Large deviation theory applied to climate physics, a new frontier of statistical physics and applied mathematics

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Outline

1. Two mathematical and computational challenges in climate dynamics related to rare events
   - Abrupt climate changes and transitions between turbulent attractors
   - Rare events with a huge impact
2. Rare transitions and Jupiter's abrupt climate changes
   - Large deviations in the weak noise regime
   - Rare transitions for zonal jets
   - Averaging and large deviations for zonal jet slow dynamics
3. Probability and dynamics of extreme heat waves
   - The jet stream, blocking events, and heat waves
   - Sampling extreme heat waves using a large deviation algorithm

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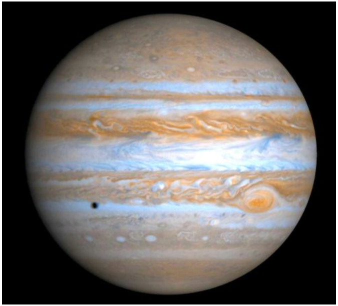
Jupiter's Zonal Jets

An example of a geophysical turbulent flow (Coriolis force, huge Reynolds number, ...)

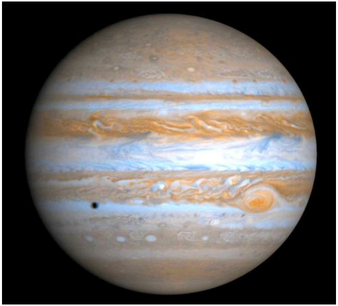


Jupiter's troposphere

Jupiter's motions (Voyager)

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



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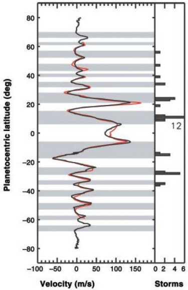
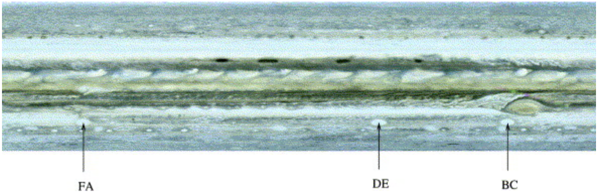
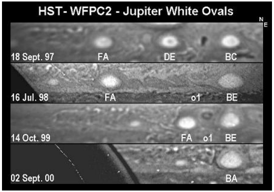


Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

Jupiter's Abrupt Climate Change

Have we lost one of Jupiter's jets ?



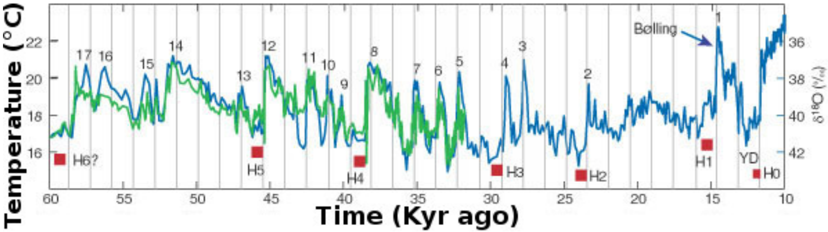


Jupiter's white ovals (see Youssef and Marcus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of one of the zonal jets?

Abrupt Climate Changes (Last Glacial Period)

Long times matter

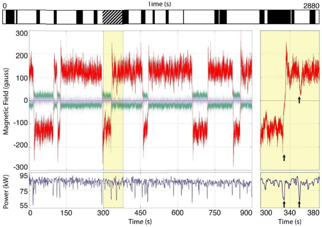


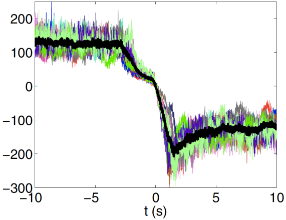
Temperature versus time: Dansgaard–Oeschger events (S. Rahmstorf)

- What is the dynamics and probability of abrupt climate changes?

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)





Magnetic field timeseries   Zoom on transition paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another (reactive paths) often occur through a predictable path.

