Conformal mapping technique for a supercavitating flow around a wedge or a hydrofoil

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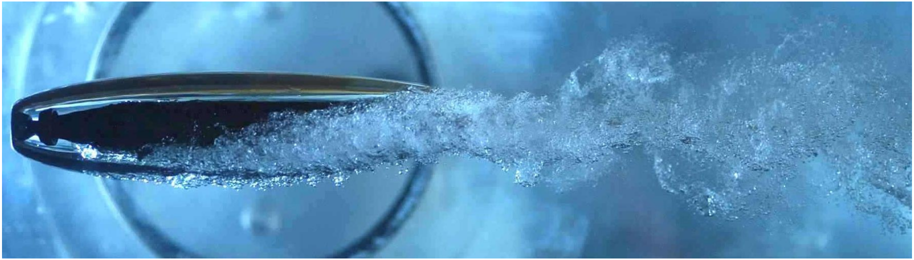
Cavitation



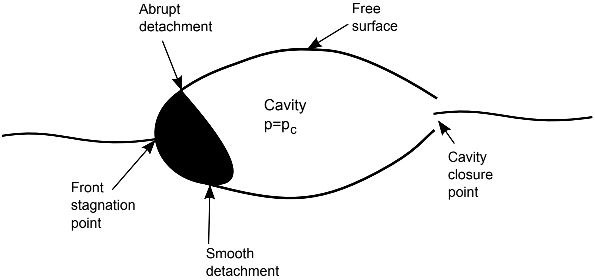
Cavitation is the formation of vapor bubbles in a region of a flowing liquid where the pressure falls below its vapor pressure.

Motivation for a study of cavitation:

- May cause a great deal of noise, damage to components, vibrations, and a loss of efficiency.
- May reduce the drag on an object, allowing it to travel at great speed by being wholly enveloped by the bubble (supercavitation).

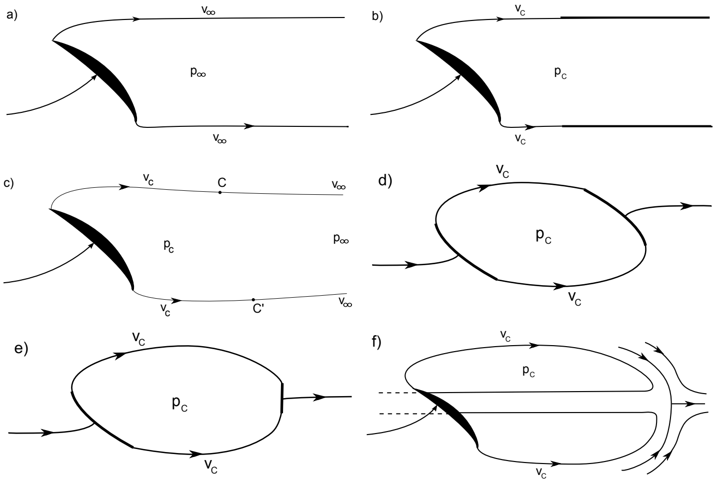
Picture by St. Anthony Falls Laboratory, University of Minnesota, Twin Cities http://cav.safl.umn.edu

Cavity closure and the Brillouin's paradox



Steady inviscid irrotational incompressible flow around the object.
Main question: How does the cavity end?
The speed is prescribed and constant on the boundary of the cavity.
This condition is violated at the rear stagnation point (the Brillouin's paradox).

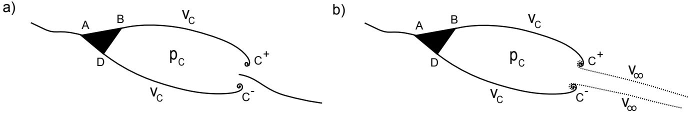
Cavity closure models



a) Kirchhoff model; b) Joukowski-Roshko model; c) Wu model;

d) Ryabushinski model; e) Modified Ryabushinsky model; f)Efros model.

Tulin's models



a) Tulin's single-spiral-vortex model; b) Tulin's double-spiral-vortex model.

