Efficient Iterative Methods for Quantitative Susceptibility Mapping

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Introduction

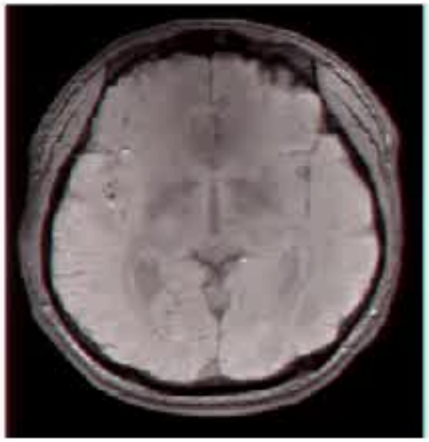
Quantitative Susceptibility Mapping

Magnetic susceptibility

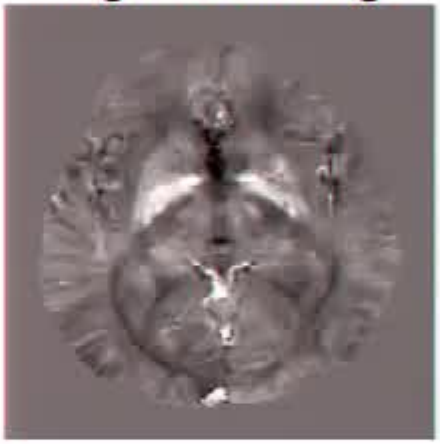
- relation between magnetization of tissue and external magnetic field
- sensitive to changes in blood oxygen level
- useful for tissue classification (iron vs. calcium)

Important biomarker for

- neurodegenerative and inflammatory diseases
- metabolic consumption of oxygen
- ...



