Energy scaling law for a single disclination in a thin elastic sheet

Heiner Olbermann

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Overview

Energy focusing in thin elastic sheets

- Experimental setup: Take a thin elastic sheet and subject it to boundary conditions that force it to form sharp folds or vertices.

- Rigorous formulation:
  simply connected Lipschitz domain $\Omega \subset \mathbb{R}^2$: elastic sheet in the undeformed configuration;
  $y : \Omega \to \mathbb{R}^3$: deformation map;
  elastic energy:

$$I_h(y,\Omega) := \int_\Omega \left| Dy^T Dy - g_0 \right|^2 + h^2 \left| D^2 y \right|^2$$
$$= \int_\Omega \|g_y - g_0\|^2 + h^2 \left| D^2 y \right|^2$$

($g_y = y^* \mathrm{e}^{(3)}$, $\mathrm{e}^{(3)}$ =Euclidean metric on $\mathbb{R}^3$, g_0=reference metric.)

Tensile boundary conditions

- Looking for lower und upper bounds, both as expansions in the small parameter h
- Crumpling: *too hard*. Alternative problem: Impose *tensile boundary conditions.* I.e., choose Dirichlet boundary conditions such that there exists a unique (Lipschitz) isometric immersion, which is singular (infinite bending energy)
- Every deviation from the singular configuration is penalized by the membrane term; the balance between the two leads to an (optimal) lower bound
- If *short maps* (i.e., maps y with $Dy^T Dy < \mathrm{Id}$) are permissible by the boundary conditions, this method of proof breaks down

Why are lower bounds difficult? (for non-tensile b.c.)

Theorem (Nash '54, Kuiper '59)

Let (M, g) be a Riemannian manifold of dimension n, $m \geq n+1$. Any short immersion of M into $\mathbb{R}^m$ can be uniformly approximated by isometric immersions of class C^1.

Thus: For a given reference metric g, there exists a huge set of maps that are very close to isometric immersions. The theory behind this is Gromov's h-principle. How to exclude these degrees of freedom?

Isometric embedding of the flat torus into $\mathbb{R}^3$ (Borrelli, Jabrane, Lazarus, Thibert, PNAS '12)

The single disclination

- Experimental setup: Remove a sector from a circular thin sheet and glue the edges back together
- Instead of "removing a sector" we may equivalently "shorten the reference metric in the angular direction",

$$\hat{y}(x) = \sqrt{1-\Delta^2}x + \Delta|x|e_z$$
$$g_\Delta = D\hat{y}^T D\hat{y}$$

$$\boxed{\Rightarrow g_\Delta(x) = \mathrm{Id}_{2\times 2} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp}$$

$0 < \Delta < 1$, domain $= B_1 = B(0,1) \subset \mathbb{R}^2$ with $\hat{x} = x/|x|$,

$$\hat{x}^\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \hat{x}$$

Main result

$$g_\Delta = \mathrm{Id} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp, \quad g_y = Dy^T Dy$$

$$I_{h,\Delta}(y) = \int_{B_1} |g_y - g_\Delta|^2 + h^2 |D^2 y|^2 \mathrm{d}x$$

Theorem (O. '15)

There exists a constant $C = C(\Delta) > 0$ such that

$$2\pi\Delta^2 h^2 \left(|\log h| - \frac{3}{2} \log |\log h| \right) \leq \inf_{y \in W^{2,2}(B_1;\mathbb{R}^3)} I_{h,\Delta}(y) \leq 2\pi\Delta^2 h^2 |\log h| + C$$

for all small enough h.

An ansatz for lower bounds for the single disclination

The "right" curvature like quantity is $\mathcal{K}(y) := \sum_{i=1}^{3} \det D^2 y_i$.

Main idea

- Use $\mathcal{K}(y)$ as a control variable for lower bounds of both, the membrane and bending term.
- To deal with the membrane term, use the formulation of the determinant $\det D^2 v$ as a *very weak Hessian*, $\det D^2 v = (v_{,1} v_{,2})_{,12} - \frac{1}{2}(|v_{,1}|^2)_{,22} - \frac{1}{2}(|v_{,2}|^2)_{,11}$
- Hence, we have

$$\sum_{i=1}^{3} \det D^2 y_i = (y_{,1} \cdot y_{,2})_{,12} - \frac{1}{2}(|y_{,1}|^2)_{,22} - \frac{1}{2}(|y_{,2}|^2)_{,11}$$

Main idea continued

- $\implies \sum_i \det D^2 y_i - \det D^2 \hat{y}_i =$
  $(g_y - g_\Delta)_{12,12} - \frac{1}{2}(g_y - g_\Delta)_{11,22} - \frac{1}{2}(g_y - g_\Delta)_{22,11}$
- Explicit calculation yields $\det D^2 \hat{y}_i = \pi\Delta^2\delta_0$
- $\implies \left\|\sum_i \det D^2 y_i - \pi\Delta^2\delta_0\right\|_{W^{-2,1}} \lesssim$ membrane energy, i.e., *we obtain control over* $\sum_i \det D^2 y_i - \pi\Delta^2\delta_0$ *in* $W^{-2,1}$ *through the membrane energy*

