A TOPOLOGICAL VIEW OF COLLECTIVE BEHAVIOR

Chad Topaz (Williams College)

NSF DMS-1813752

MY FIRST CONFERENCE

Fifth SIAM Conference on

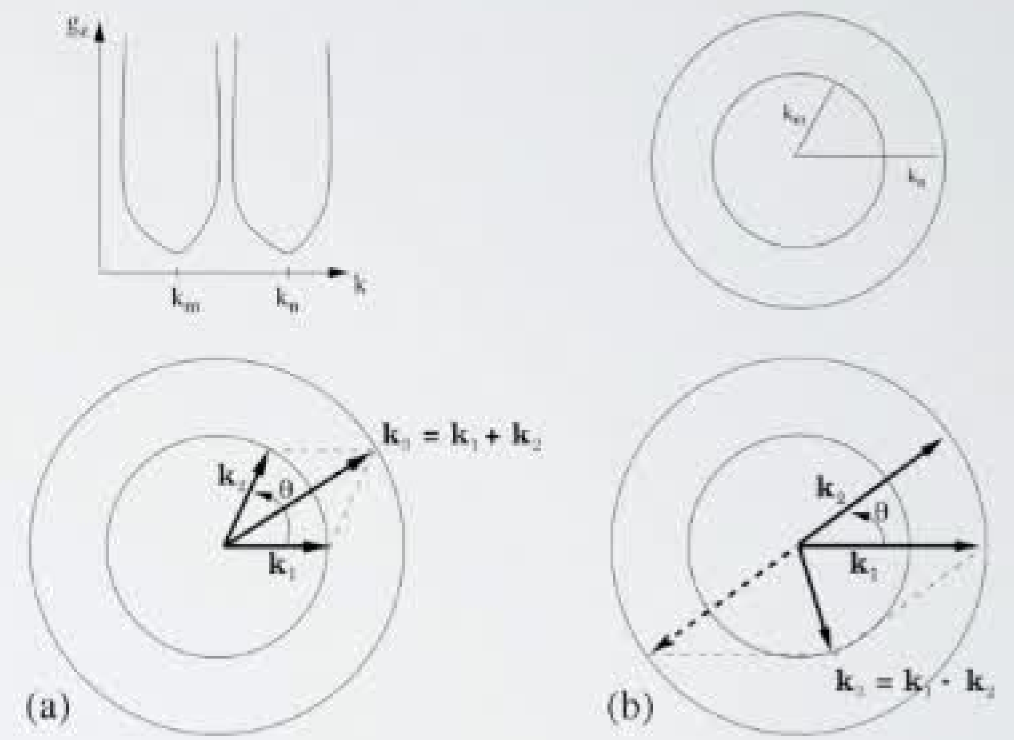
APPLICATIONS of DYNAMICAL SYSTEMS

May 23 - 27, 1999
Snowbird Ski and Summer Resort
Snowbird, Utah

Q: Something something co-dimension blah blah blah?

A: ¯_(ツ)_/¯

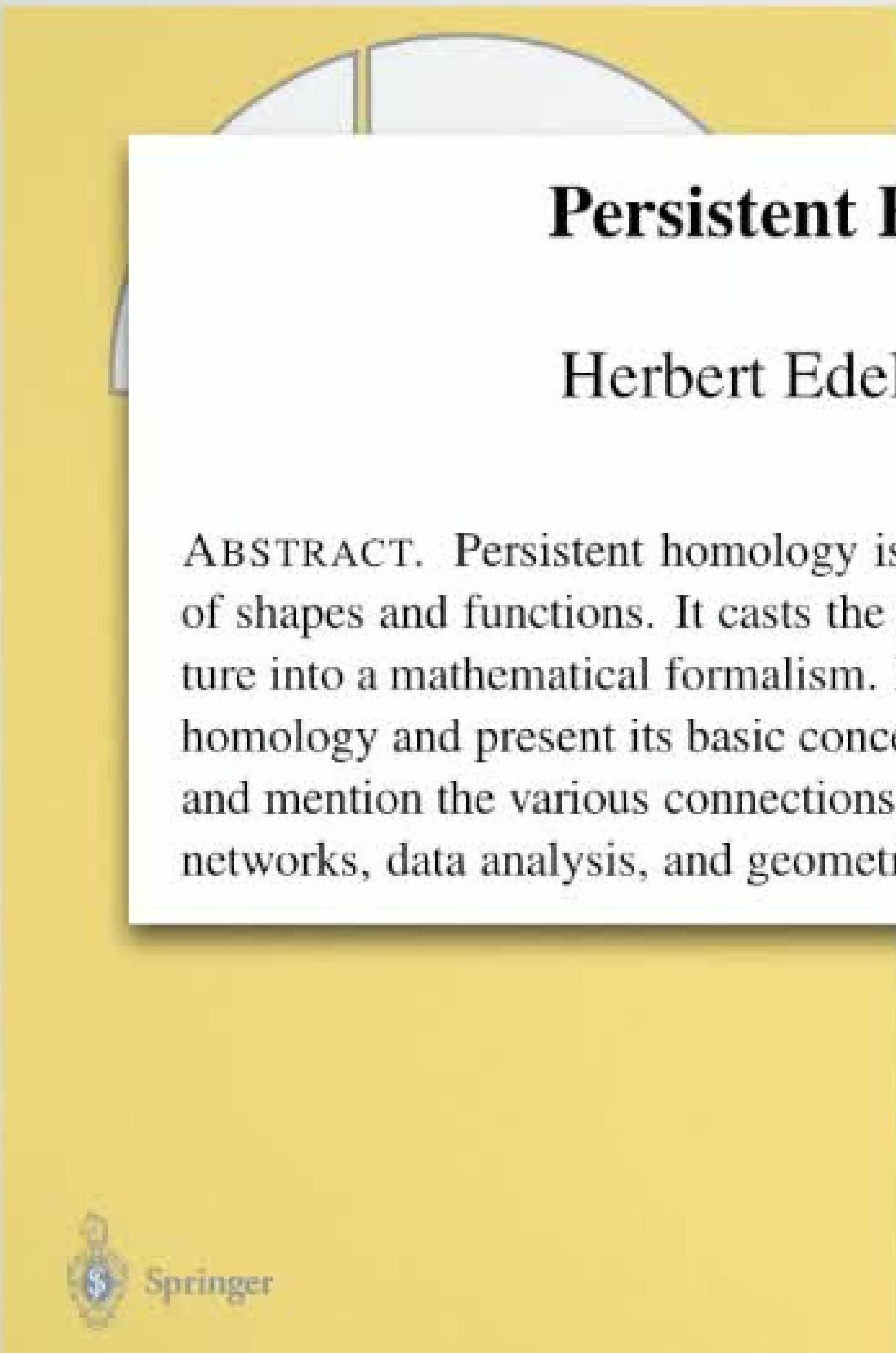
Pattern formation background:
Resonant triad interactions



Example (a) $\rightarrow h = z_1 e^{i\vec{k_1}\cdot\vec{x}} + z_2 e^{i\vec{k_2}\cdot\vec{x}} + z_3 e^{i\vec{k_3}\cdot\vec{x}} + c.c.$

$$\begin{aligned}
\dot{z}_1 &= \lambda z_1 + \alpha \bar{z}_2 z_3 + (a|z_1|^2 + b|z_2|^2 + c|z_3|^2) z_1 \\
\dot{z}_2 &= \lambda z_2 + \alpha \bar{z}_1 z_3 + (a|z_2|^2 + b|z_1|^2 + c|z_3|^2) z_2 \\
\dot{z}_3 &= \mu z_3 + \beta z_1 z_2 + (d|z_1|^2 + d|z_2|^2 + e|z_3|^2) z_3
\end{aligned}$$

TOPOLOGICAL DATA ANALYSIS

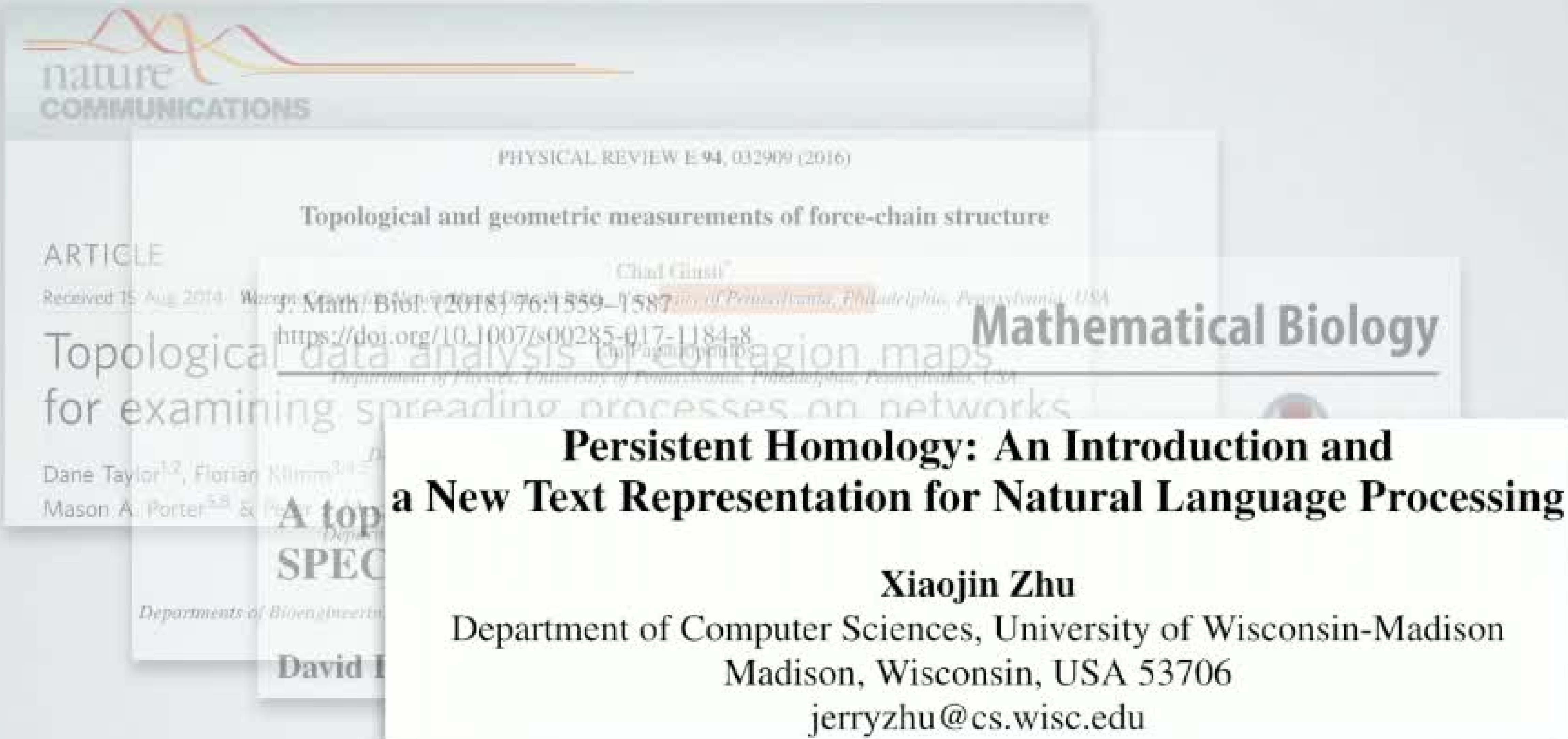


Persistent Homology — a Survey

Herbert Edelsbrunner and John Harer

ABSTRACT. Persistent homology is an algebraic tool for measuring topological features of shapes and functions. It casts the multi-scale organization we frequently observe in nature into a mathematical formalism. Here we give a record of the short history of persistent homology and present its basic concepts. Besides the mathematics we focus on algorithms and mention the various connections to applications, including to biomolecules, biological networks, data analysis, and geometric modeling.

TDA AT WORK


nature
COMMUNICATIONS

ARTICLE

Topological data analysis of contagion maps for examining spreading processes on networks

Dane Taylor, Florian Klimm, Mason A. Porter

PHYSICAL REVIEW E 94, 032909 (2016)

Topological and geometric measurements of force-chain structure

Chad Giusti

J. Math. Biol. (2018) 76:1559–1587
https://doi.org/10.1007/s00285-017-1184-8

Mathematical Biology

Persistent Homology: An Introduction and a New Text Representation for Natural Language Processing

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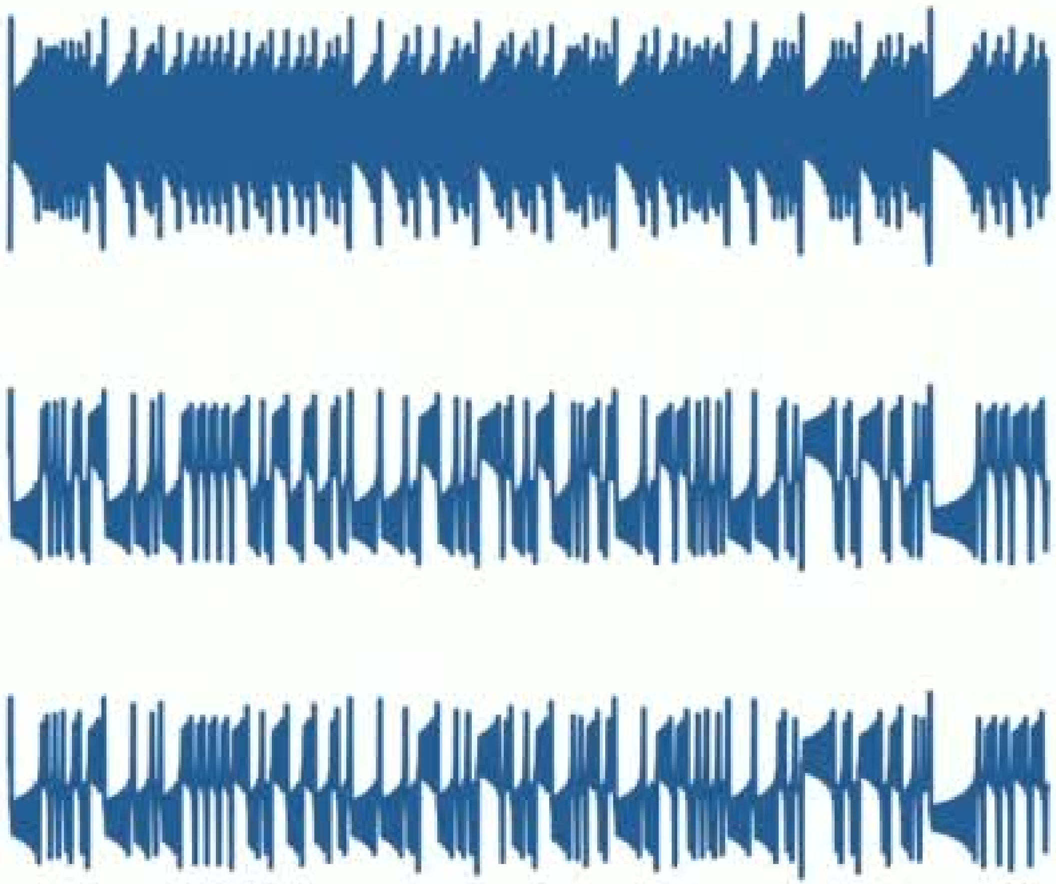
jerryzhu@cs.wisc.edu