magnitute image

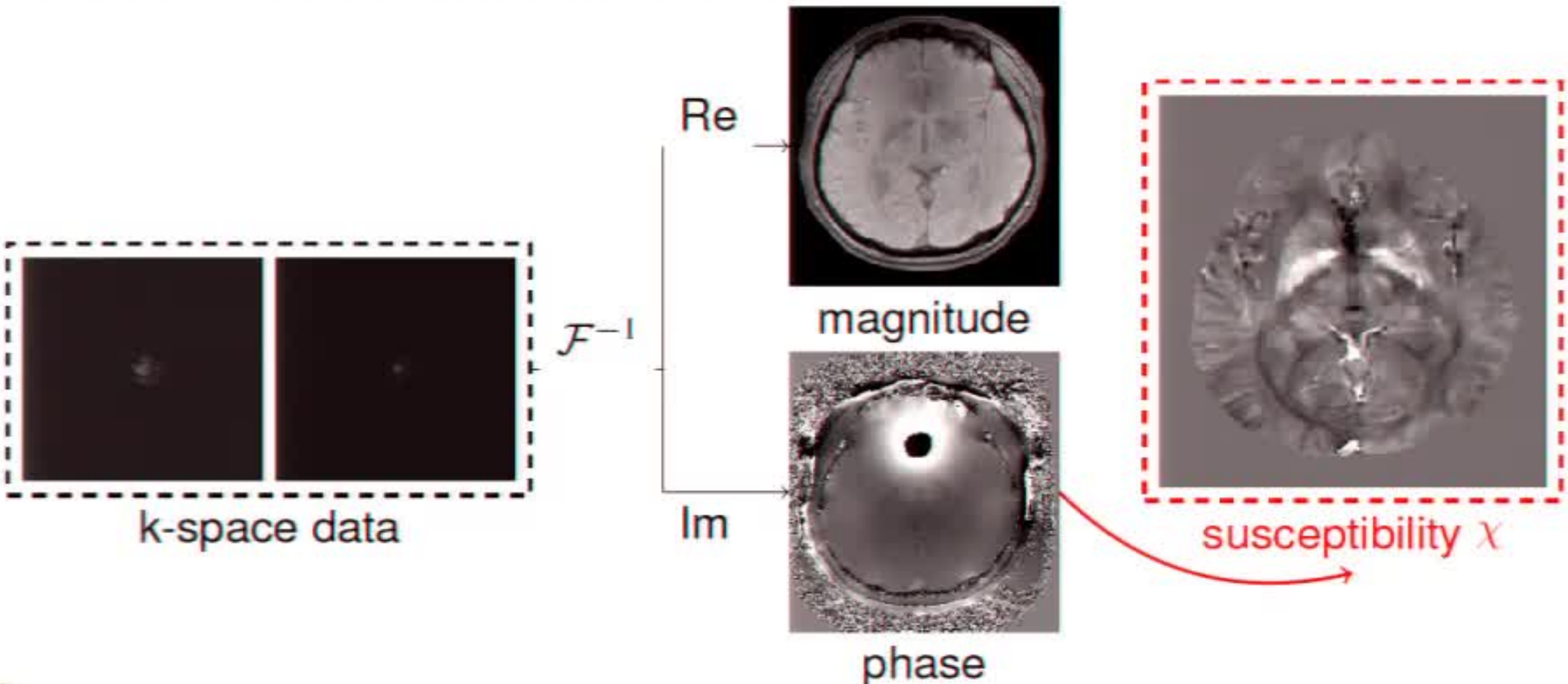


susceptibility map

Y Wang and T Liu
Quantitative susceptibility mapping (QSM): Decoding MRI data for a tissue magnetic biomarker.
Magnetic Resonance in Medicine; 73(1): 82–101, 2014.

Overview: Quantitative Susceptibility Mapping

- with Julianne Chung and Maximilian März



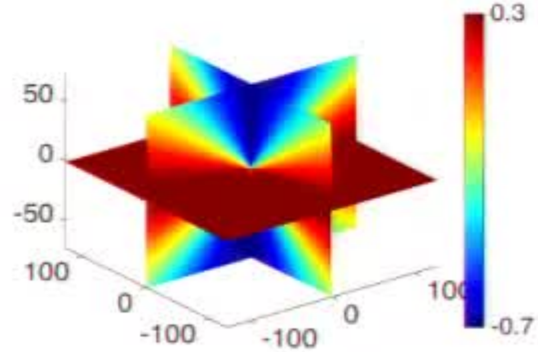
L Li, J S Leigh
Quantifying Arbitrary Magnetic Susceptibility Distributions with MR.
Magnetic Resonance in Medicine; 51: 1077–1082, 2004.

J Liu, T Liu, L de Rochefort et al.
Morphology Enabled Dipole Inversion for Quantitative Susceptibility Mapping. . . .
NeuroImage; 59:2560–2568; 2012.

Field to Source Inversion

Forward Problem

true image, $\mathbf{x}_{\text{true}}$



dipole kernel, $\mathbf{d}$

noisy data, $\mathbf{b}$

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \mathbf{n} = \mathbf{F}\ \mathrm{diag}(\mathbf{d})\ \mathbf{F}^*\mathbf{x}_{\text{true}} + \mathbf{n},$$

