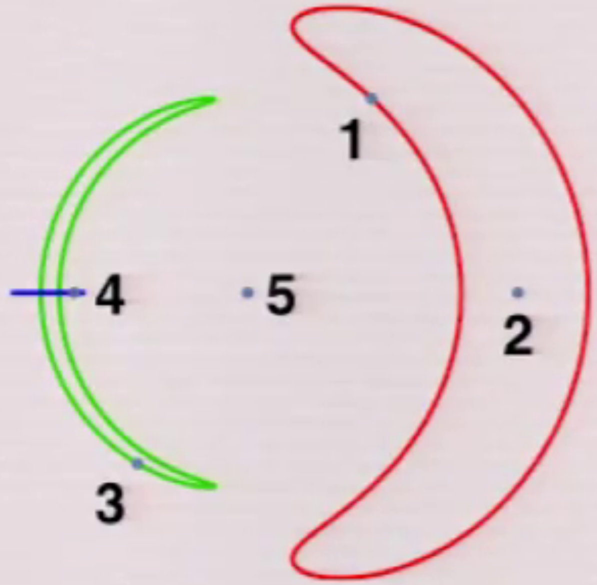
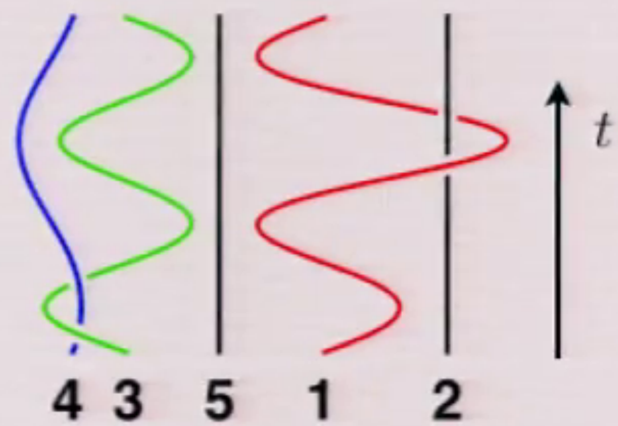