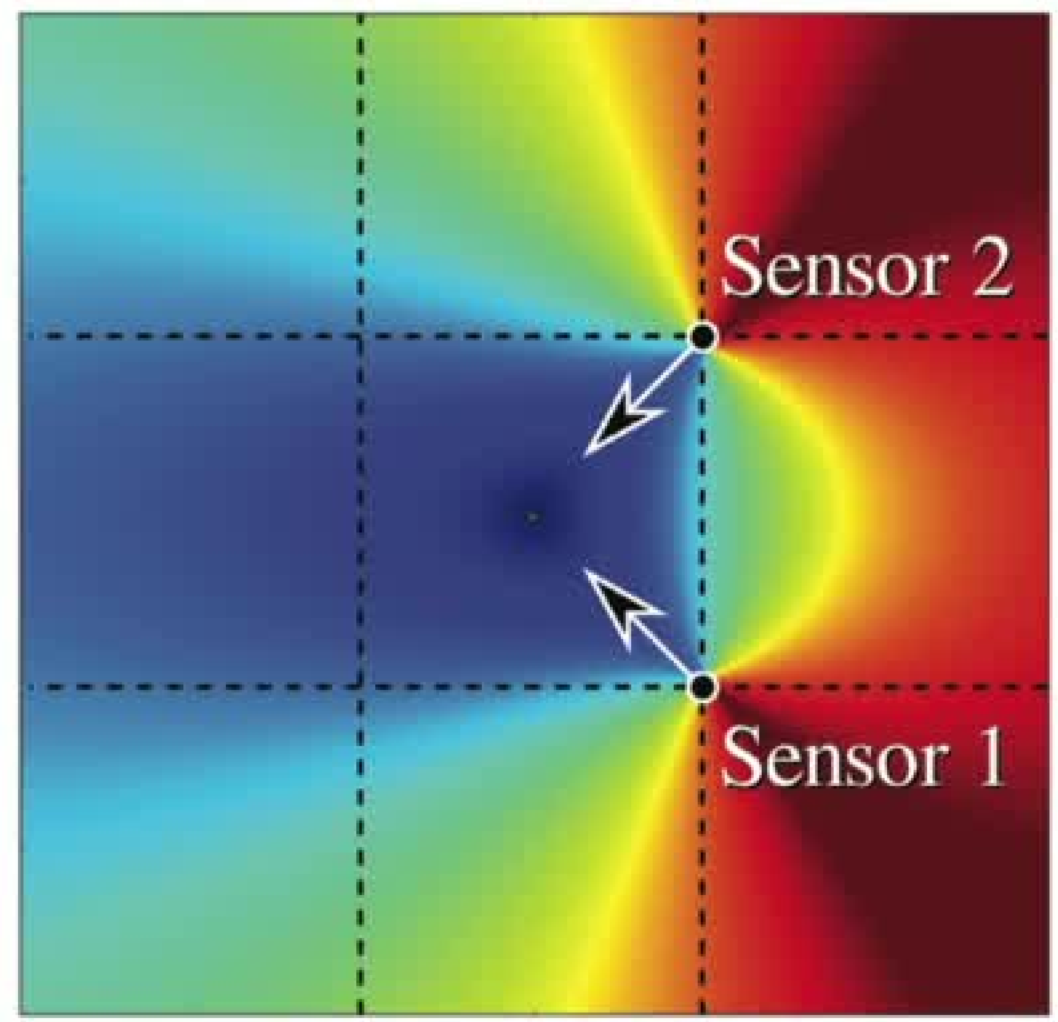
AMS NOTICES, MAY 2019

Topological Time Series Analysis



Jose A. Perea

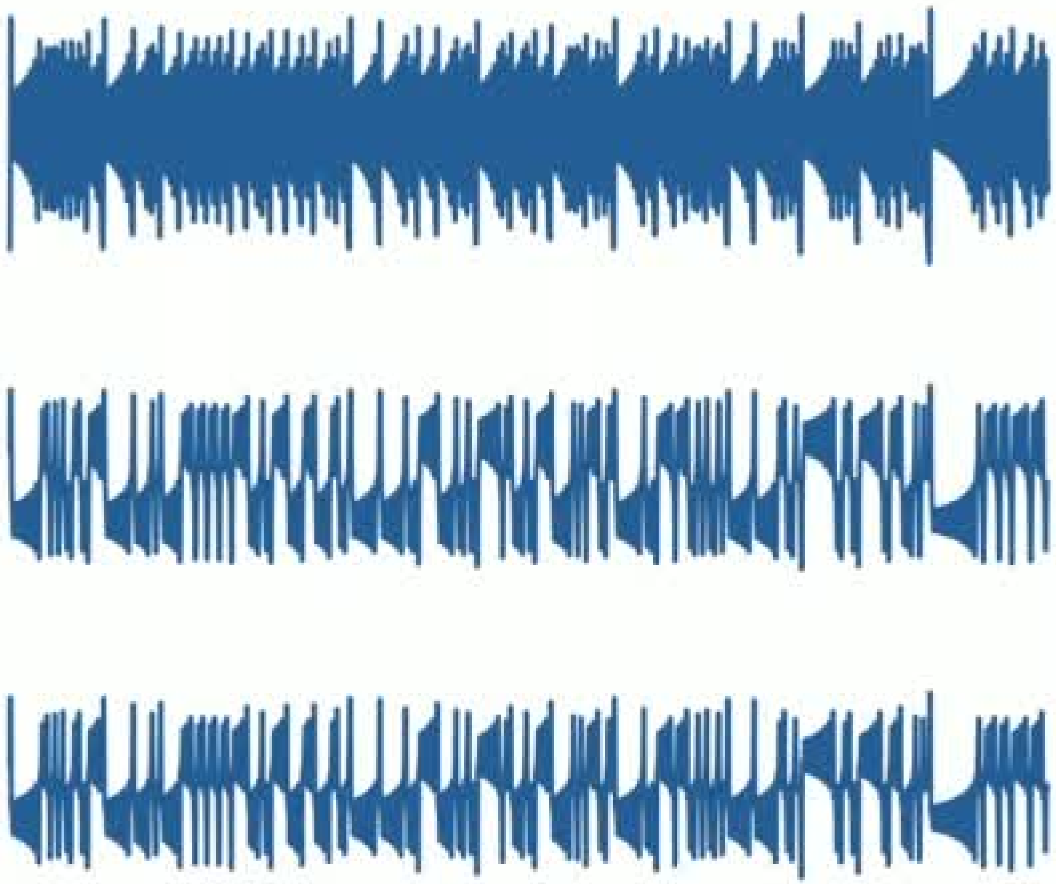
Hunting for Foxes with Sheaves



Michael Robinson

AMS NOTICES, MAY 2019

Topological Time Series Analysis



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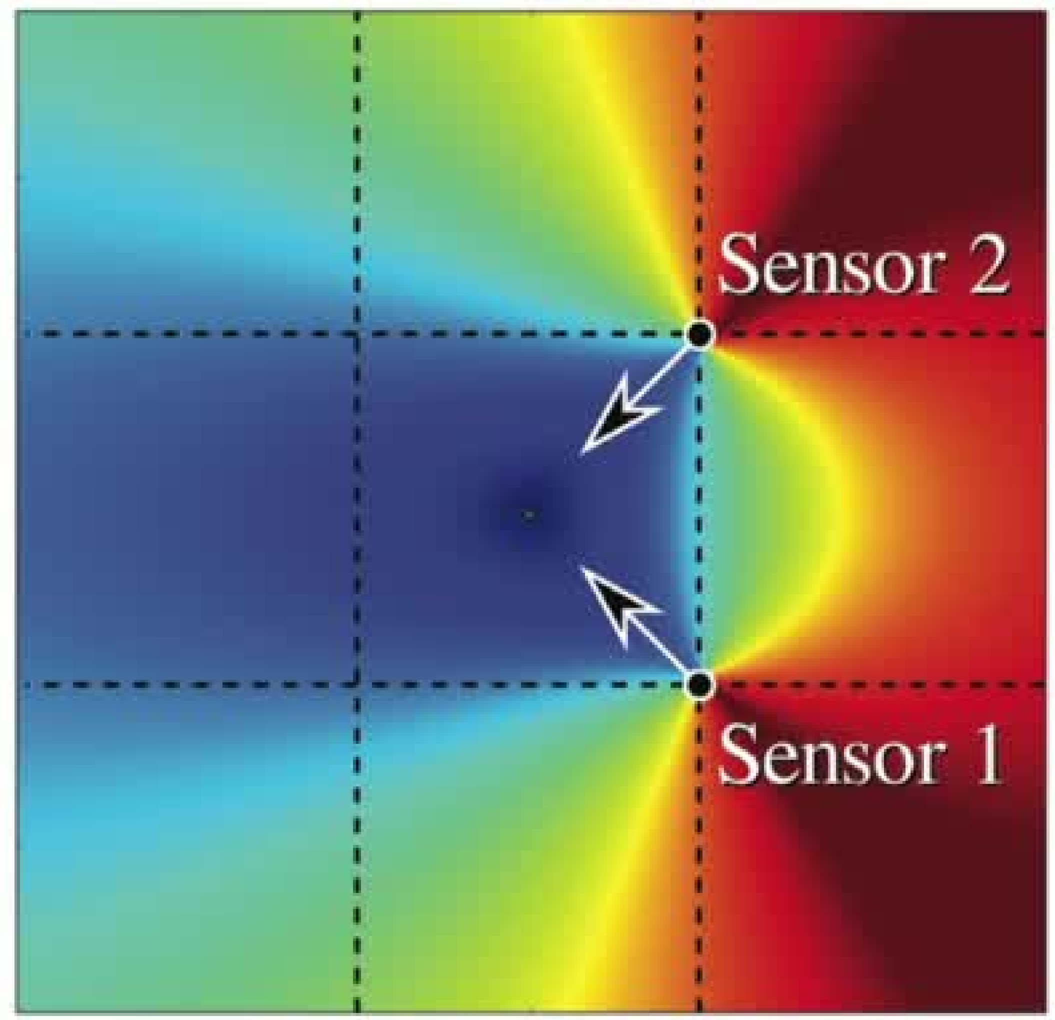
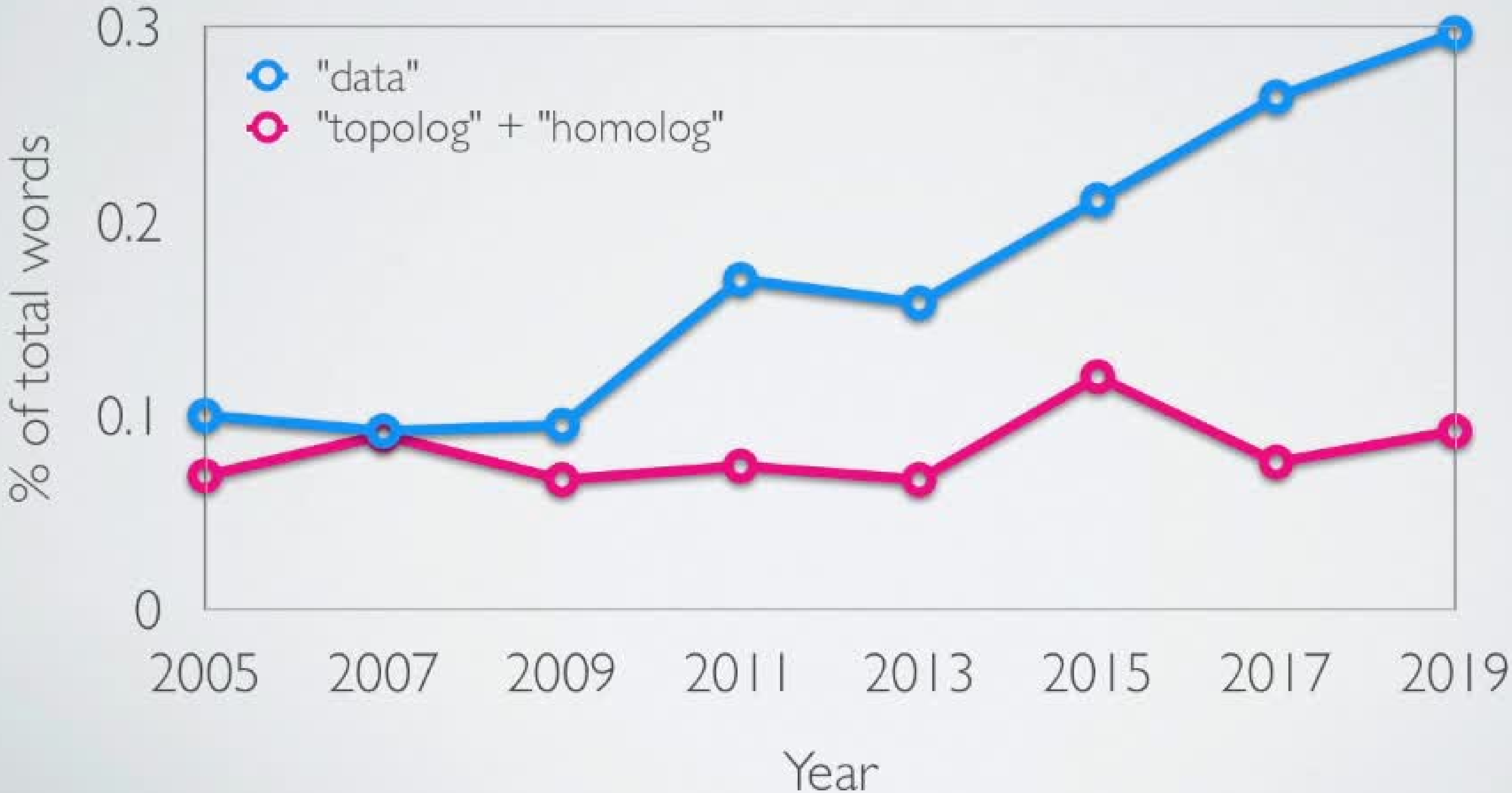


Michael Robinson

TOPOLOGY AT SIAM DS

Occurrences of specific words as percentage of total words in SIAM DS abstract booklet



BACK TO 2009

Theorem 2.1 (structure) If D is a PID, then every finitely generated D-module is isomorphic to a direct sum of cyclic D-modules. That is, it decomposes uniquely into the form

$$D^\beta \oplus \left(\bigoplus_{i=1}^{m} D/d_i D \right), \qquad (1)$$

for $d_i \in D, \beta \in \mathbb{Z}$, such that $d_i | d_{i+1}$. Similarly, every graded module M over a graded PID D decomposes uniquely into the form

$$\left(\bigoplus_{i=1}^{n} \Sigma^{\alpha_i} D \right) \oplus \left(\bigoplus_{j=1}^{m} \Sigma^{\gamma_j} D/d_j D \right), \qquad (2)$$

where $d_j \in D$ are homogeneous elements so that $d_j | d_{j+1}$, $\alpha_i, \gamma_j \in \mathbb{Z}$, and Σ^α denotes an α-shift upward in grading.