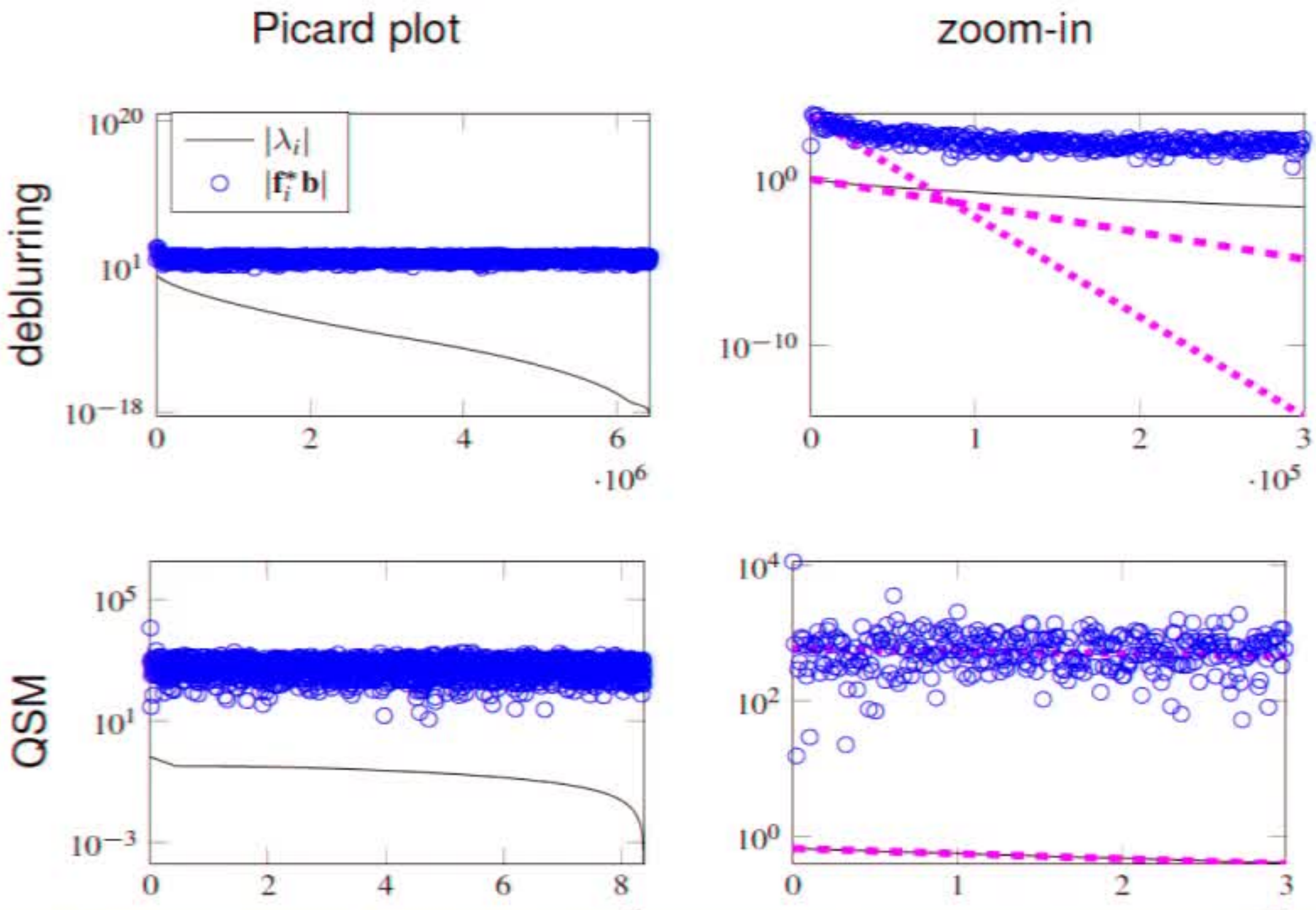
where

- $\mathbf{b} \in \mathbb{R}^n$ - discrete data
- $\mathbf{x}_{\text{true}} \in \mathbb{R}^n$ - magnetic susceptibility
- $\mathbf{F} \in \mathbb{C}^{n\times n}$ - unitary discrete Fourier transform matrix
- $\mathbf{d} \in \mathbb{R}^n$ - dipole kernel (Salomir et al. 2003)

$$\mathbf{d}(k) = 1/3 - k_z^2/(k_x^2 + k_y^2 + k_z^2)$$

- $\mathbf{n} \in \mathbb{R}^n$ - noise vector, $\mathbf{n} \sim N(0, \Sigma)$

Discrete Picard Condition



Inverse Problem (Liu et al., Neuroimage 2012)