The Main Scientific Issues

- How to characterize and predict the attractors of turbulent geophysical flows?
- Can we compute the transition paths and the transition rates?
- For most geophysical problems, an approach through direct numerical simulations is impossible (trade off between realistic turbulence representation and physical time - here one needs both).
- Can we devise new theoretical and numerical tools to tackle these issues?

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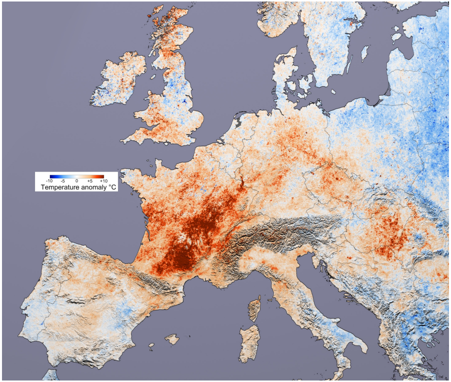
Extreme Heat Waves

Example: the 2003 heat wave over western Europe



July 20 2003-August 20 2003 land surface temperature minus the average for the same period for years 2001, 2002 and 2004 (TERRA MODIS).

Rare Events with a Huge Impact

- Heat waves, rogue waves, floods, droughts, extreme precipitations, and so on

The scientific questions:

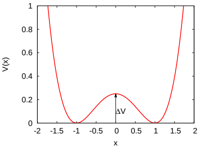
- What is the probability and the dynamics of those rare events?
- Is the dynamics leading to such rare events predictable?
- How to sample rare events, their probability, and their dynamics.
- Are direct numerical simulations a reasonable approach?
- Can we devise new theoretical and numerical tools to tackle these issues?

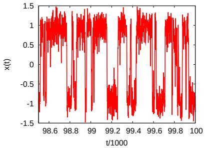
Outline

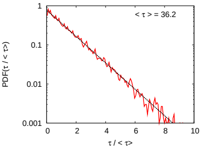
Kramers' Problem: a Pedagogical Example for Bistability

Historical example: Computation by Kramer of Arrhenius' law for a bistable mechanical system with stochastic noise

$$\frac{dx}{dt} = -\frac{dV}{dx}(x) + \sqrt{2k_B T_e}\eta(t) \text{ Rate : } \lambda = \frac{1}{\tau}\exp\left(-\frac{\Delta V}{k_B T_e}\right).$$







The problem was solved by Kramer (30'). Modern approach: path integral formulation (instanton theory, physicists) or large deviation theory (Freidlin–Wentzell, mathematicians).

Freidlin–Wentzell Theory

- For dynamical systems with weak noises

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t).$$

- Path integral representation of transition probabilities (Onsager–Machlup, 53'):

$$P(x_{-1}, T; x_1, 0) = \int_{x(0)=x_1}^{x(T)=x_{-1}} e^{-\frac{1}{4\varepsilon}\int_0^T [\dot{x}-b(x)]^2} \mathscr{D}[x].$$

- We consider a saddle point approximation (WKB), and obtain the Arrhenius law as a large deviation result $\lambda \underset{\varepsilon\downarrow 0}{\asymp} e^{-\frac{\Delta V}{\varepsilon}}$ with

$$\Delta V = \inf_{T\geq 0} \inf_{\{x(t)|x(0)=x_1 \text{ and } x(T)=x_{-1}\}} \left\{ \frac{1}{4}\int_0^T [\dot{x}-b(x)]^2 \, dt \right\}$$

Most Transition Paths Follow the Instanton

- In the weak noise limit, most transition paths follow the most probable path (instanton)

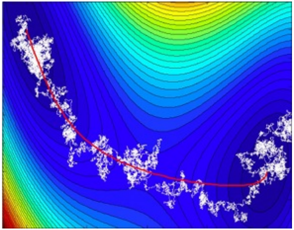


Figure by Eric Van den Eijnden

- Arrhenius law then follows, for both gradient (reversible) and non gradient (irreversible) dynamics

$$\lambda \underset{\varepsilon \to 0}{\asymp} e^{-\frac{\Delta V}{\varepsilon}}.$$