BACK TO 2009

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TOPOLOGY 1, CHAD 0

Theorem 2.1 (structure) If [redacted] is a [redacted] then every [redacted] is [redacted] to a [redacted] of [redacted] That is, it [redacted] into the form

[redacted] (1)

for [redacted] such that [redacted] Similarly, every [redacted] over a [redacted] into the form

[redacted] (2)

where [redacted] are [redacted] so that [redacted] denotes an [redacted] in [redacted].

MY OUTLOOK, CHANGED

"Think of a bird flock as a bunch of data points moving in six-dimensional position-velocity space. At every moment in time, build an object by forming connections between points that are close. Ask about the topology of that object, and how it changes as your notion of closeness changes."

– Lori Ziegelmeier, 2014
Assistant Professor of Mathematics
Macalester College

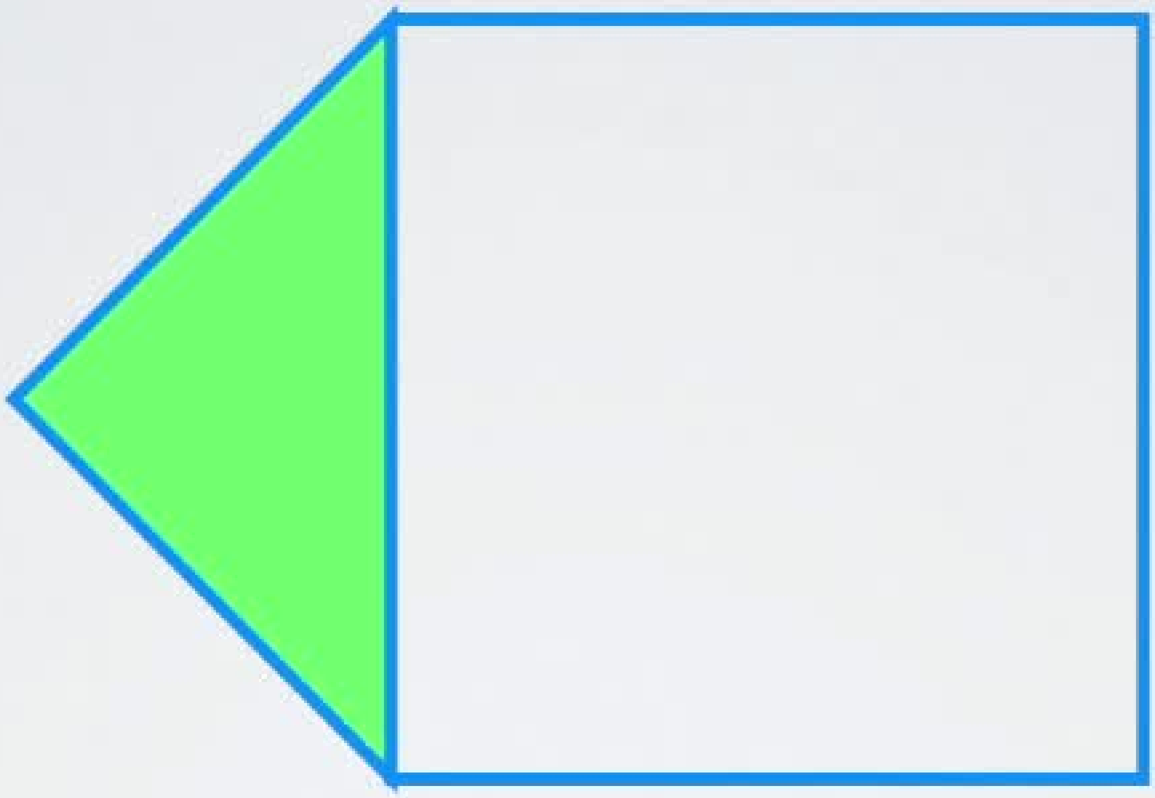
(T)APOLOGY

Main idea: Betti numbers of simplicial complexes are useful topological invariants that you can calculate with linear algebra.

TOPOLOGY

TOPOLOGY

SIMPLICIAL COMPLEXES



loops $b_1 = 3 - 1$ (homologous) $- 1$ (boundary) $= 1$

loops $b_1 = \dim \mathrm{Ker}\, \partial_{1\to 0} - \dim \mathrm{Im}\, \partial_{2\to 1}$

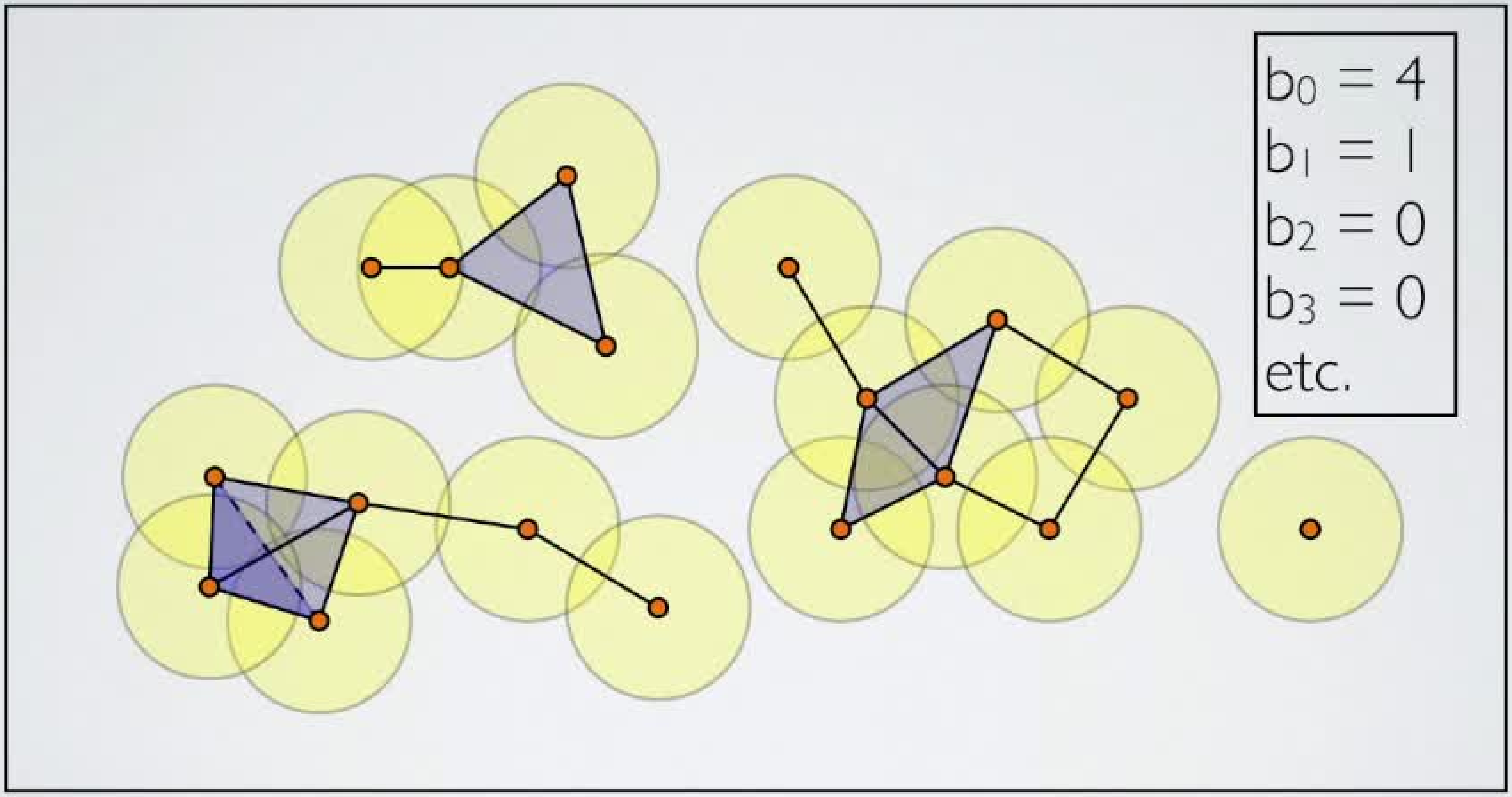
IT'S LINEAR ALGEBRA

Chad's Self-Help
Homology Tutorial
For the Simple(x)-Minded

If you don't love topology then how in the hell are you gonna love yourself?

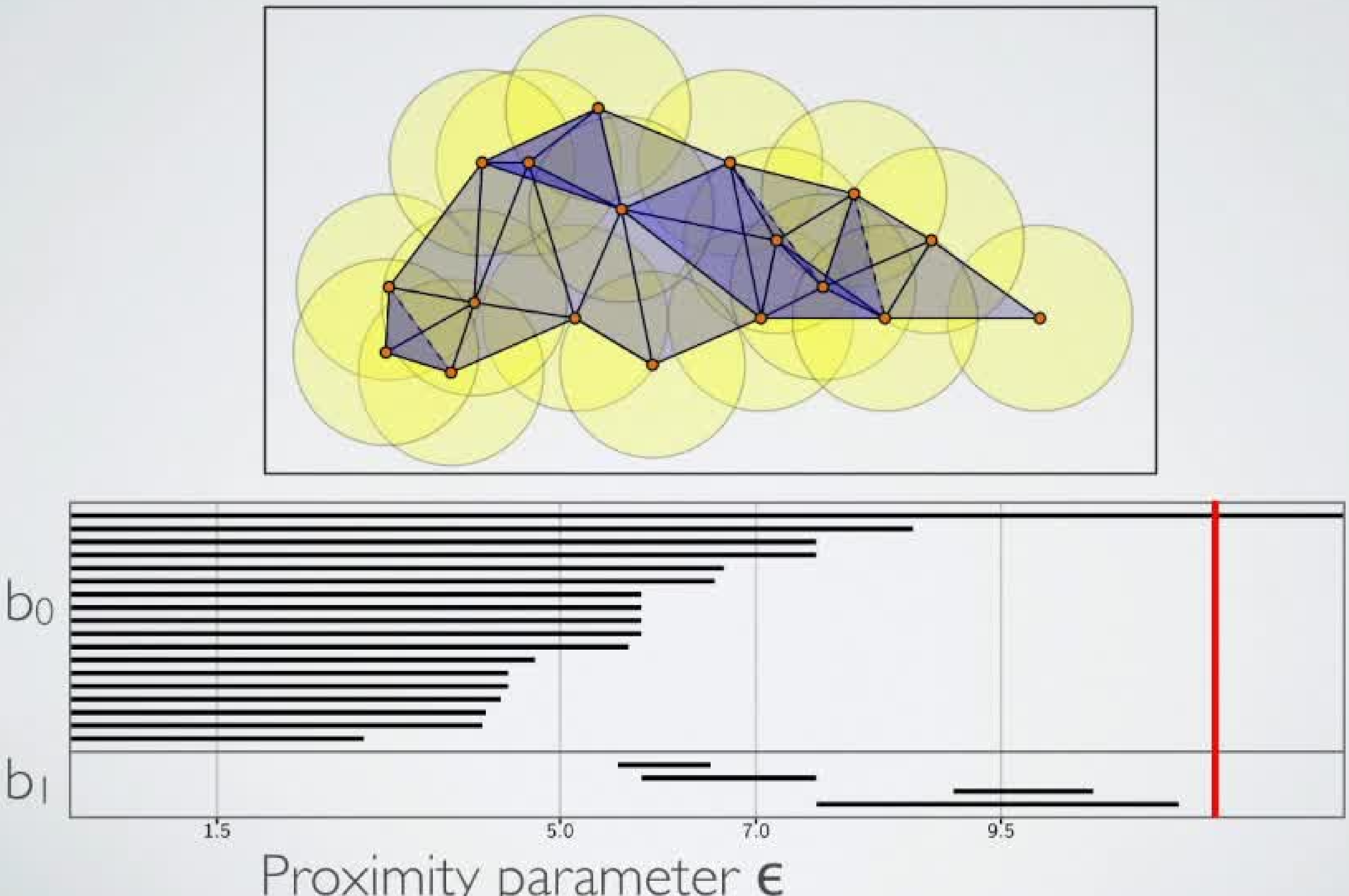
A Full-Color Extravaganza

CALCULATE BETTI NUMBERS



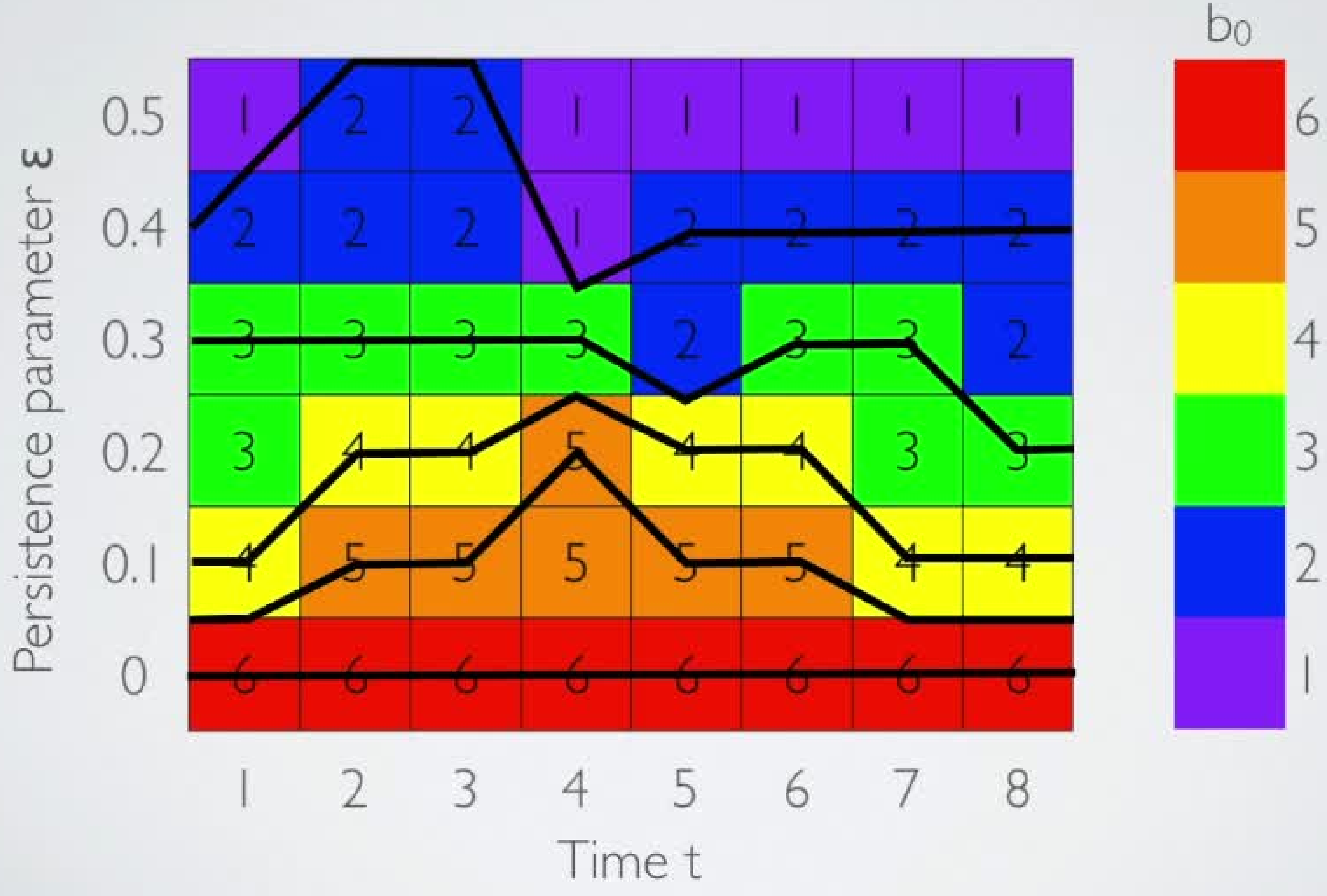
Vietoris-Rips Complex

FIND PERSISTENT HOMOLOGY



START AGAIN WITH PERSISTENT HOMOLOGY

EVOLVE IN TIME



EVOLVE IN TIME

OTHER TOPOLOGICAL SIGNATURES MAY INCLUDE

- Persistence diagrams
- Persistence landscapes
- Persistence images
- Vineyards
- Crocker videos
- Who knows what else