mapped plane

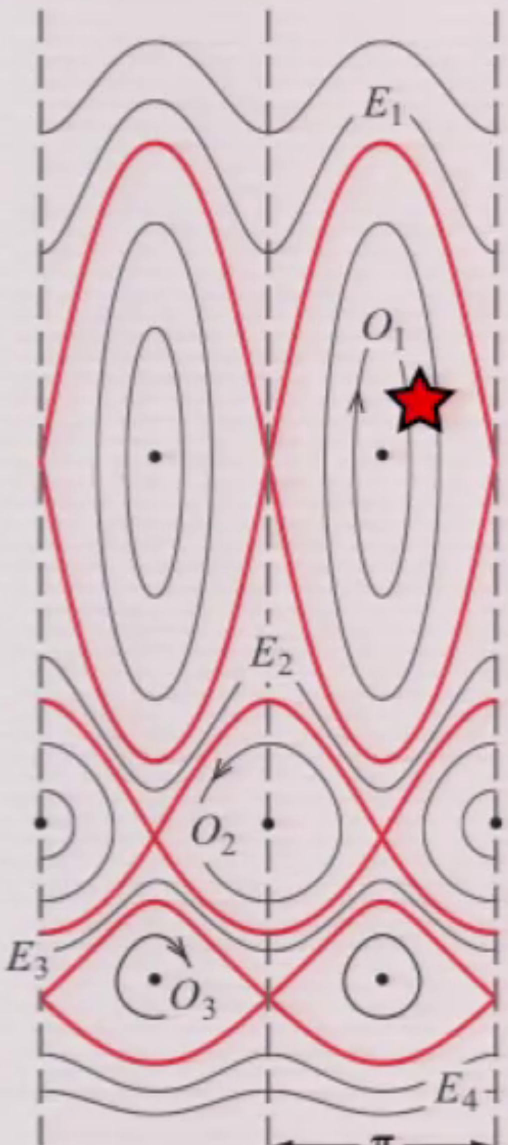
braid representation



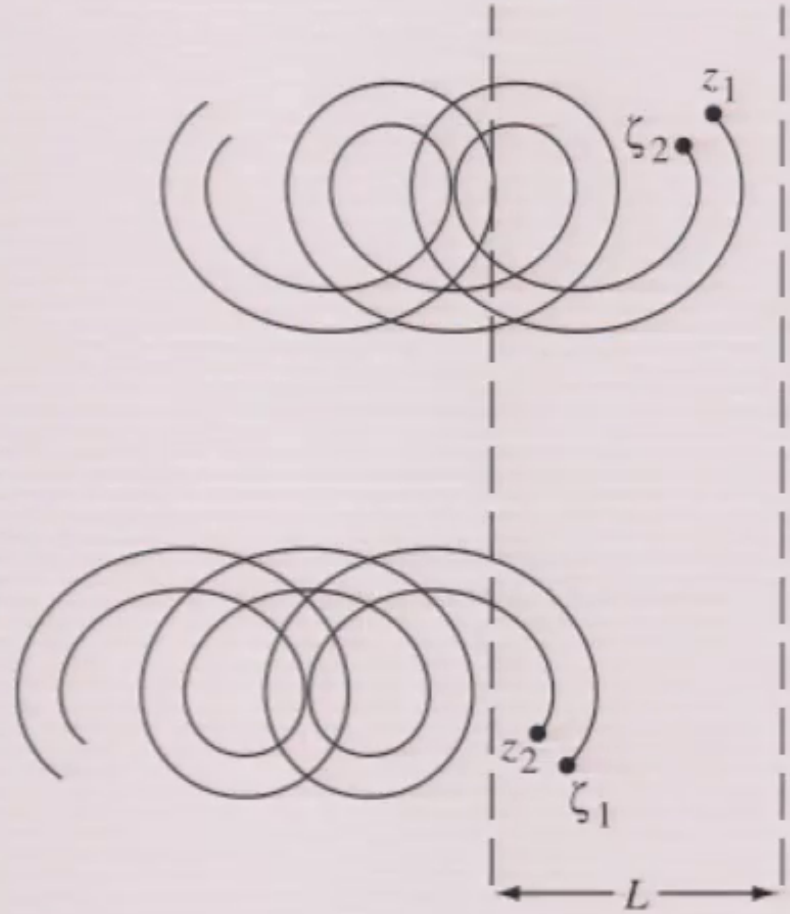
"orbiting mode" O_2

this braid is *reducible* with all parts *finite order*