SINGLE-SPIRAL-VORTEX MODEL (Tulin, 1964; modified by Terent'ev, 1976): the complex potential $w(z)$ has a singularity

$$\ln \frac{dw}{dz} \sim -K(w - w_0)^{-1/2}, \quad z \to C^{\pm}, \tag{1}$$

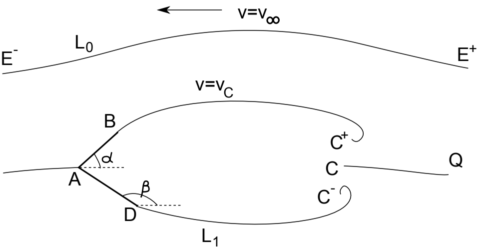
where $K = const > 0$, $w_0 = w(C^{\pm})$.
DOUBLE-SPIRAL-VORTEX MODEL (Tulin, 1964):

$$\ln \frac{dw}{dz} \sim iK \ln(w - w_0), \quad K = const. \tag{2}$$

Previous results

- A wedge in an infinite stream of liquid (Terent'ev 1981, Larock & Street 1965).
- Symmetric flows around wedge in a channel or in a jet (Terent'ev 1981).
- A hydrofoil in an infinite stream of liquid (single- and double-spiral-vortex models, Gurevich 1979, Larock & Street 1967).
- A cascade of cavitating hydrofoils (Terent'ev 1981, Brennen 1995, Wen & Ingham 1991).
- Two hydrofoils or two slender wedges under a free surface (double-spiral-vortex model, Green & Street 1965, 1967).
- A curvilinear body in an infinite domain (single-spiral-vortex model, Larock & Street 1968).
- A polygonal obstacle in an infinite body of liquid (double-spiral-vortex model, Bassanini & Elcrat 1988, 1993).
- Two hydrofoils in a channel (no numerical results, Antipov & Silvestrov 2007).
- Arbitrary number of hydrofoils (numerical results for one hydrofoil, Antipov & Silvestrov 2009).

A supercavitating wedge under a free surface



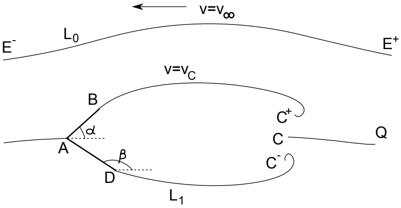
Find an analytic function $w(z) = \varphi(z) + i\psi(z)$ from the conditions

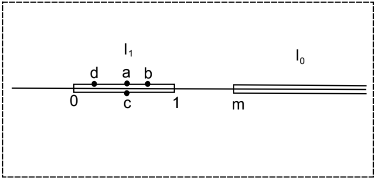
$$\operatorname{Im} w(z) = \begin{cases} \psi_1, & z \in ABCDA, \\ \psi_0, & z \in E^-E^+, \end{cases} \tag{3}$$

$$\arg \frac{dw}{dz} = \begin{cases} -\alpha, & z \in AB, \\ \pi - \beta, & z \in AD, \end{cases} \qquad \left|\frac{dw}{dz}\right| = \begin{cases} v_c, & z \in BC \cup DC, \\ v_\infty, & z \in E^-E^+. \end{cases} \tag{4}$$

Here $\alpha = \alpha_0 + \delta$, $\beta = \beta_0 + \delta$, δ is the yaw angle.

Conformal mapping





$z = f(\zeta)$ is a conformal mapping from the auxiliary domain $\mathcal{D}$ in the ζ-plane onto the flow domain $\tilde{\mathcal{D}}$.
The conformal mapping $z = f(\zeta)$ can be found from the formula:

$$f'(\zeta) = \frac{dz}{d\zeta} = \frac{dw}{d\zeta} : \frac{dw}{dz}. \tag{5}$$

Function $\frac{dw}{dz}$

Introduce a logarithmic hodograph variable:

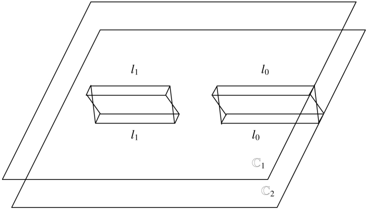
$$\omega_1(\zeta) = \ln \frac{1}{v_\infty} \frac{dw}{dz}.$$

The function $\omega_1(\zeta)$ is analytic in the domain $\mathcal{D}$ and satisfies the following boundary conditions:

$$\operatorname{Re} \omega_1(\zeta) = \begin{cases} \frac{1}{2} \ln(\sigma + 1), & \zeta \in bcd, \\ 0, & \zeta \in l_0, \end{cases}$$

$$\operatorname{Im} \omega_1(\zeta) = \begin{cases} -\alpha, & \zeta \in ab, \\ \pi - \beta, & \zeta \in da, \end{cases}$$

Additionally, $\omega_1(\zeta)$ has a simple pole at the point $\zeta = c$ and $\omega_1(\zeta) \to 0$ as $\zeta \to \infty$.