OTHER TOPOLOGICAL SIGNATURES MAY INCLUDE

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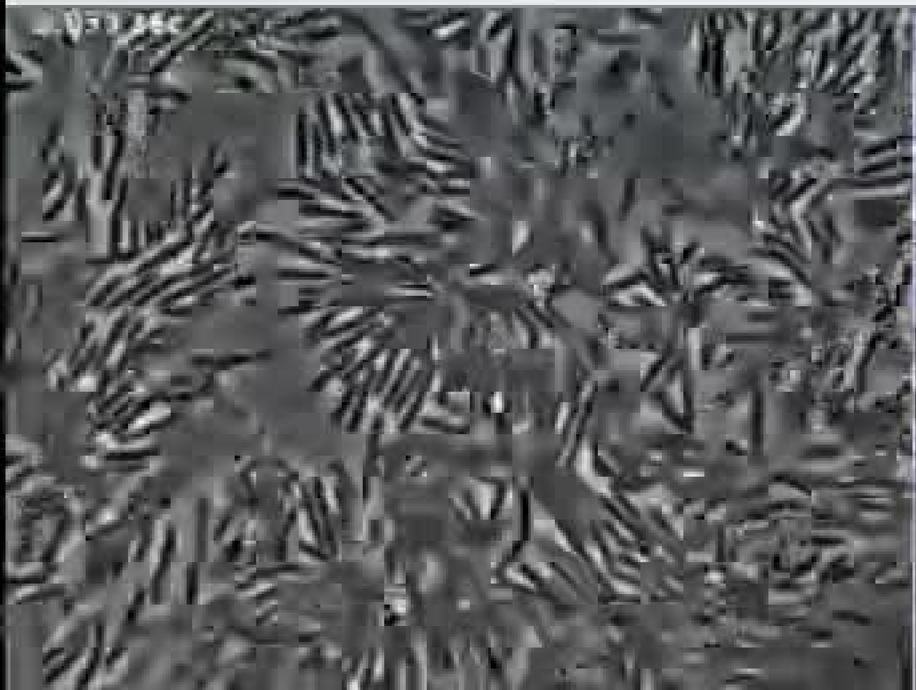
COLLECTIVE MOTION





COLLECTIVE MOTION





COLLECTIVE MOTION



A SEMINAL MODEL OF ALIGNING AGENTS

Novel type of phase transition in a system of self-driven particles

T Vicsek, A Czirók, E Ben-Jacob, I Cohen, O Shochet - Physical review letters, 1995 - APS

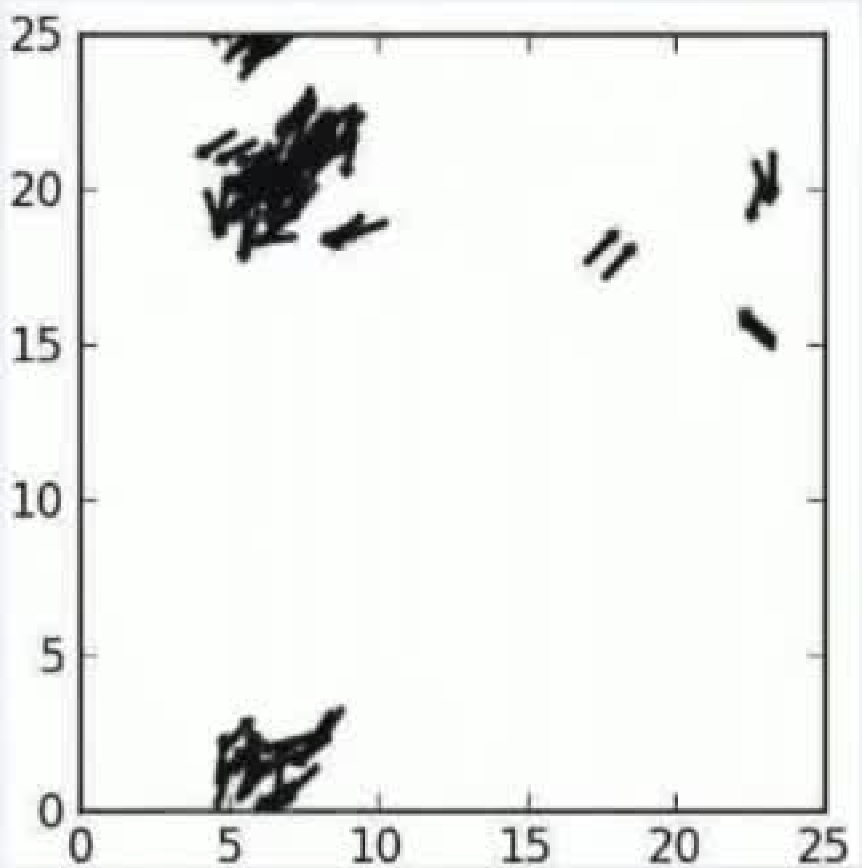
☆ Cited by 5296 Related articles All 30 versions

alignment noise

$$\theta_i \rightarrow \boxed{\langle \theta_j \rangle_{|\mathbf{x}_i - \mathbf{x}_j| \leq R}} + \boxed{U(-\eta/2, \eta/2)}$$

$$\mathbf{v}_i \rightarrow v_0 \left(\cos \theta_i, \sin \theta_i \right)$$

$$\mathbf{x}_i \rightarrow \mathbf{x}_i + \mathbf{v}_i \, \Delta t$$



http://youtu.be/jphRZV3oCaI

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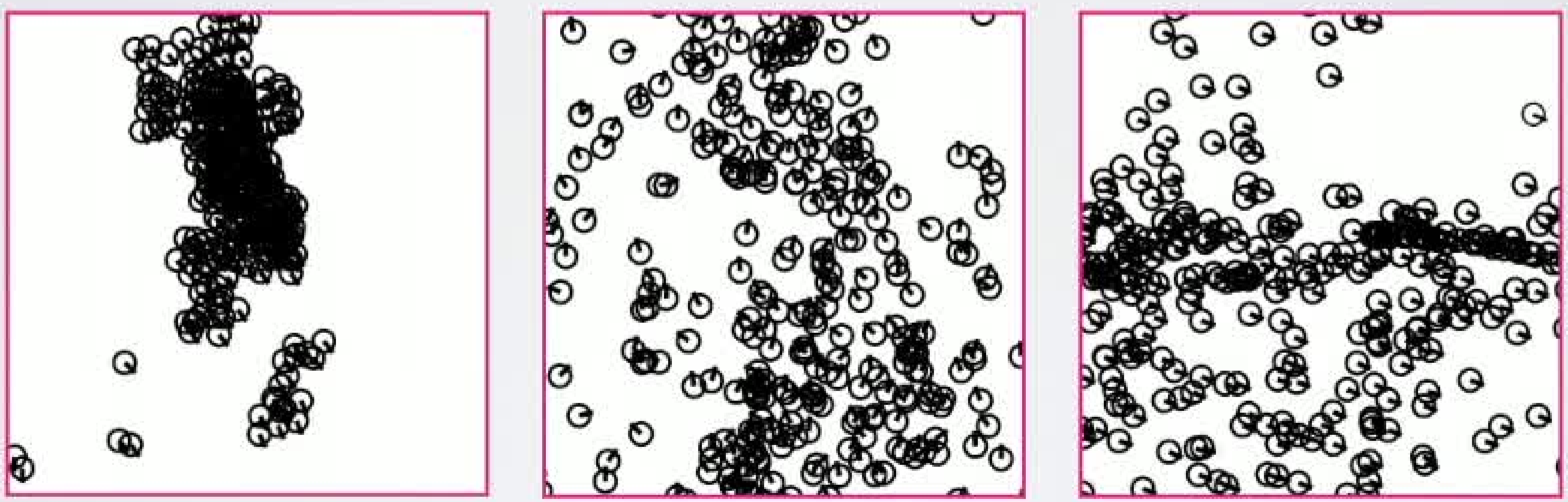
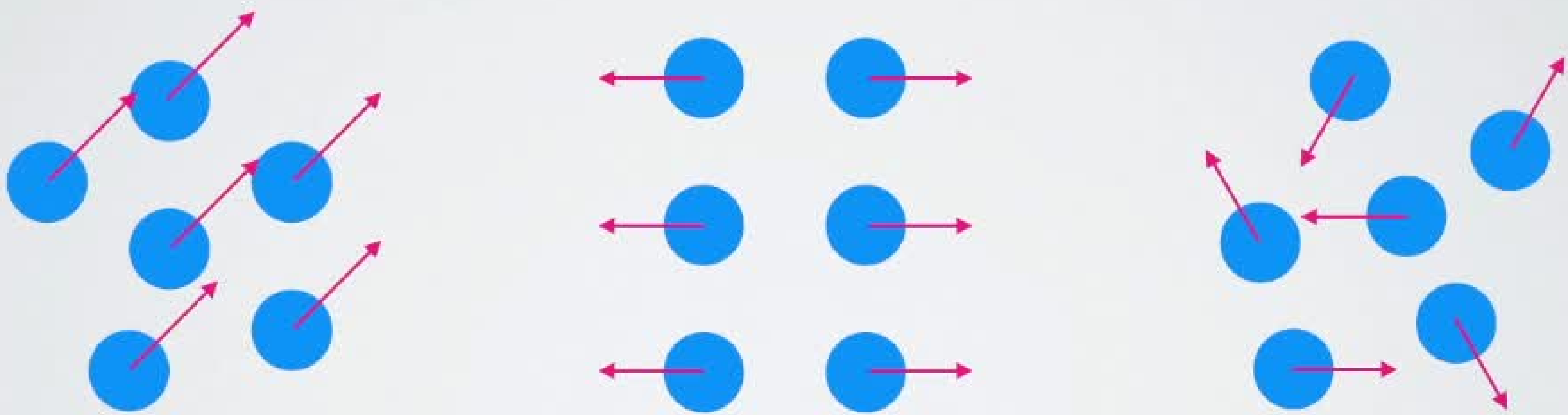


clusters loose alignment polarization

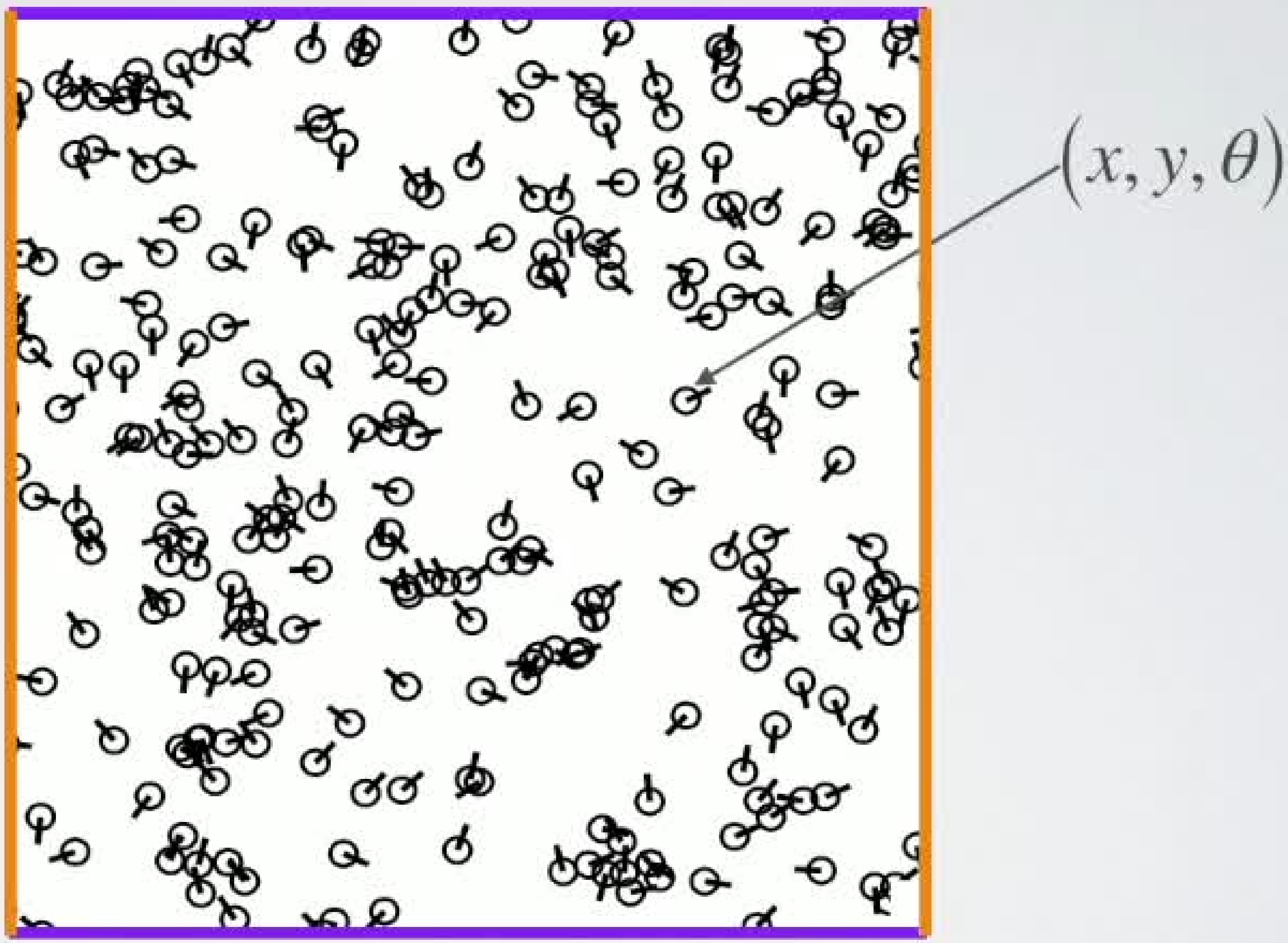
SUMMARIZE DATA WITH ORDER PARAMETER

$$\text{Alignment } \phi(t) = \frac{1}{N v_0} \left| \sum_{i=1}^{N} \mathbf{v}_i(t) \right|$$