results for $\gamma = 2/5 \quad \pi\mathbb{P} = -1$



phase space



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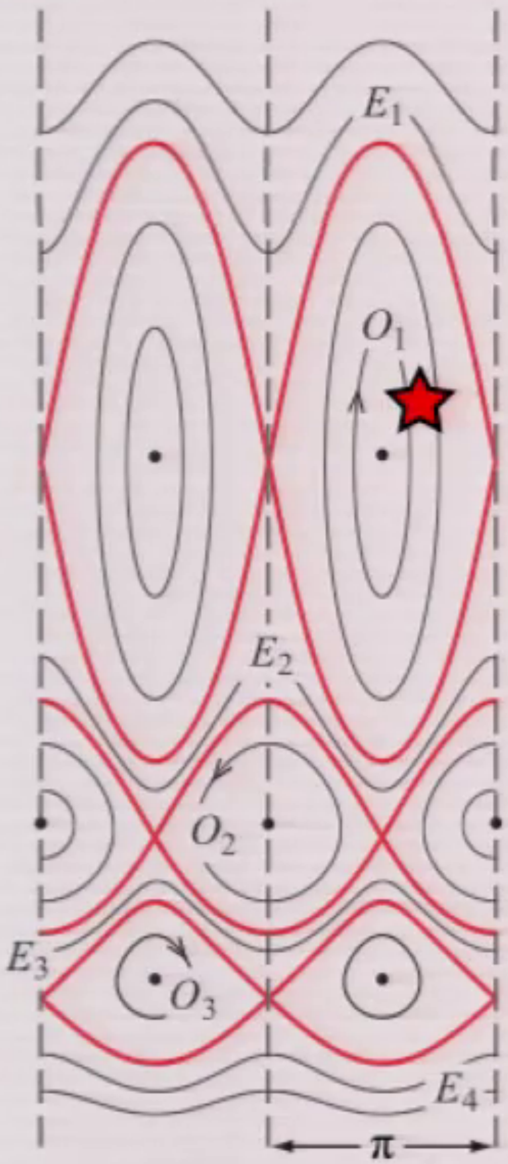
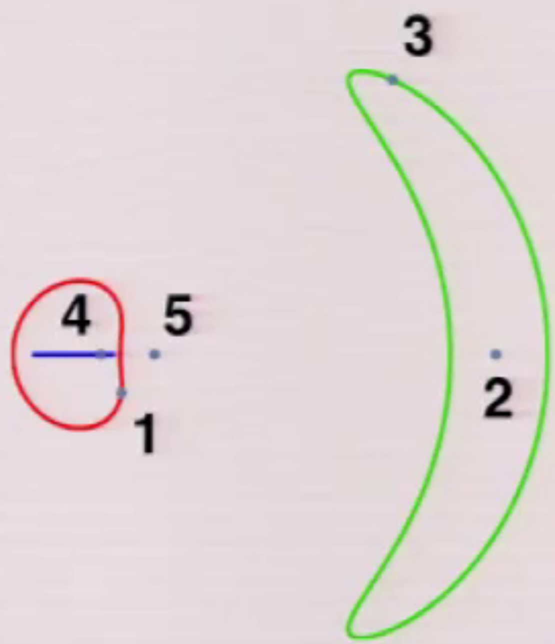
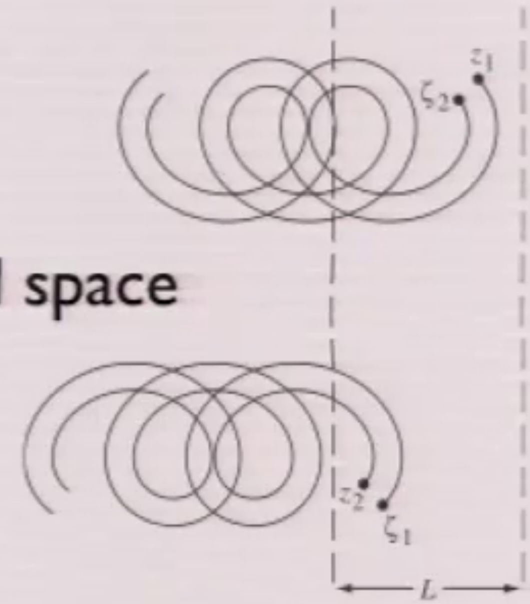