$\mathrm{diag}(\mathbf{M})$ magnitude image $\mathrm{diag}(\mathbf{W}_1)$ $\mathrm{diag}(\mathbf{W}_2)$ $\mathrm{diag}(\mathbf{W}_3)$

Given noisy data $\mathbf{b}$, reconstruct susceptibility map $\mathbf{x}$ by solving

$$\min_{\mathbf{x}} \|\mathbf{M}(\mathbf{A}\mathbf{x} - \mathbf{b})\|_2^2 + \alpha \|\mathbf{W}\mathbf{G}\mathbf{x}\|_1.$$

Here,

- $\mathbf{A}$ - QSM forward operator (convolution with dipole kernel)
- $\mathbf{M}$ - Cholesky factor of inverse covariance matrix
- $\mathbf{G}$ - discrete image gradient
- $\mathbf{W} = \mathrm{diag}(\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3)$ - diagonal weighting matrix
- α - regularization parameter

Implementation Details

Most resources go into solving least-squares problem

$$\min_{\mathbf{x}} \left\| \begin{pmatrix} \mathbf{MA} \\ \sqrt{\mu}\mathbf{WG} \end{pmatrix} \mathbf{x} - \begin{pmatrix} \mathbf{Mb} \\ \sqrt{\mu}(\mathbf{p}_k - \mathbf{z}_k) \end{pmatrix} \right\|_2^2$$