$\phi = 1$ $\phi = 0$ $\phi = 0$

INITIAL CONDITION COVERS A THREE TORUS

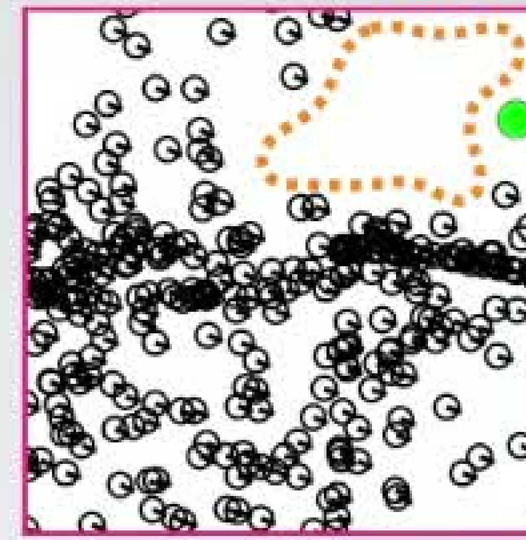


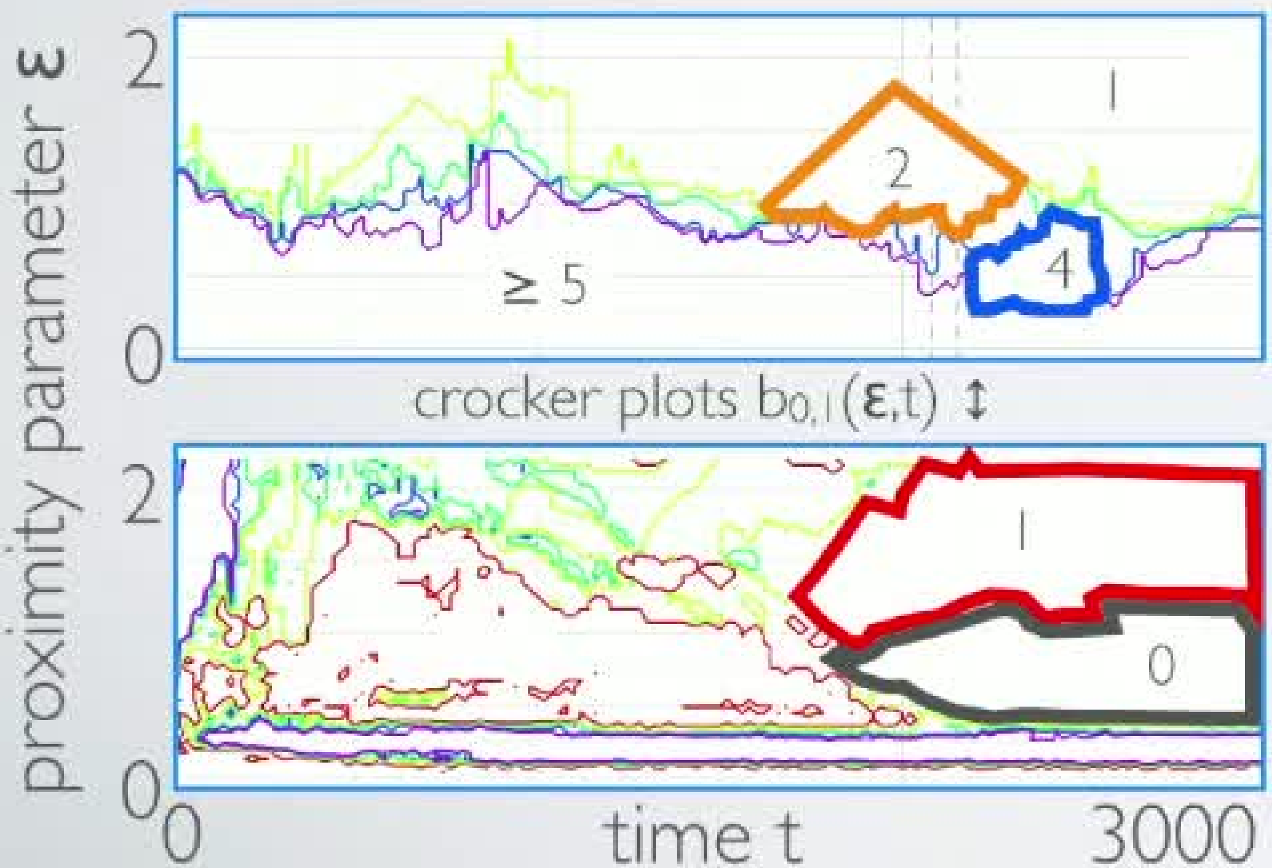
b = (1,3,3,1,0,...)

TOPOLOGICAL ORDER PARAMETER

Parameter Set #1
Clusters?

Parameter Set #2
Polarization?





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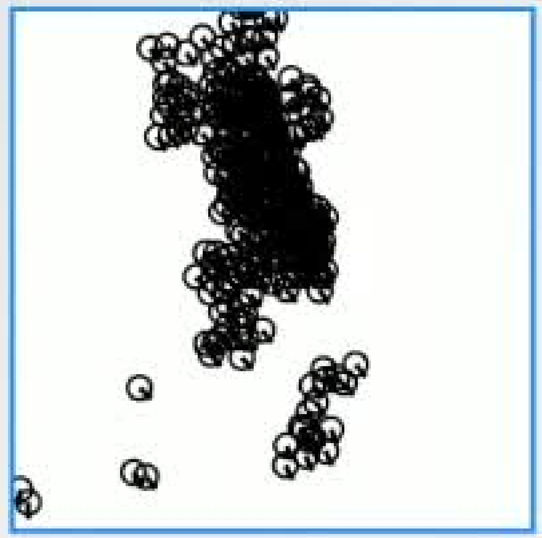
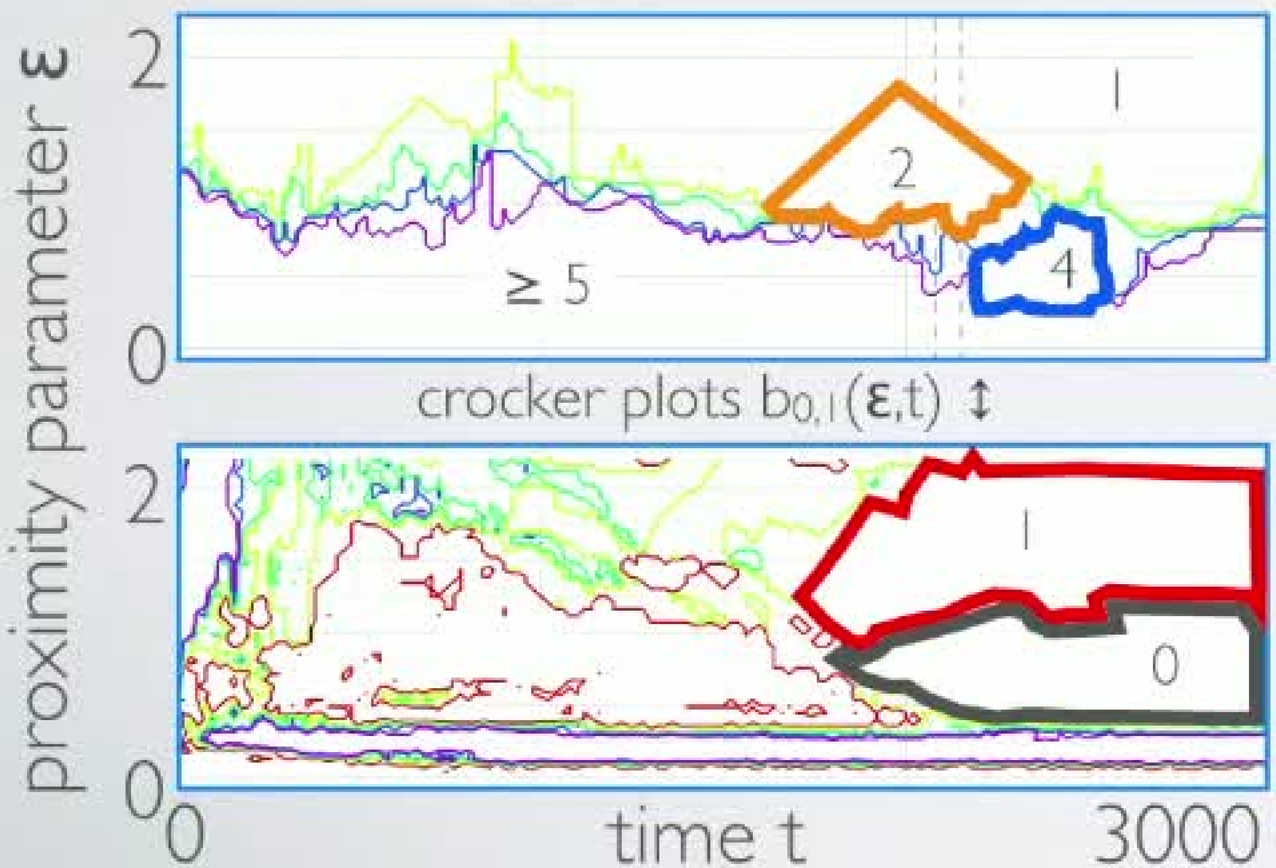
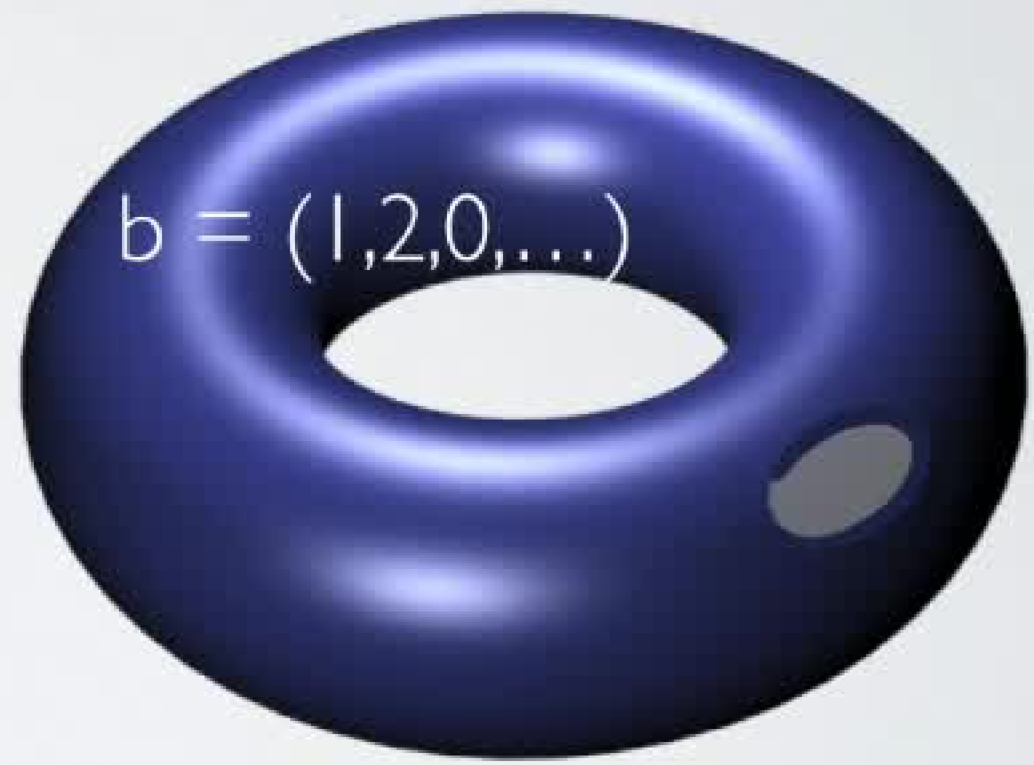
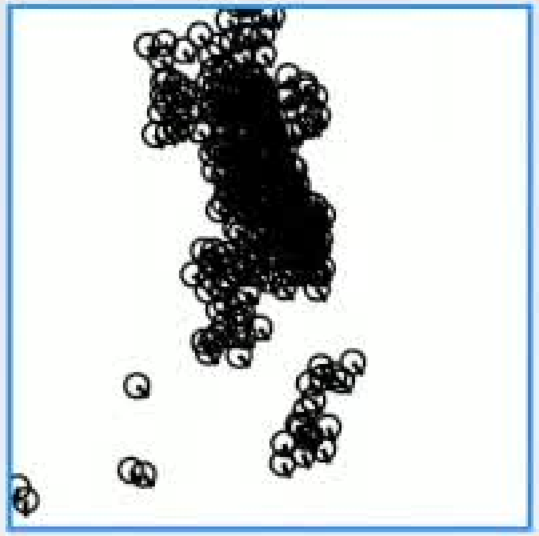