phase space



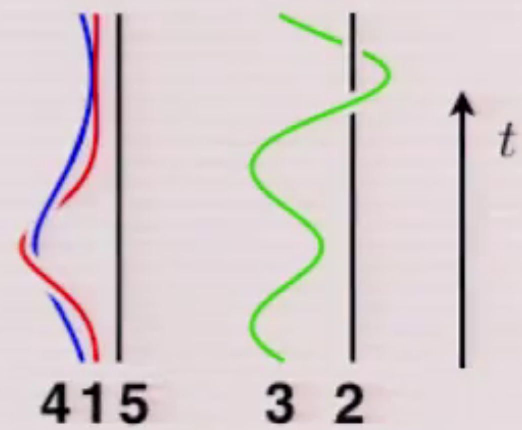
mapped plane

"orbiting mode" O_1



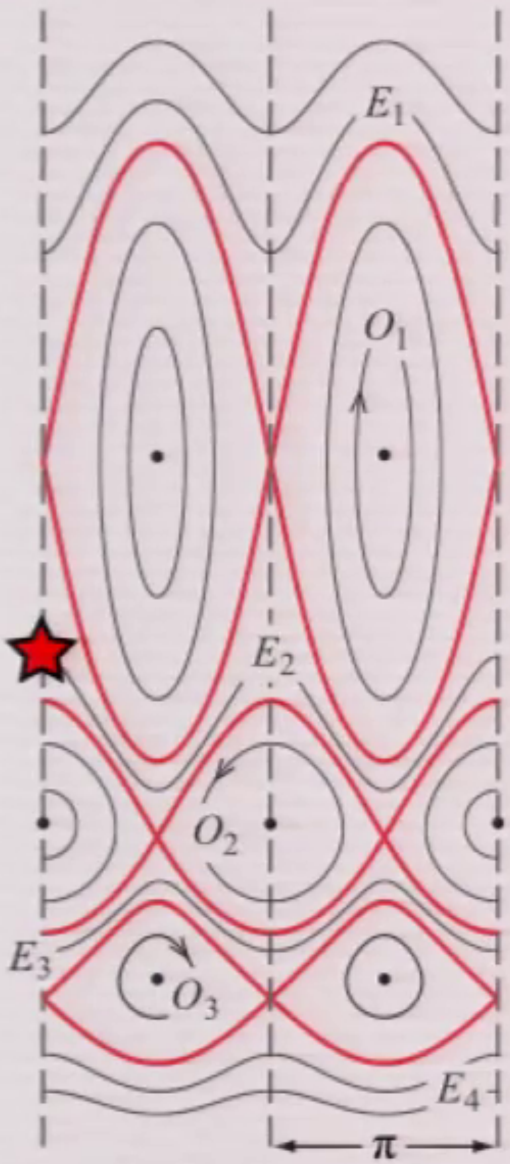
physical space

braid representation

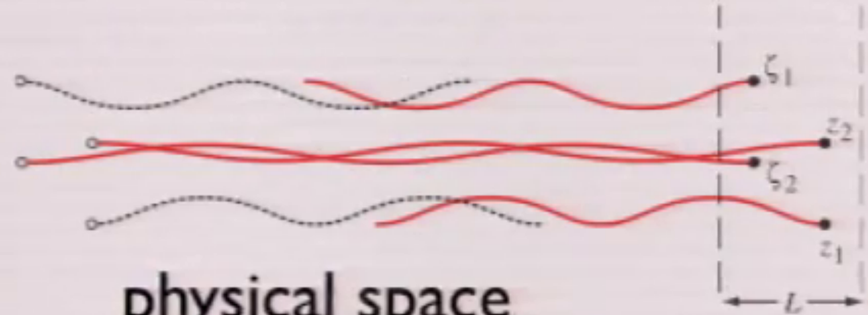


this braid is *reducible* with all parts *finite order*

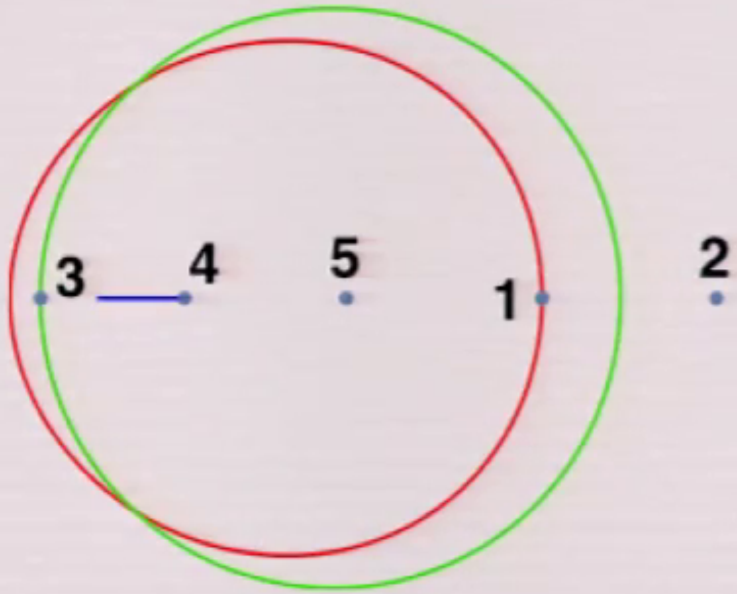
results for $\gamma = 2/5 \quad \pi\mathbb{P} = -1$