Random Transitions in Turbulence Problems

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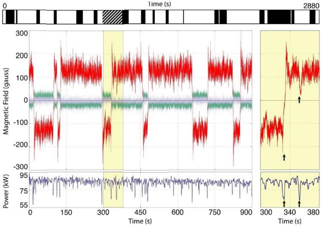
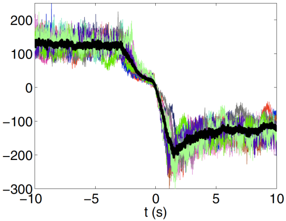


Magnetic field timeseries



Zoom on transition paths

(VKS experiment)

In turbulent flows, transitions from one attractor to another (reactive paths) often occur through a predictable path.

Transition Rates Beyond Large Deviations: the Eyring–Kramers Formula

- Large deviation theory gives the exponential factor for the transition rate $\lambda = 1/\tau \exp(-\Delta V/\varepsilon)$:

$$\lim_{\varepsilon \to 0} \varepsilon \log \lambda = -\Delta V.$$

- But the prefactor $1/\tau$ is also essential in giving the time scale.
- For gradient dynamics $\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla V + \sqrt{2\varepsilon}\eta(t)$, the Eyring–Kramers formula (Landauer and Swanson, 1961, Langer, 1969?) gives

$$\lambda \underset{\varepsilon \to 0}{\sim} \frac{|\lambda_*|}{2\pi} \sqrt{\frac{\det \mathrm{Hess}\, V(x_1)}{|\det \mathrm{Hess}\, V(x_*)|}} \exp\left(-\frac{\Delta V}{\varepsilon}\right),$$

where λ_* is the unstable direction eigenvalue, at the saddle point.

Proof of the Eyring–Kramers Formula for Reversible (Gradient) Dynamics

- First proof (using potential theory and capacities): Bovier, Eckhoff, Gayrard, and Klein (2004).
- Generalizations to non quadratic saddle points, SPDE in dimension $d = 1$. See the review: Berglund (2011).
- An Eyring–Kramers formula for SPDE in dimension $d = 2$: Berglund, Di Gesu, and Weber (arXiv 2016).
- What is the prefactor for irreversible (non-gradient) dynamics? Formal results by Maier and Stein (1997) (for 2 degrees of freedom), Schuss (2009).

Transition Rates for Irreversible (Non-Gradient) Dynamics

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta\left(t\right).$$

- We assume that there exists a transverse decomposition in the instanton neighborhood

$$\mathbf{b}(\mathbf{x}) = -\nabla V(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \text{ with for all } \mathbf{x},\ \nabla V(\mathbf{x}).\mathbf{G}(\mathbf{x}) = 0.$$

- The transition rate then reads

$$\lambda \underset{\varepsilon \to 0}{\sim} \frac{|\lambda_*|}{2\pi}\sqrt{\frac{\det \mathrm{Hess}\, V(x_1)}{|\det \mathrm{Hess}\, V(x_*)|}} \exp\left(-\frac{\Delta V}{\varepsilon}\right) \exp\left\{-\int_{-\infty}^{+\infty} \mathrm{d}t\, [\nabla.G(X(t))]\right\},$$

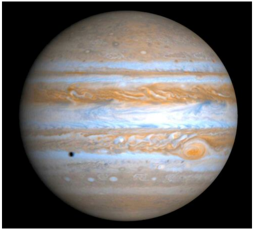
where λ_* is the negative eigenvalue corresponding to the unstable direction at the saddle point, for the dynamics (and not for V) and $\{X(t)\}$ is the instanton.

F. Bouchet and J. Reygner, AHP 2016

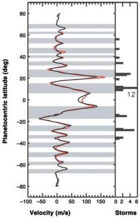
Outline

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



Jupiter's atmosphere



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), β is the Coriolis parameter, f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t') \rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$.

- A reasonable model for Jupiter's zonal jets.

The 2D Stochastic Navier-Stokes Equations ($\beta = 0$)

$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla \omega = \nu \Delta \omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Bricmont, Debussche, Hairer, Kuksin, Kupiainen, Mattingly, Shirikyan, Sinai, ...
    - Existence of a stationary measure μ_ν. Existence of $lim_{\nu \to 0}\ \mu_\nu$,
    - In this limit, almost all trajectories are solutions of the 2D Euler equations.