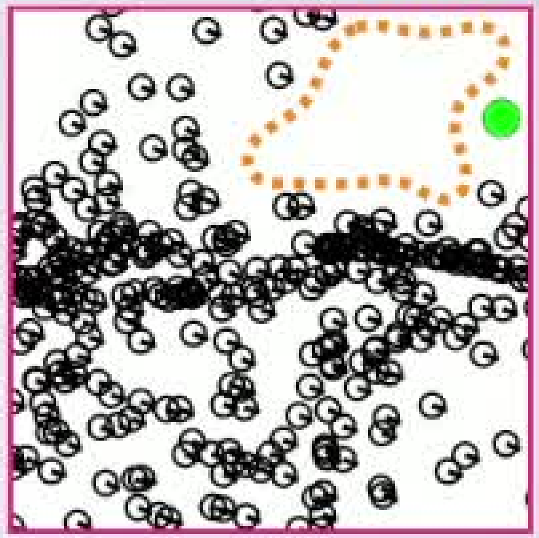


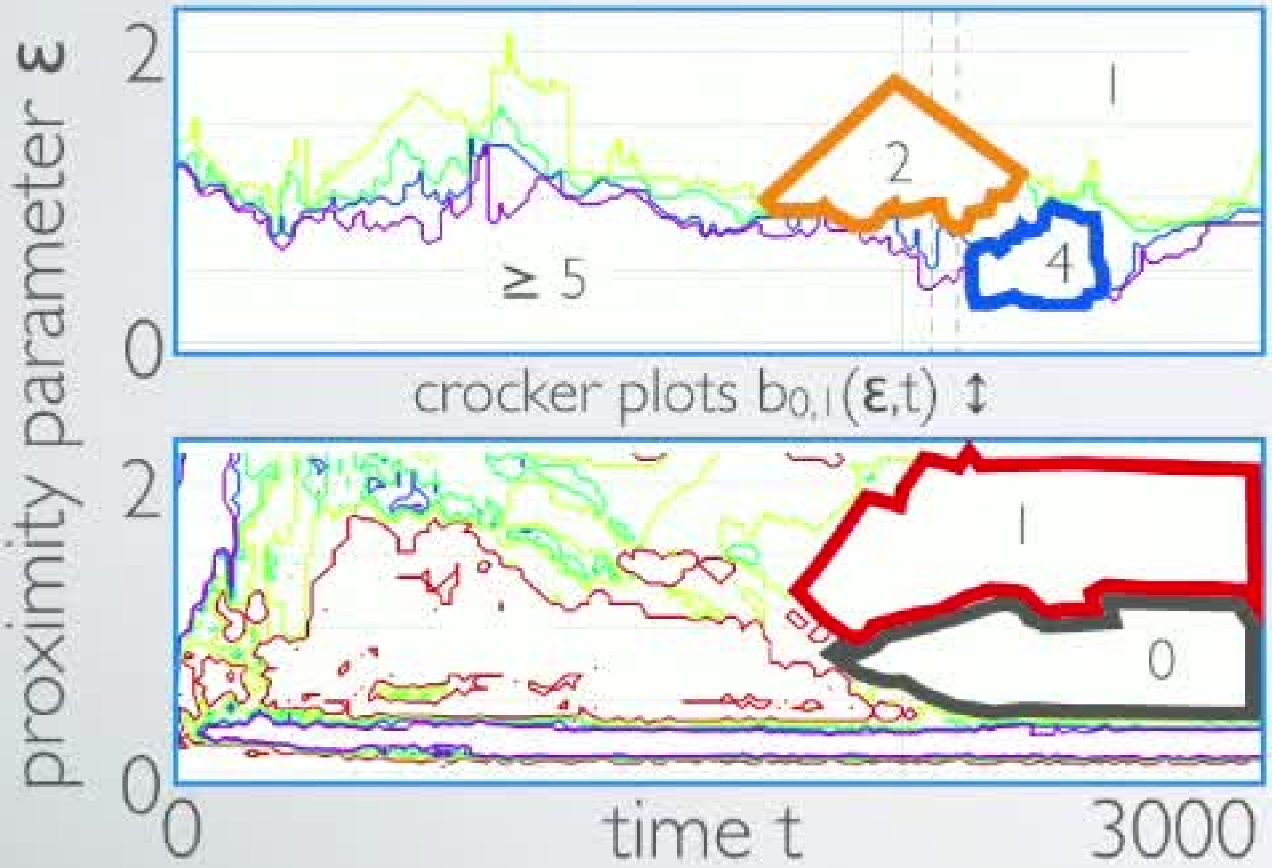
TOPOLOGICAL ORDER PARAMETER

Parameter Set #1
Clusters?



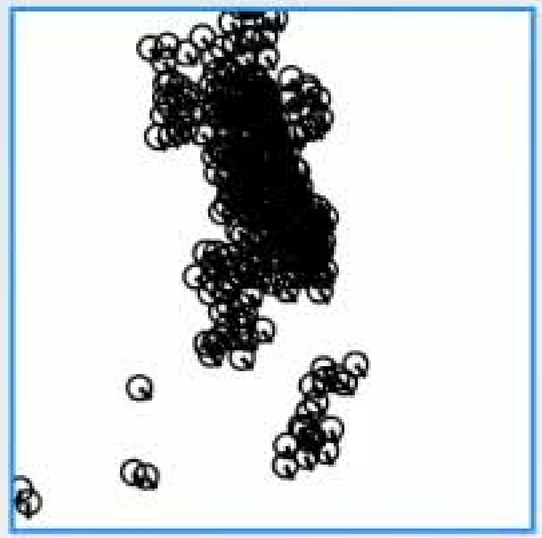
Parameter Set #2
Polarization?

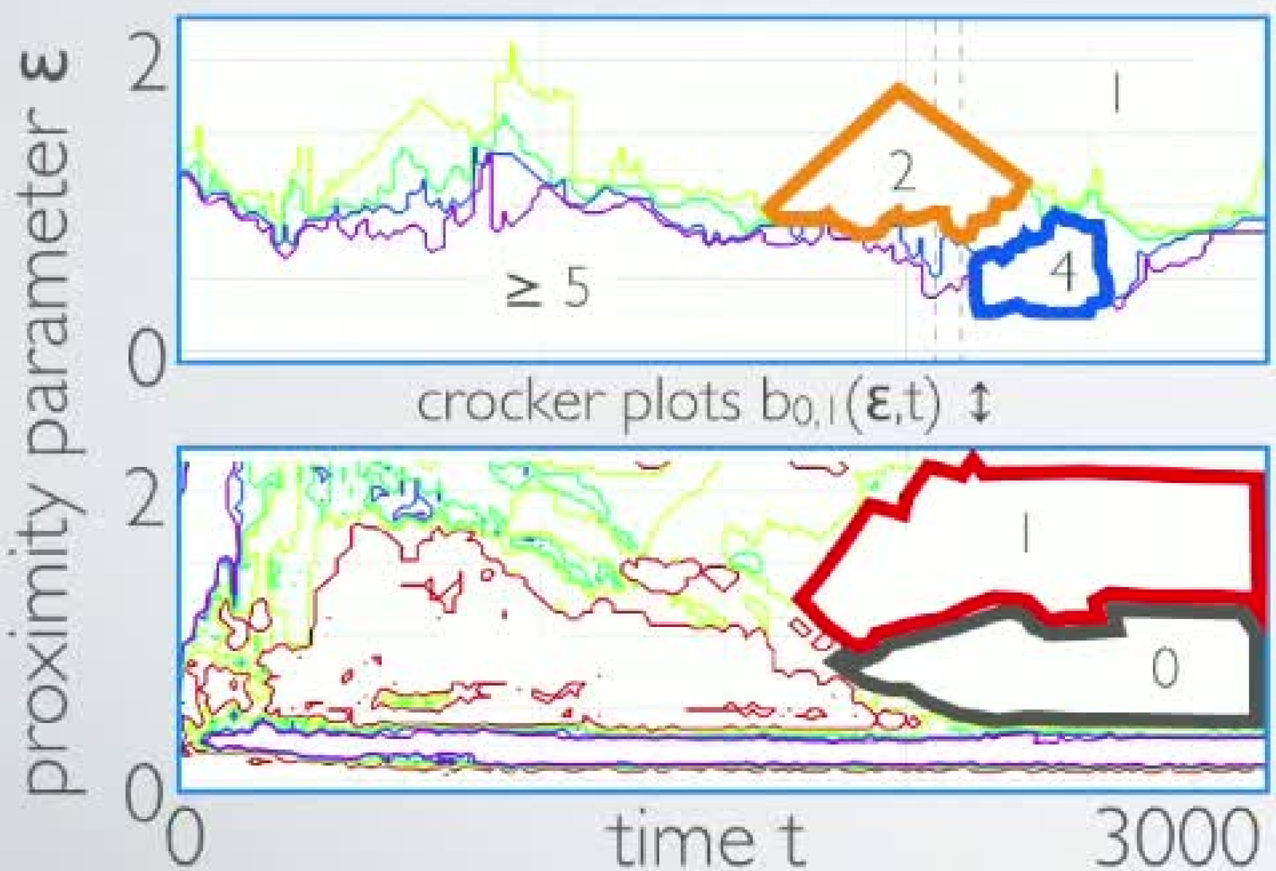


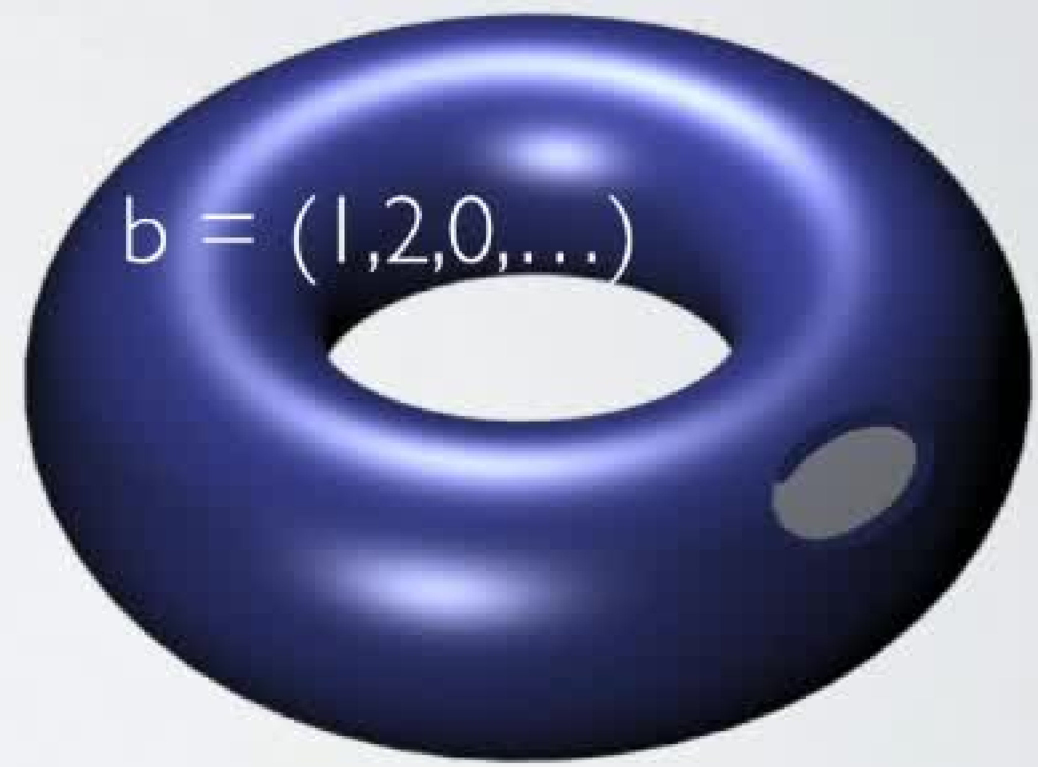


TOPOLOGICAL ORDER PARAMETER

Parameter Set #1
Clusters?