phase space



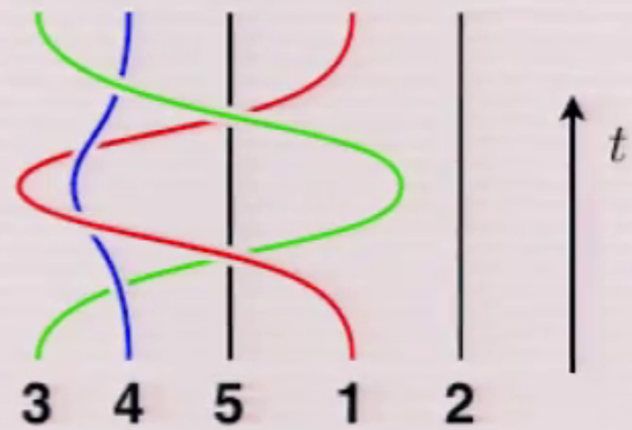
physical space



mapped plane

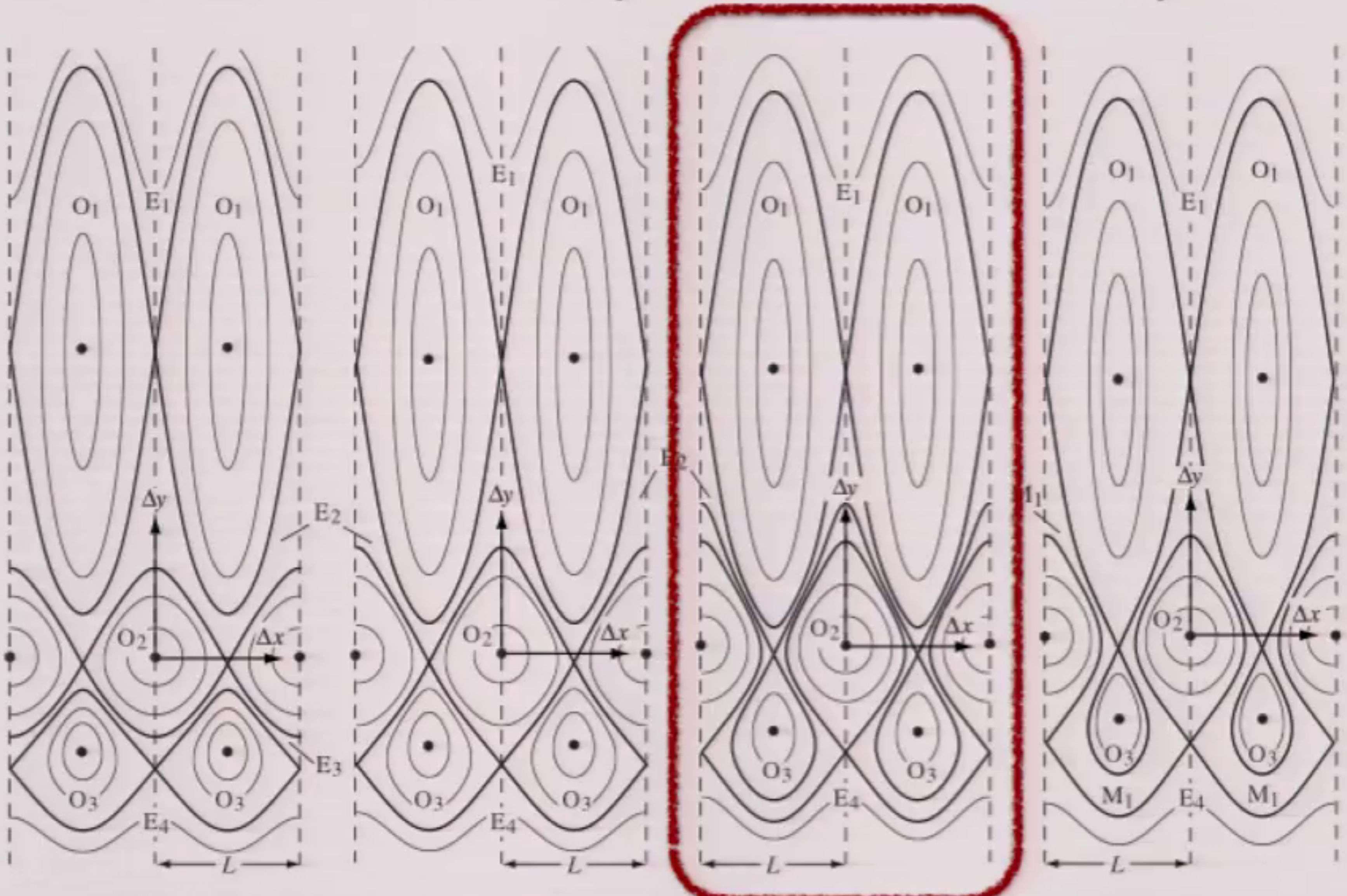
"exchanging mode" E_2