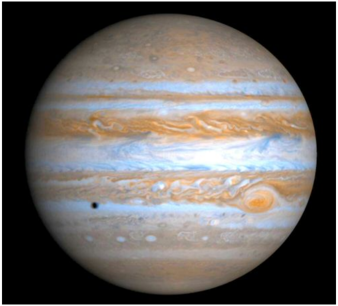
Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

- We would like to describe the invariant measure

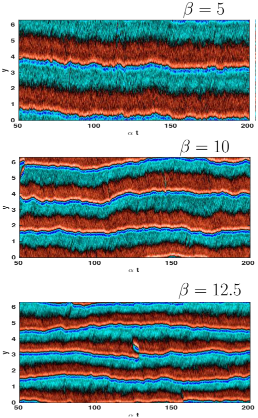
Dynamics of the Barotropic Quasi-Geostrophic Equations

Top: Zonally averaged vorticity (Hovmöller diagram and red curve) and velocity (green). Bottom: vorticity field

Multistability for Quasi-Geostrophic Jets



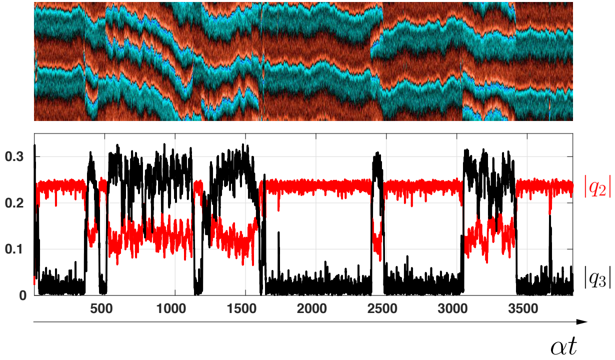
Jupiter's atmosphere



QG zonal turbulent jets

- Multiple attractors had been observed previously by B. Farrell and P. Ioannou.

Rare Transitions Between Quasigeostrophic Jets



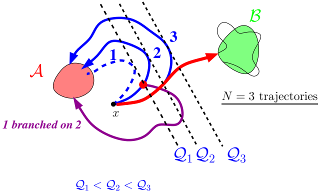
Rare transitions for quasigeostrophic jets (with E. Simonnet)

- This is the first observation of spontaneous transitions.
- How to predict those rare transitions? What is their probability? Which theoretical approach?

Rare Events and Adaptive Multilevel Splitting (AMS)

AMS: an algorithm to compute rare events, for instance rare transition paths

- Rare event algorithms: Kahn and Harris (1953), Chandler, Vanden-Eijnden, Schuss, Del Moral, Dupuis, ...
- The adaptive multilevel splitting algorithm:



AMS algorithm

Strategy: selection and cloning.
Probability estimate:

$$\hat{\alpha} = (1 - 1/N)^K, \text{ where}$$

N is the clone number and K the iteration number.

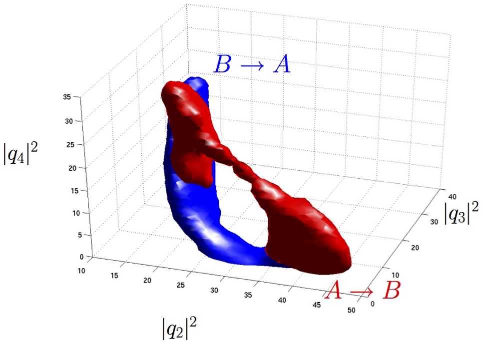
Cérou, Guyader (2007). Cérou, Guyader, Lelièvre, and Pommier (2011).