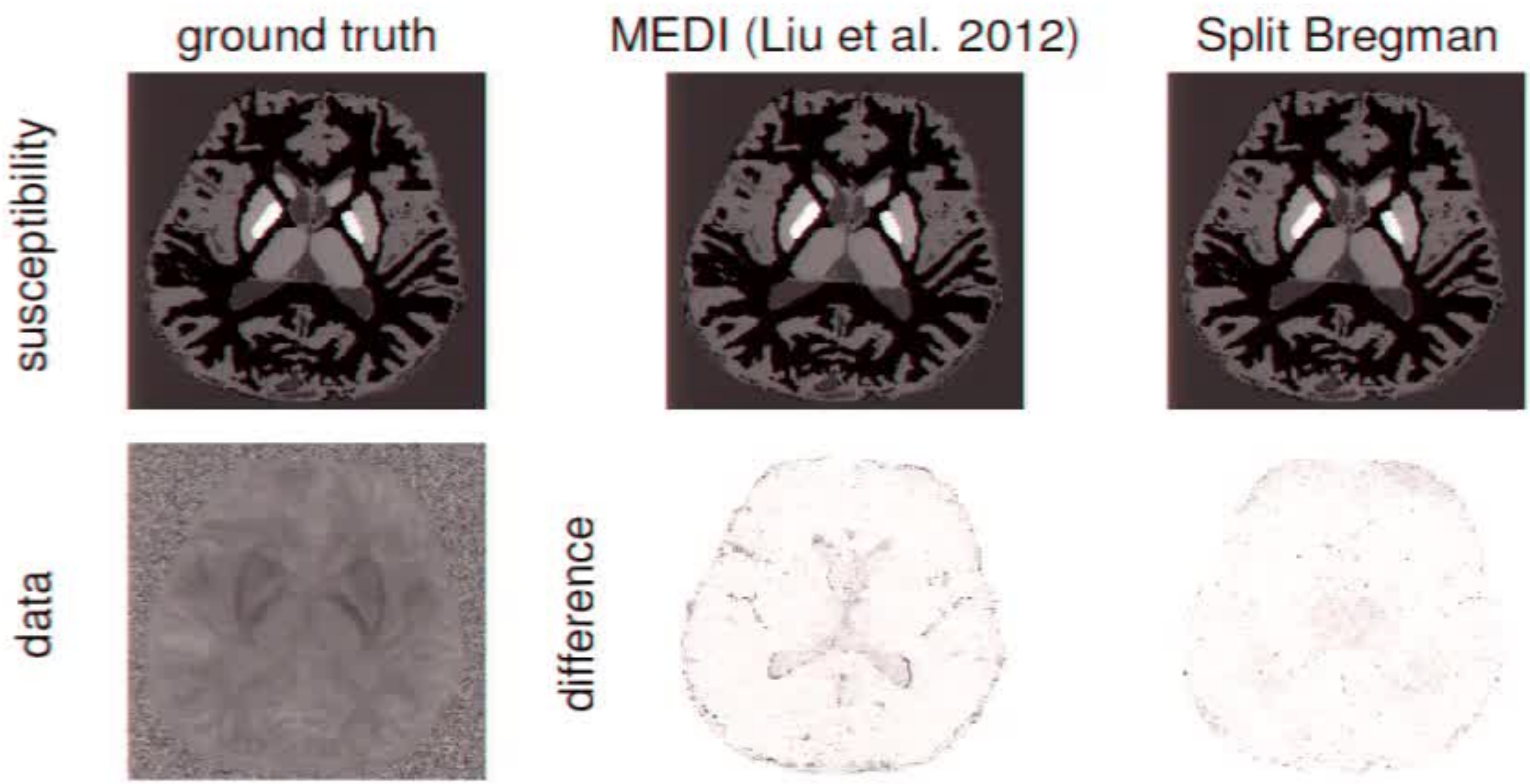
solve with

- LSQR with relatively large tolerance (`tol=0.01`)
- fixed parameter $\mu = 10\sqrt{\alpha}$.

Synthetic test problem: image size $256 \times 256 \times 96$

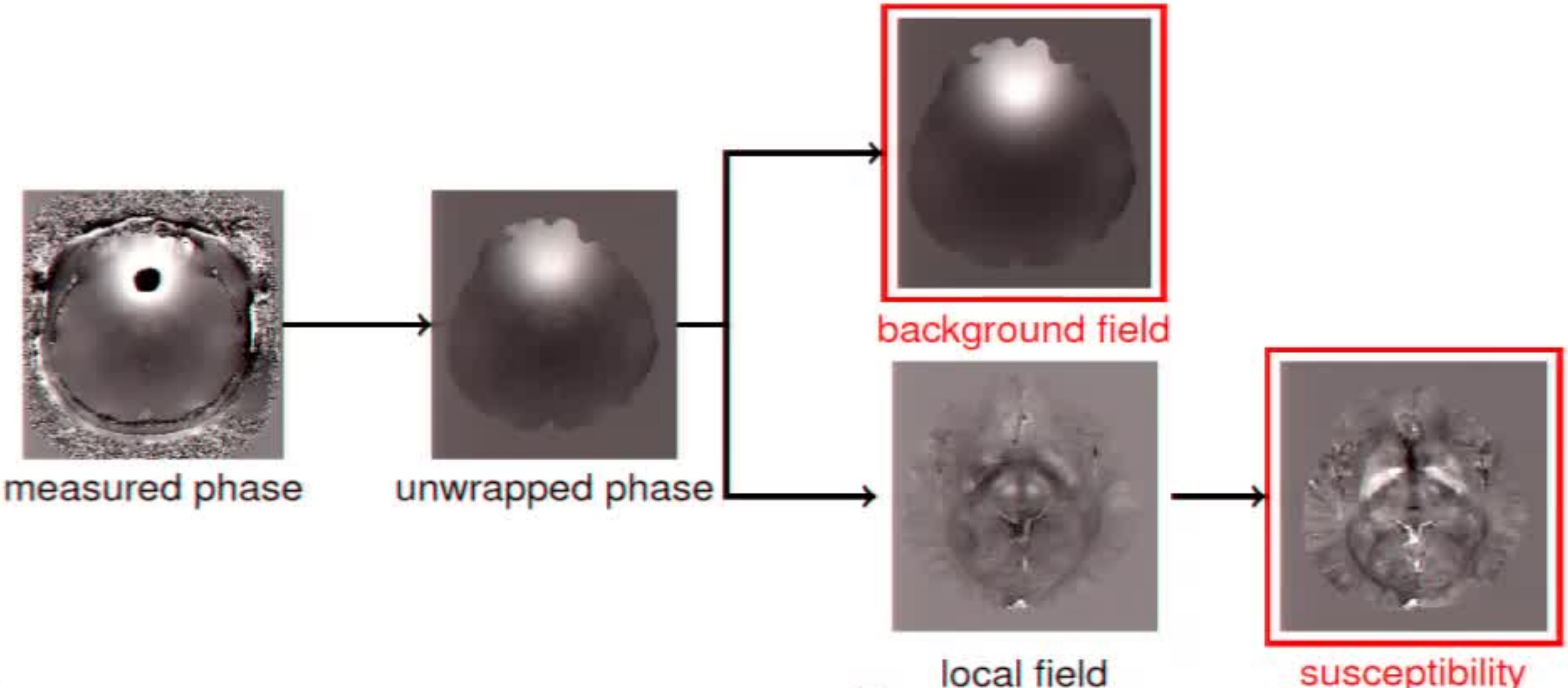
	MEDI (Liu et al. 2012)	Split Bregman
runtime	287.4 sec	71.3 sec
rel. error	15.8%	11.7%
# $(\mathbf{Av})$	600	122