braid representation

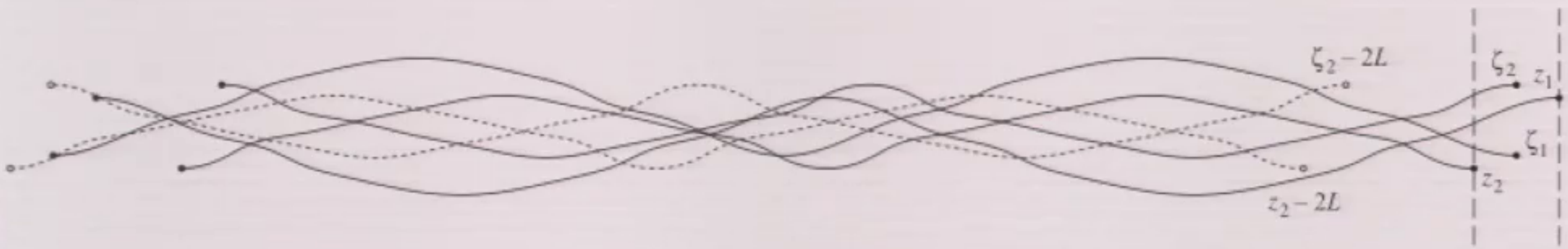


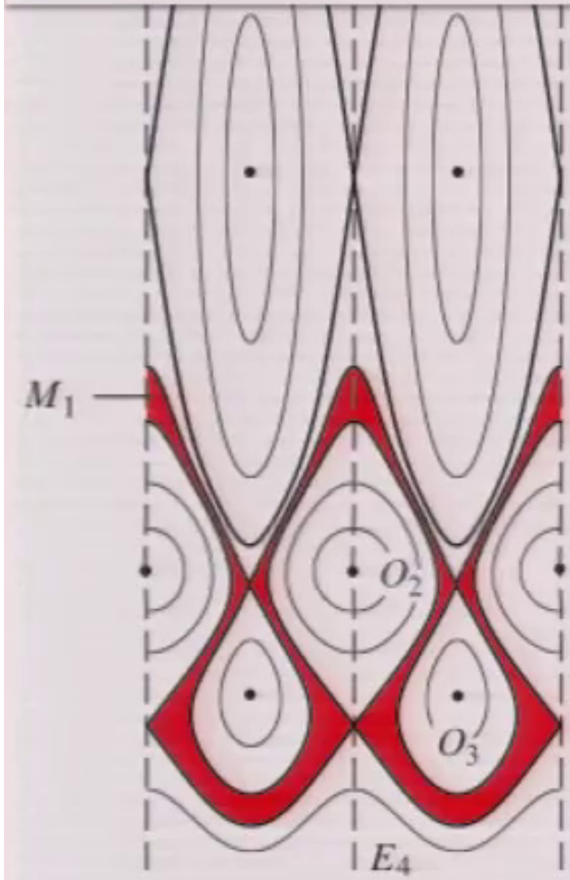
this braid is *reducible* with all parts *finite order*