A Transition from 2 to 3 Jets

Top: Zonally averaged vorticity (Hovmöller diagram and red curve) and velocity (green). Bottom: vorticity field

Atmosphere Jet "Instantons" Computed using the AMS

AMS: an algorithm to compute rare events, for instance rare reactive trajectories

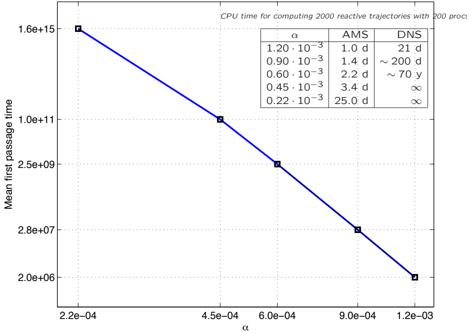


Transition trajectories between 2 and 3 jet states

- The dynamics of turbulent transitions is predictible.
- Asymmetry between forward and backward transitions.

Transition Rates for Unreachable Regimes Through DNS

With the AMS we can estimate huge average transition times



α	AMS	DNS
$1.20 \cdot 10^{-3}$	1.0 d	21 d
$0.90 \cdot 10^{-3}$	1.4 d	~ 200 d
$0.60 \cdot 10^{-3}$	2.2 d	~ 70 y
$0.45 \cdot 10^{-3}$	3.4 d	∞
$0.22 \cdot 10^{-3}$	25.0 d	∞

Average transition time versus α

- With the AMS algorithm, we study transitions that would require an astronomical computation time using direct numerical simulations.

Outline

Which Mathematical Framework for the Inertial Limit?

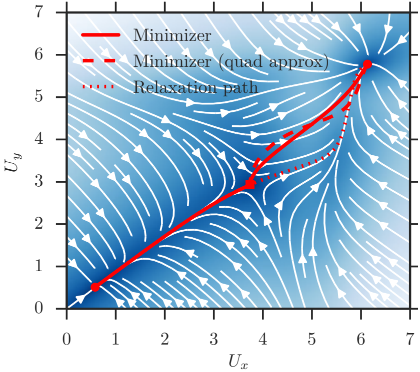
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s.$$

- Inertial limit: spin up or spin down time $= 1/\alpha \gg 1 =$ jet inertial time scale (a relevant assumption for Jupiter).
- This is an averaging problem for an Hamiltonian system perturbed by weak non Hamiltonian forces.
- The Hamiltonian system is an infinite dimensional one with an infinite number of conserved quantities.
- We will need to consider large deviations for the slow process.

Gaussian Fluctuations Do Not Describe Rare Transitions

$$\frac{\partial U}{\partial t} = F(U) + \sqrt{\alpha}\sigma(U,t)$$



(Figure from F. Bouchet, T. Grafke, T. Tangarife, and E. Vanden-Eijnden, J. Stat. Phys. 2016)

Zonal Jet Conclusions

- We have computed rare transitions between zonal jets, similar to Jupiter's abrupt climate changes, that can not be computed using direct numerical simulations (with E. S.).
- We have partial results for the justification of averaging (ergodicity, etc ...), (with C.N., and T.T.).
- For small scale forces, the average Reynolds stress can be computed explicitly and is universal. We have a good qualitative agreement with Jupiter's jets. (with E.W.).
- The rare transitions involve non-Gaussian fluctuations of the Reynolds stress.
- A theory based on large deviations can be derived for the computation of transition rates and transition paths between zonal jets (with T.G., B..M., T.T., and E. V-E).

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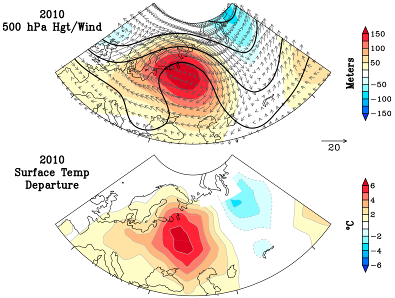
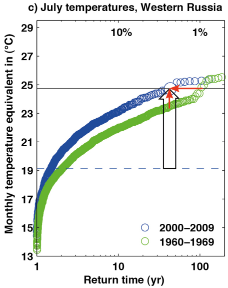
Outline

Anthropogenic Causes of the 2010 Heat Wave



(Dole et al., 2011)



Return time of monthly temperature

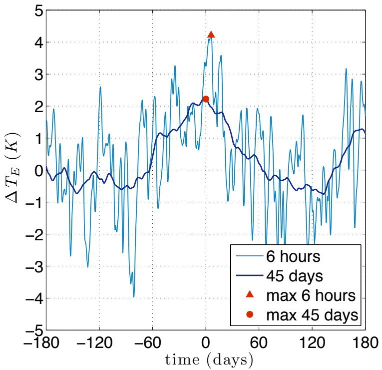
(Otto et al., 2012)

- A clear anthropogenic impact.
- What are the dynamical mechanisms for such extreme events?