Example: ℓ_1 QSM Reconstruction



QSM Background Field Removal

- Maximilian März, Emory / Berlin Mathematical School



Y Wen, D Zhou, T Liu, P Spincemaille, Y Wang
An Iterative Spherical Mean Value Method for Background Field Removal in MRI.
Magnetic Resonance in Medicine; 72(4), 2013.

D Zhou, T Liu, P Spincemaille, Y Wang.
Background Field Removal by Solving the Laplacian Boundary Value Problem.
NMR in Biomedicine; 2014.

Background Field is Harmonic

total field — background field, $\mathbf{b}_{\text{out}}$ — local field, $\mathbf{b}_{\text{in}}$

Let $\Omega \subset \mathbb{R}^3$ denote the region of interest. Then background field b_{out}, caused by sources in $\mathbb{R}^3 \setminus \Omega$, satisfies

$$\Delta b_{\text{out}}(x) = 0 \quad \text{for} \quad x \in \Omega \quad \text{and} \quad b_{\text{out}}(x) = g(x) \quad \text{for} \quad x \in \partial\Omega.$$

Zhou et al. NMR 2014: Discretize and compute background field by solving

$$\mathbf{L}_{\text{in}}\mathbf{b}_{\text{out}} = -\mathbf{L}_{\text{bc}}\mathbf{g}, \quad \text{and} \quad \mathbf{b}_{\text{in}} = \mathbf{b} - \mathbf{b}_{\text{out}}.$$

Note: $\mathbf{b}_{\text{out}} \gg \mathbf{b}_{\text{in}}$ / How to get $\mathbf{g}$?

Laplacian Boundary Value Problem

Problem: Boundary conditions may not be accurate close to boundary (no MRI signal in background).

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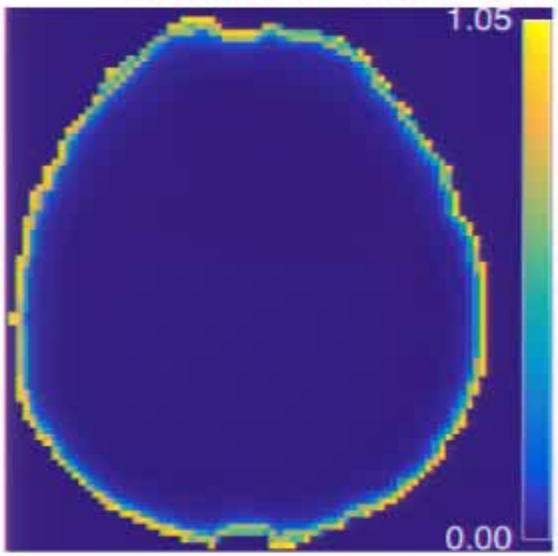
Experiment: Vary position of unit dipole inside the brain (only local field) and perform background field removal.