Lower bound for bending through $\int \sum_i \det D^2 y_i$

Bending: Note that by the coarea formula, $\left|\int_{B_\rho} \sum_i \det D^2 y_i \mathrm{d}x\right| \leq \sum_i \mathcal{L}^2(Dy_i(B_\rho))$. By the isoperimetric inequality in $\mathbb{R}^2$, we can get a lower bound for $\sum_i \int_{\partial B_\rho} |D^2 y_i|$:

Lemma

For $v \in C^2(B_1)$ and $0 \leq r \leq 1$,

$$\int_{\partial B_r} |D^2 v| \mathrm{d}\mathcal{H}^1 \geq \sqrt{4\pi \left|\int_{B_r} \det D^2 v \mathrm{d}x\right|}.$$

More results: 3d elasticity

$$\Omega_h := B_1 \times [-h/2, h/2]\,, \quad \hat{Y}(x,z) = \hat{y}(x) + z\nu_{\hat{y}}$$
$$g^{(3)}_\Delta(x,z) = \mathrm{Id}_{3\times 3} - \Delta^2 \hat{x}^\perp \otimes \hat{x}^\perp$$
$$E_{h,\Delta}(Y) = \int_{\Omega_h} \mathrm{dist}^2\left(DY(x), SO(3)\sqrt{g^{(3)}_\Delta(x)}\right) \mathrm{d}\mathcal{L}^3(x)\,.$$

Theorem (O., '15)

Let $0 < \Delta < 1$. There exists a constant $C = C(\Delta)$ with the following property: For h small enough,

$$\frac{1}{C}|\log h| \le \inf_{y \in W^{2,2}(\Omega_h;\mathbb{R}^3)} h^{-2} E_{h,\Delta}(y) \le 2\pi\Delta^2 (|\log h| + C)\,.$$

Proof: Translate lower bounds of $I_{h,\Delta}(y)$ into bounds in 3d elasticity through Geometric Rigidity by Friesecke, James, Müller.

More results: The Föppl-von Kármán model

$$I_{h,\Delta}^{\mathrm{vK}} = \int_{B_1} \left| \mathrm{sym}\, Du + \frac{1}{2} Dv \otimes Dv + \Delta\, e_\varphi \otimes e_\varphi \right|^2 + h^2 |D^2 v|^2 \mathrm{d}x$$

Theorem (O. '15)

There exists a constant $C = C(\Delta) > 0$ with the following property: For h small enough,

$$\begin{aligned} &4\pi\Delta\left(|\log h| - 2\log|\log h|\right) - C \\ &\le h^{-2} \inf_{(u,v)\in W^{1,2}(B_1;\mathbb{R}^2)\times W^{2,2}(B_1)} I_{h,\Delta}^{\mathrm{vK}}(u,v) \le 4\pi\Delta|\log h| + C\,. \end{aligned}$$

More results: Convergence of almost minimizers to the singular cone

$$\hat{y} : B_1 \to \mathbb{R}^3, \quad \hat{y}(x) = \sqrt{1 - \Delta^2} x + \Delta |x| e_3 \,.$$

Theorem (Müller, O., '15)

Let $y^h \in W^{2,2}(B_1; \mathbb{R}^3)$ *be a sequence with* $I_{h,\Delta}(y^h) \leq 2\pi \Delta^2 h^2 (|\log h| + C)$*. Then up to Euclidean motions, we have for every* $0 < \rho < 1$*,*

$$y^h \rightharpoonup \hat{y} \quad \text{in } W^{2,2}(B_1 \setminus B_\rho; \mathbb{R}^3)\,.$$

Important tools in the proof: Lower bounds for $I_{h,\Delta}(y)$+ Structure result for flat $W^{2,2}$ surfaces (Pakzad'04, Hornung '11)

Summary/Outlook

- Main results: Lower bounds for a single disclination in a geometrically fully nonlinear plate model, in 3-dimensional nonlinear elasticity and in the Föppl-von Kármán model. Moreover, minimizers of the elastic energy converge to the singular cone as $h \to 0$ weakly in $W^{2,2}_{\mathrm{loc}}(B_1 \setminus \{0\}; \mathbb{R}^3)$
- Key to the first result is to look at a curvature-like quantity, that gives control over certain properties of the graph of Dy. The lower bound follows by an isoperimetric inequality
- Can this idea be applied to other problems of a similar flavor? (D-cones? Non-conical geometries?)

Literature:

H. O., Energy scaling law for a single disclination in a thin elastic sheet. Preprint, 2015. Available at arXiv:1509.07378.

H. O., Energy scaling law for the regular cone. *J. Nonlinear Sci.*, 2015. Available at arXiv:1502.07013.