The Jet Stream, Rossby Waves, and Blocking Events

Higher troposphere winds

45-Day Averaged Temperature over Europe (Plasim Model)



Temperature and 45-day averaged temperature

$$a = \frac{1}{T}\int_0^T \int_{\text{Europe}} Temp(t)\,\mathrm{d}t\mathrm{d}\mathbf{x} \text{ with T=45 days}$$

Outline

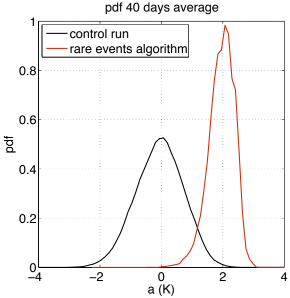
Numerical Computation of Donsker–Varadhan Large Deviations

- Importance sampling: how to sample efficiently the tilted distribution

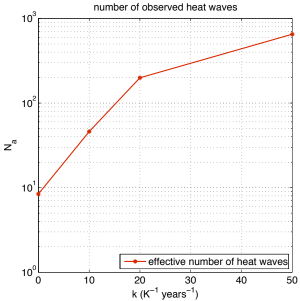
$$\tilde{P}_k\left(\{X(t)\}_{0\leq t\leq T}\right) = \frac{1}{\exp(T\lambda(k))} P_0\left(\{X(t)\}_{0\leq t\leq T}\right)\exp\left[k\int_0^T A(X(t))\,\mathrm{d}t\right]?$$

- We use the Giardina–Kurchan–Leconte–Tailleur algorithm (Giardina et al 2006).

Importance Sampling of Extreme Heat Waves in a Climate Model



PDF of time averaged temperature



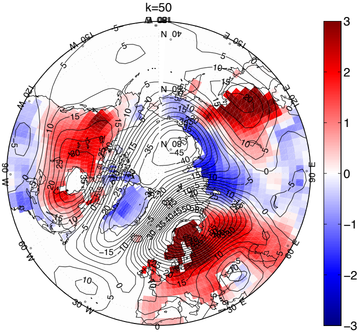
Heat wave number

- At fixed numerical cost, we get hundreds more heat waves with the large deviation algorithm than with the control run.
- We can consider interesting dynamical studies.

A Typical Heat Wave

500 HPa geopotential height and temperature anomalies

Heat Wave Conditional Statistics and Teleconnection Patterns



500 HPa geopotential height anomalies and temperature anomalies

Heat wave statistics defined as statistics conditioned with $\frac{1}{T}\int_0^T Temp(X(t))\,\mathrm{d}t > 2°\mathrm{C}$, with $T = 40\,\mathrm{days}$.

Collaborators

- An Eyring-Kramers formula for transition rates of non-reversible stochastic differential equations (with J. Reygner).
- Numerical simulation of abrupt transitions for Jupiter zonal jets using Adaptive Multilevel Splitting algorithms (with J. Rolland and E. Simonnet (Nice)).
- Ergodicity and averaging for the quasi-geostrophic dynamics (with C. Nardini (post-doc) and T. Tangarife (PHD)).
- Averaging, large deviations, and transitions for Jupiter jets (with T. Grafke and E. Vanden-Eijnden).
- Sampling extreme heat waves using large deviation algorithms (with J. Wouters and F. Ragone, project AXA).

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Summary and Perspectives

- Large deviation theory can be applied to geophysical turbulence and climate.
- This is the main approach for non-equilibrium statistical mechanics applied to climate dynamics.
- With rare event algorithms, we can compute probability of rare events that can not be sampled using direct numerical simulations. This should have a huge impact on future computations of climate extremes and rare transitions.
- The dynamics leading to rare events is usually predictable, even for turbulent flows.

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