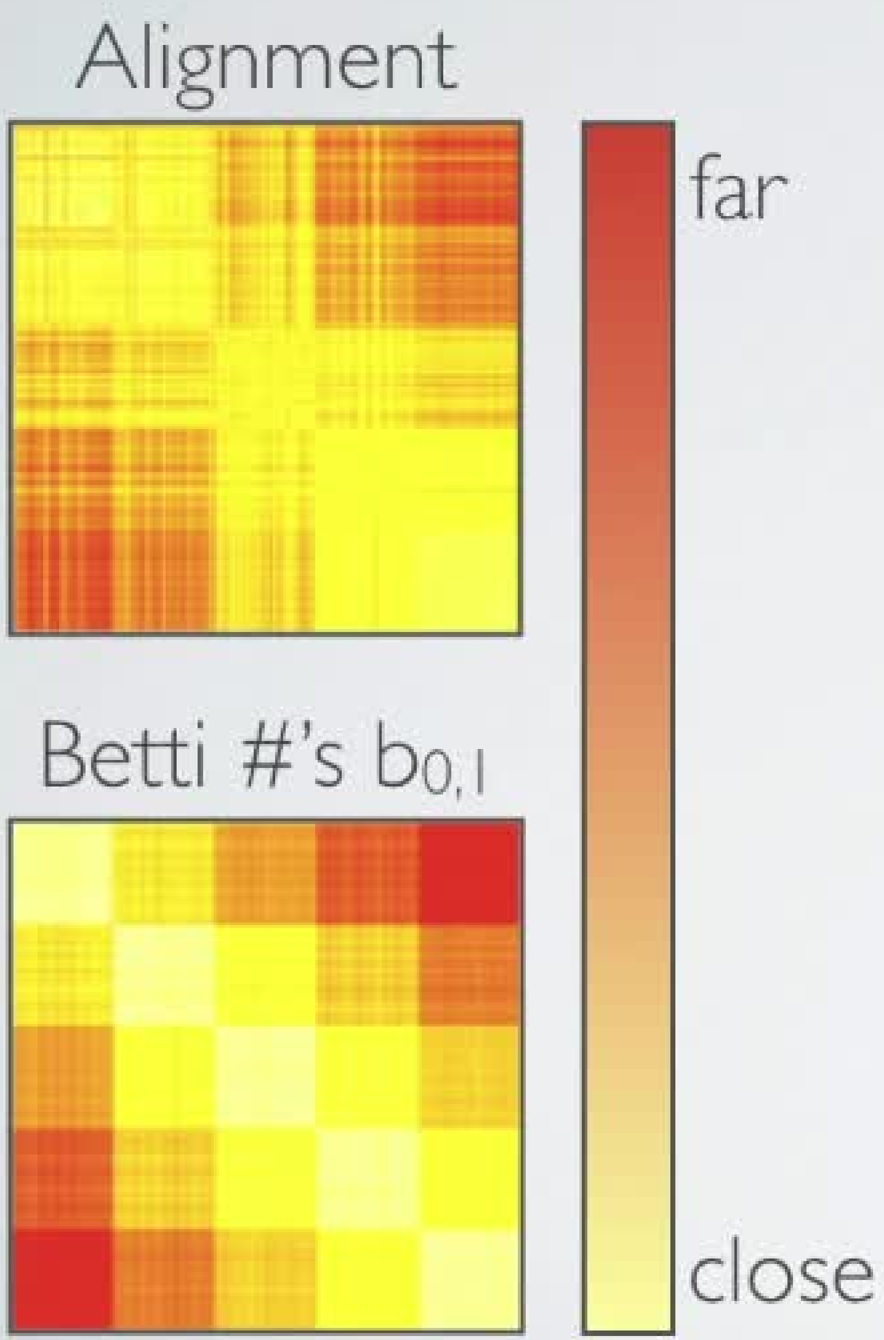




all the data | order parameter | topology

DATA EXPLORATION

MACHINE LEARNING RESULTS

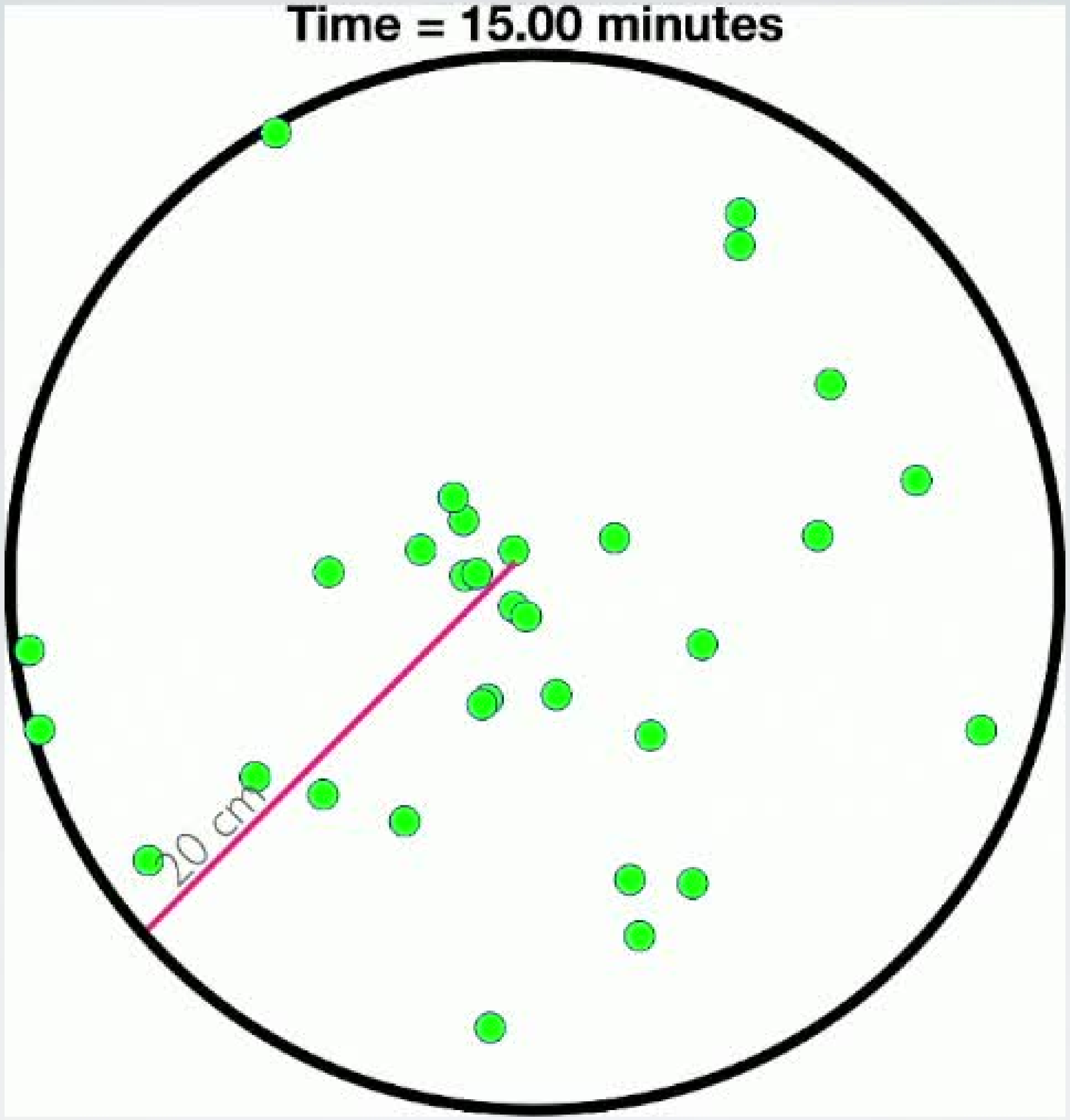


k-Medoids Clustering		
Metric	PCA 3d?	Accuracy
Alignment	✓	51%
$b_{0,1}$	✓	100%
b_0	✓	100%
b_1	✓	99%

COLLECTIVE MOTION

Main idea: Use topology as a metric for model selection.

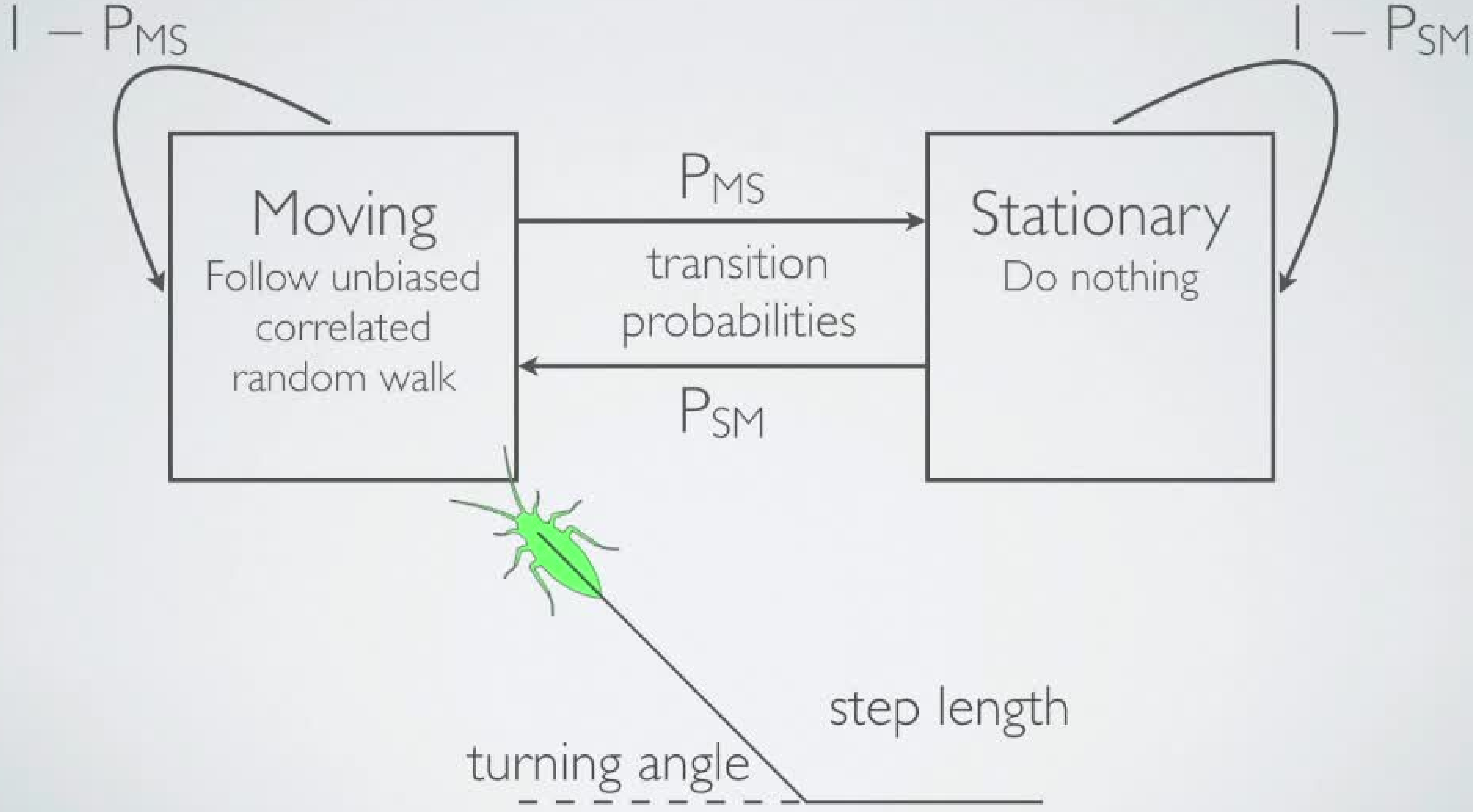
Ziegelmeier, Ulmer, Topaz, PLOS One (2019)



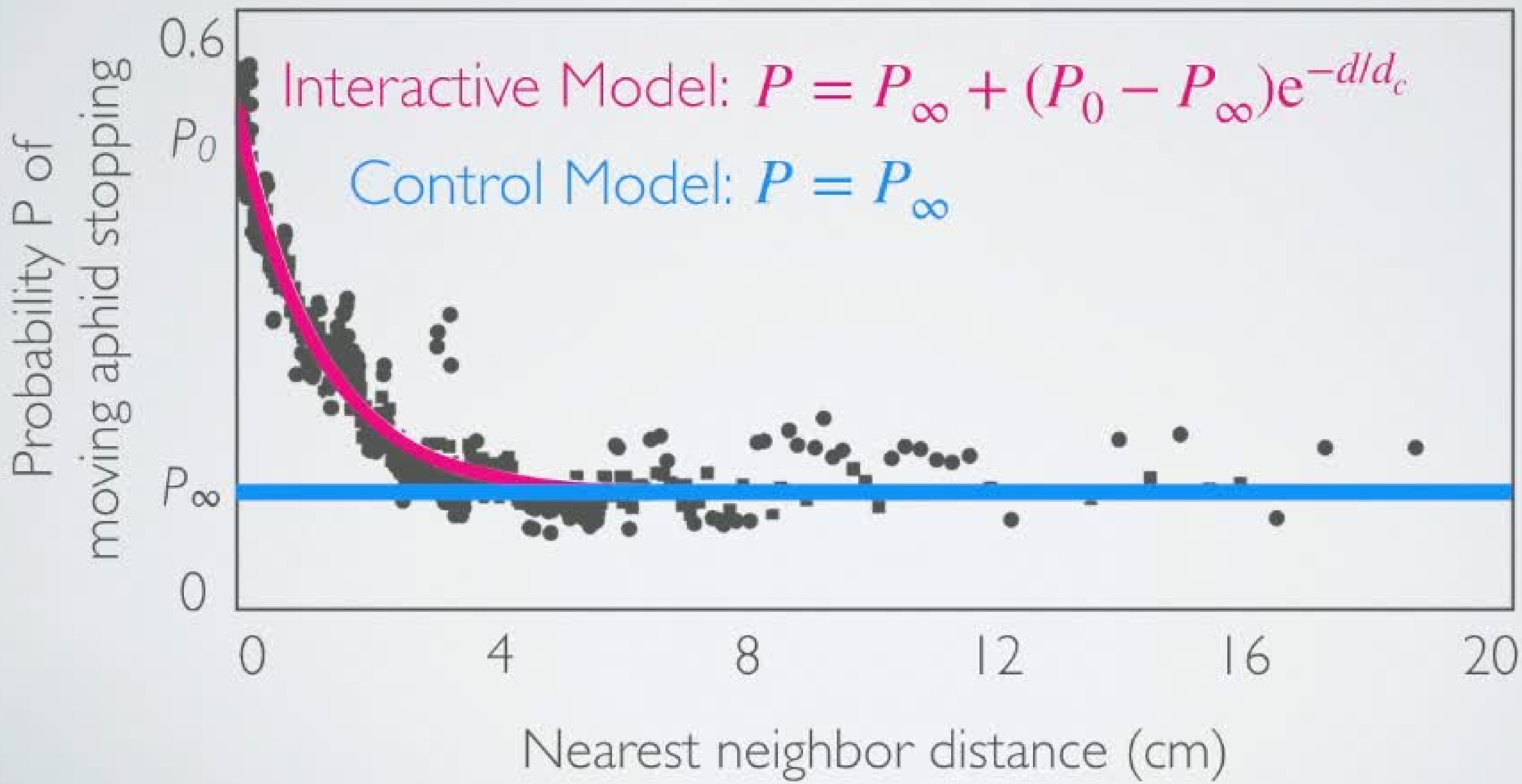
PEA APHID EXPERIMENT

Nilsen, Paige, Warner, Mayhew, Sutley, Lam, Bernoff, Topaz, PLOS One (2013)

TWO-STATE RANDOM WALK MODEL



TWO POSSIBLE MODELS



EVALUATE MODEL FITNESS

Experiment
Interactive model ×1000
Control model ×1000
} × 9

Distance between simulations and experiment according to <u>some metric</u>

TRADITIONAL			A PRIORI, A.K.A. CHEATING		TOPOLOGY			
Polariz.	Angular Mom.	Abs. Angular Mom.	Avg. Buddy Dist.	% Moving	$b_0(\underline{x})$	$b_1(\underline{x})$	$b_0(\underline{x},\underline{v})$	$b_1(\underline{x},\underline{v})$

EVALUATE MODEL FITNESS

	TRADITIONAL			A PRIORI, A.K.A. CHEATING		TOPOLOGY			
Exp.	Polariz.	Angular Mom.	Abs. Angular Mom.	Avg. Buddy Dist.	% Moving	$b_0(\underline{x})$	$b_1(\underline{x})$	$b_0(\underline{x},\underline{v})$	$b_1(\underline{x},\underline{v})$
1									
2									
3									
4									
5									
6									
7									
8									
9									

EVALUATE MODEL FITNESS

	TRADITIONAL			A PRIORI, A.K.A. CHEATING		TOPOLOGY			
Exp.	Polariz.	Angular Mom.	Abs. Angular Mom.	Avg. Buddy Dist.	% Moving	$b_0(\underline{x})$	$b_1(\underline{x})$	$b_0(\underline{x},\underline{v})$	$b_1(\underline{x},\underline{v})$
1									
2									
3									
4									
5									
6									
7									
8									
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EVALUATE MODEL FITNESS