relative error of background field removal



Zhou et al., NMR, 2014

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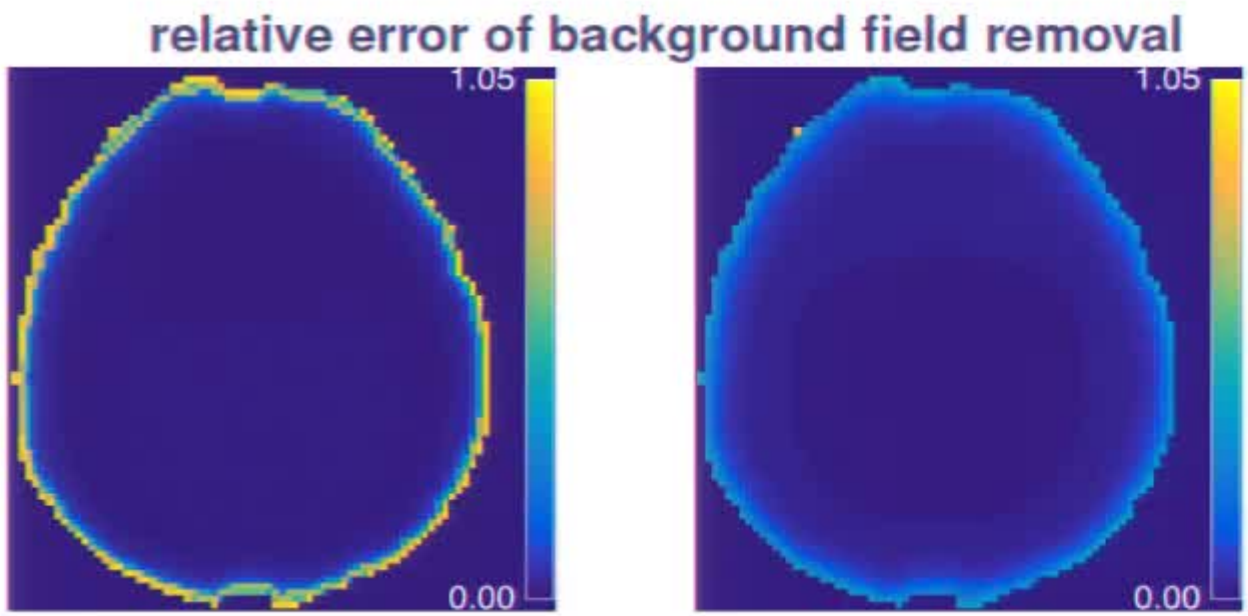

Zhou et al., NMR, 2014 optimized boundary conditions

Estimating Boundary Conditions

Given the total field $\mathbf{b}$, estimate boundary conditions by solving

$$\min_{\mathbf{g}} \left\| \mathbf{M}(-\mathbf{L}_{\text{in}}^{-1}\mathbf{L}_{\text{bc}}\mathbf{g} - \mathbf{b}) \right\|_2^2 + \beta \left\| \mathbf{g} - \mathbf{g}_{\text{ref}} \right\|_2^2 ,$$

where

- $\mathbf{L}_{\text{in}}$ - discrete Laplacian on Ω
- $\mathbf{L}_{\text{bc}}$ - boundary conditions on $\partial\Omega$
- $\mathbf{g}_{\text{ref}}$ - best guess of boundary conditions
- β - regularization parameter