relative circulation and impulse are bifurcation parameters

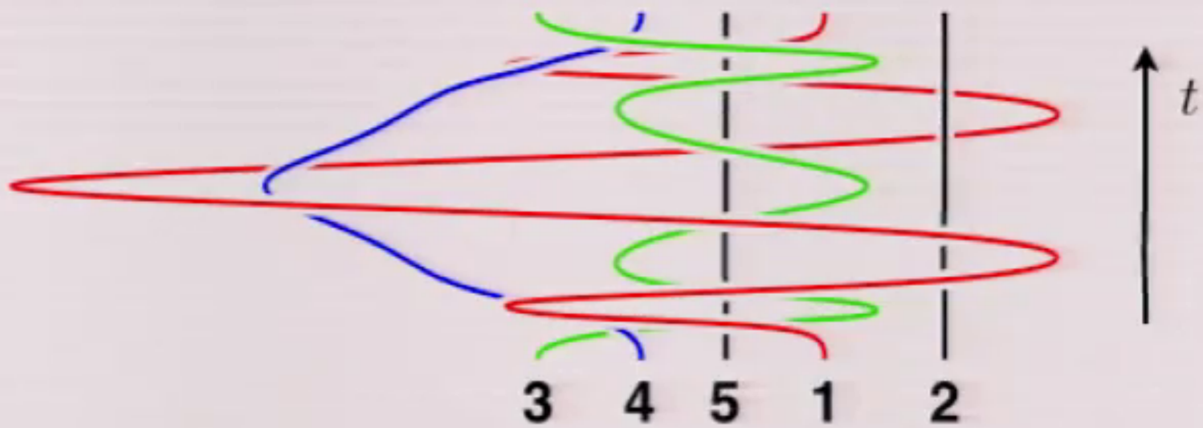


$\gamma = 3/7$ decreasing $\mathbb{P}$ →





"mixed mode" displays features of the orbiting modes and the exchanging modes

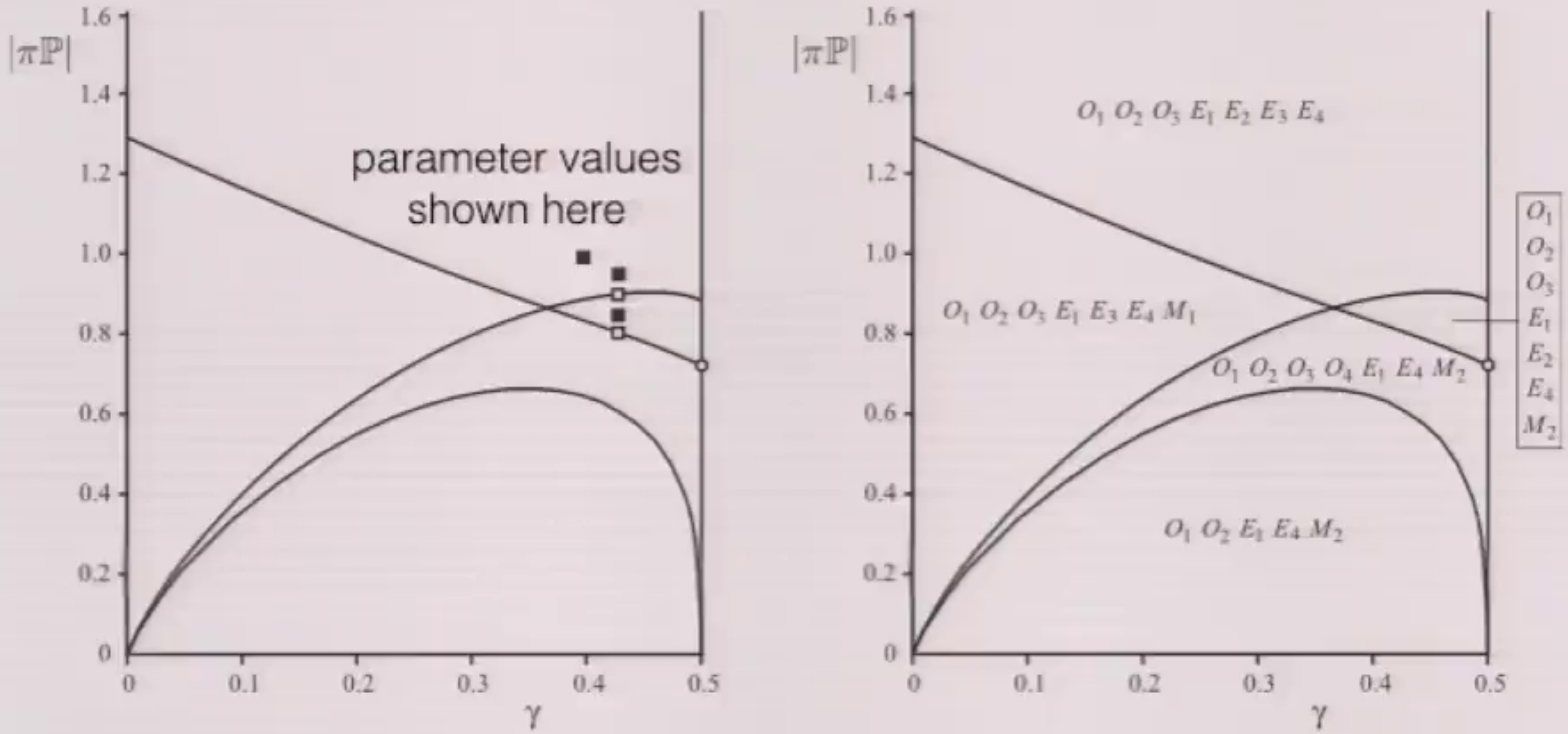


$\gamma = 3/7 \quad \pi\mathbb{P} = -0.85$

even this braid is *reducible* with all parts *finite order*

bifurcation diagram gives full representation of possible point vortex motions in this system

Basu & Stremler (in review) On the motion of two point vortex pairs with glide-reflective symmetry in a periodic strip, *Physics of Fluids*



every braid representation is *reducible* with all parts *finite order*