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Exp.	Polariz.	Angular Mom.	Abs. Angular Mom.	Avg. Buddy Dist.	% Moving	$b_0(\underline{x})$	$b_1(\underline{x})$	$b_0(\underline{x},\underline{v})$	$b_1(\underline{x},\underline{v})$
1									
2									
3									
4									
5									
6									
7									
8									
9									

metrics
that cheat

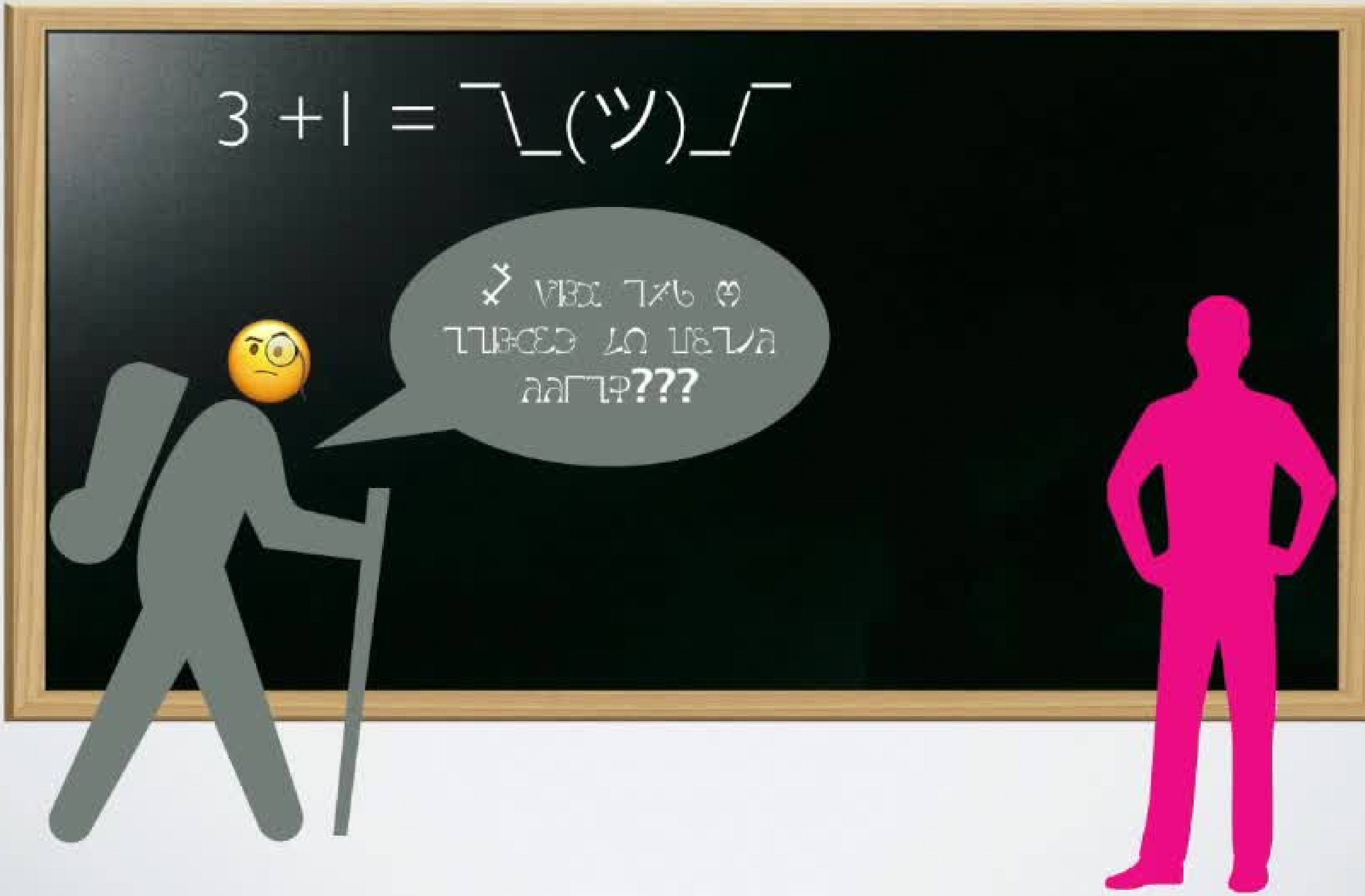
traditional
metrics

topological
signatures

MODEL SELECTION

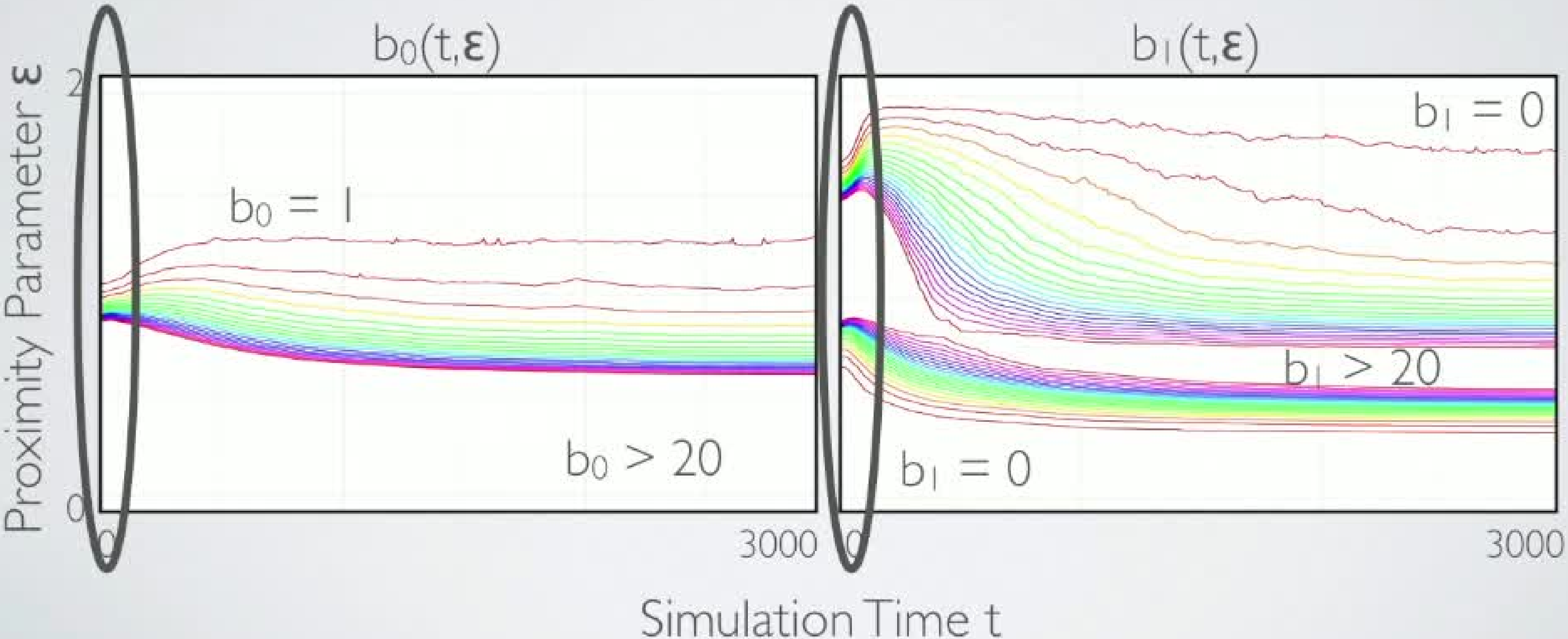
ONCE UPON A TIME

ONCE UPON A TIME



AVERAGE HOMOLOGY?

Vicsek model (naive) average over n = 1000 simulations



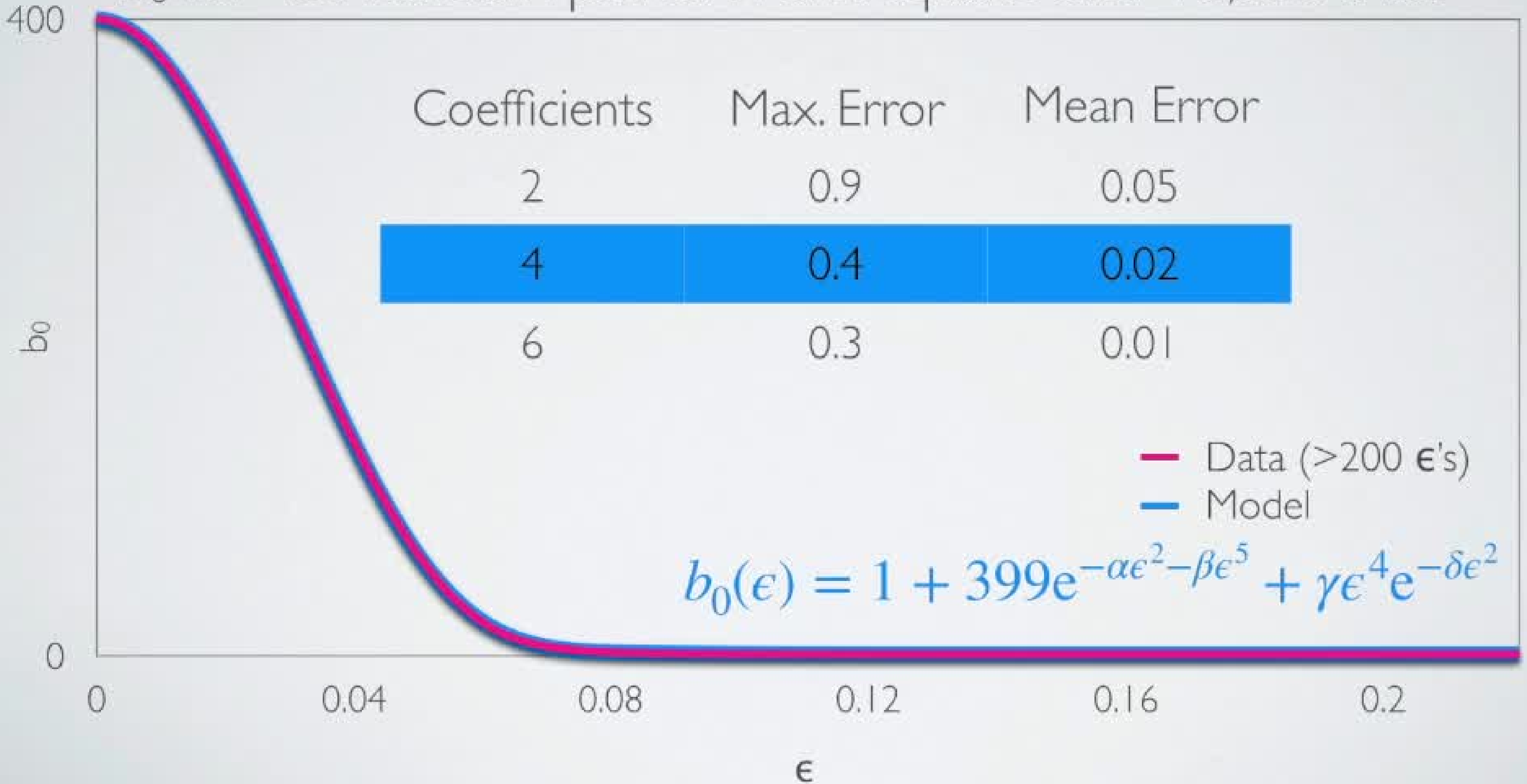
¯_(ツ)_/¯

SYMBOLIC REGRESSION

SIAM DS15, IP7, Hod Lipson

Distilling Natural Laws from Raw Data, from Robotics to Biology and Physics

b_0 for 400 random points in unit square over 10,000 trials



Coefficients	Max. Error	Mean Error
2	0.9	0.05
4	0.4	0.02
6	0.3	0.01

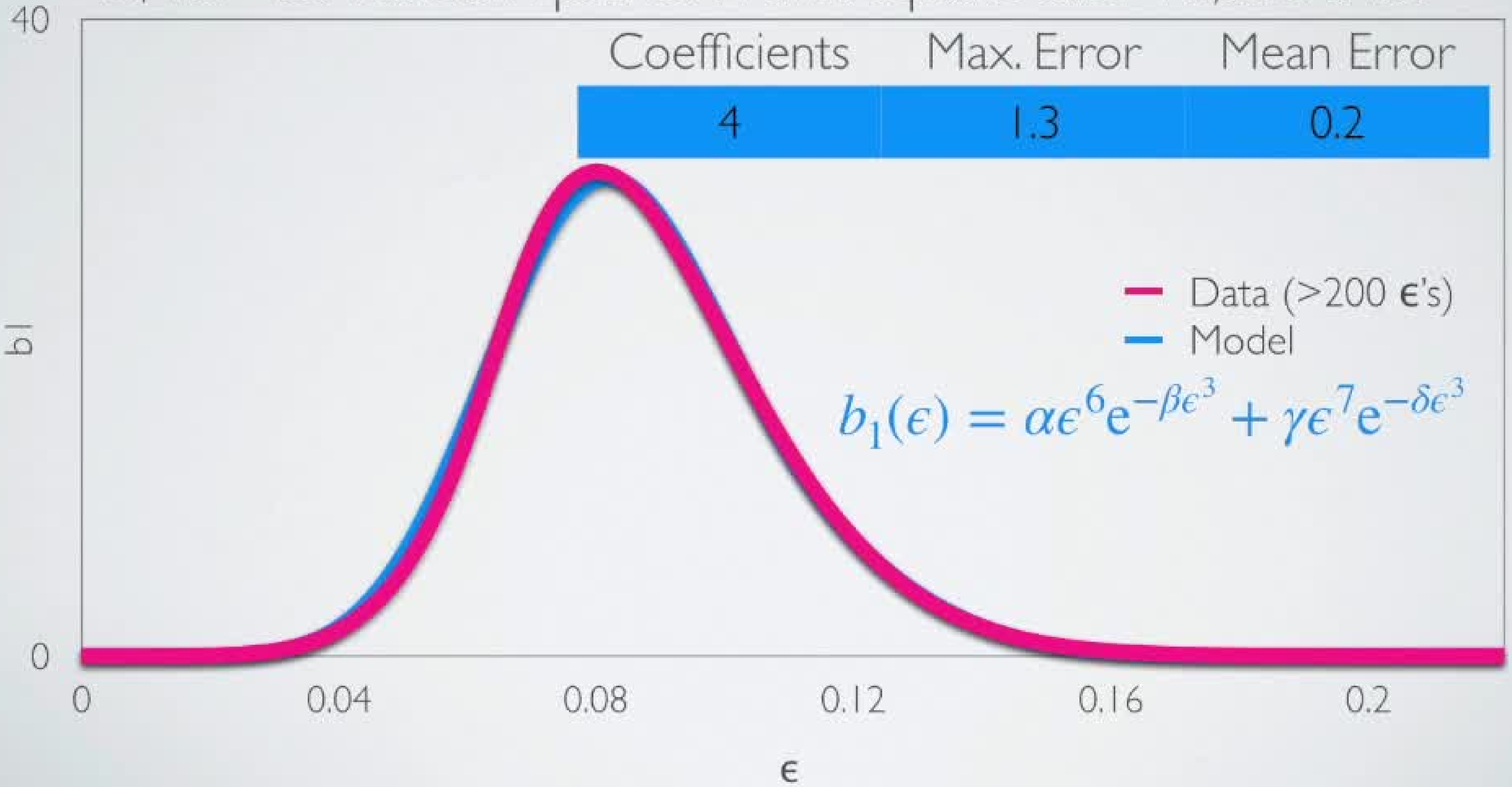
$$b_0(\epsilon) = 1 + 399\mathrm{e}^{-\alpha\epsilon^2 - \beta\epsilon^5} + \gamma\epsilon^4\mathrm{e}^{-\delta\epsilon^2}$$

SYMBOLIC REGRESSION

SIAM DS15, IP7, Hod Lipson

Distilling Natural Laws from Raw Data, from Robotics to Biology and Physics

b_1 for 400 random points in unit square over 10,000 trials



TDA

- I have found TDA valuable for
  - Exploring large data sets
  - Recovering parameters
  - Choosing between models of given data
  - Constructing reduced theoretical models??
- We may benefit from topological tools
- Topologists may benefit from understanding our needs
- …and from applied approaches to theoretical questions