Consider a Riemann surface $\mathcal{R}$ of the function $u^2 = p(\zeta)$, $p(\zeta) = \zeta(1-\zeta)(\zeta - m)$.
Introduce a new function on the whole Riemann surface $\mathcal{R}$:

$$\Phi(\zeta, u) = \begin{cases} -i\omega_1(\zeta), & \zeta \in \mathbb{C}_1, \\ \overline{i\omega_1(\bar{\zeta})}, & \zeta \in \mathbb{C}_2. \end{cases} \tag{6}$$

Then

$$i(\Phi^+(\xi, v) - \Phi^-(\xi, v)) = 2\,\mathrm{Re}\,\omega_1(\xi),$$

$$\Phi^+(\xi, v) + \Phi^-(\xi, v) = 2\,\mathrm{Im}\,\omega_1(\xi), \quad (\xi, v) \in l_0 \cup l_1.$$

Riemann-Hilbert problem for the function $\Phi(\zeta, u)$

Find all functions $\Phi(\zeta, u)$ analytic in $\mathcal{R} \setminus (l_0 \cup l_1)$, Hölder continuous up to the boundary $l_0 \cup l_1$ apart from the singular points a, b, c and d and satisfying the boundary condition:

$$\Phi^+(\xi, v) = G(\xi, v)\Phi^-(\xi, v) + g(\xi, v), \quad (\xi, v) \in l_0 \cup l_1, \tag{7}$$

$$G(\xi, v) = \begin{cases} 1, & (\xi, v) \in bcd \cup l_0, \\ -1, & (\xi, v) \in dab, \end{cases}$$

$$g(\xi, v) = \begin{cases} -i\log(\sigma + 1), & (\xi, v) \in bcd, \\ -2\alpha, & (\xi, v) \in ab, \\ 2(\pi - \beta), & (\xi, v) \in da, \\ 0, & (\xi, v) \in l_0, \end{cases}$$

and the symmetry condition

$$\Phi(\zeta, u) = \overline{\Phi(\bar{\zeta}, -u(\bar{\zeta}))},$$

where $\Phi(\zeta, u) = O((\zeta - c)^{-1})$ as $\zeta \to c$; $\Phi(\zeta, u) \to 0$ as $\zeta \to \infty$; $\Phi(\zeta, u)$ is bounded at the points $\zeta = b$ and $\zeta = d$ and may have a logarithmic singularity at the point $\zeta = a$.

Canonical function $X(\zeta, u)$

Find a piecewise meromorphic function $X(\zeta, u)$ which satisfies the homogeneous boundary value condition

$$X^+(\xi, v) = -X^-(\xi, v), \quad (\xi, v) \in dab, \tag{8}$$

and the symmetry condition

$$X(\zeta, u) = \overline{X(\bar{\zeta}, -u(\bar{\zeta}))}, \quad (\zeta, u) \in \mathcal{R}/L. \tag{9}$$

Take the logarithm of the condition (8):

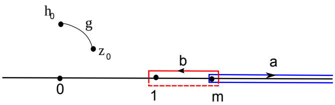
$$\ln X^+(\xi, v) = \ln(-1) + \ln X^-(\xi, v), \quad (\xi, v) \in dab, \tag{10}$$

One of the solutions:

$$\chi(\zeta, u) = \exp\left\{\frac{1}{4}\int_{dab}\left(1 + \frac{u(\zeta)}{u(\xi)}\right)\frac{d\xi}{\xi - \zeta}\right\}. \tag{11}$$

This function has essential singularity at infinity point of $\mathcal{R}$.

Boundedness at infinity



$$X(\zeta,u)=\exp\left\{\frac{1}{4}\int_{dab}\left(1+\frac{u}{v}\right)\frac{d\xi}{\xi-\zeta}-\frac{1}{2}\int_{\gamma}\left(1+\frac{u}{v}\right)\frac{d\xi}{\xi-\zeta}\right.$$
$$\left.-\frac{1}{2}\int_{\gamma}\left(1-\frac{u}{\overline{v}}\right)\frac{d\overline{\xi}}{\overline{\xi}-\zeta}-2m_a\int_{l_0^+}\frac{u}{v}\frac{d\xi}{\xi-\zeta}\right\}. \qquad (12)$$

By considering the asymptotic expansion as $\zeta\to\infty$ obtain the condition:

$$\frac{1}{4}\int_{dab}\frac{d\xi}{p^{1/2}(\xi)}-\frac{1}{2}\int_{\gamma}\frac{d\xi}{u(\xi)}+\frac{1}{2}\int_{\gamma}\frac{d\overline{\xi}}{\overline{u(\xi)}}-2m_a\int_{\mathbf{a}^+}\frac{d\xi}{p^{1/2}(\xi)}=0.$$

This is equivalent to the Jacobi inversion problem.

Jacobi inversion problem

Formulation of the Jacobi problem: Find a point $(\zeta_0, u(\zeta_0)) \in \mathcal{R}$ and two integers m_a, m_b which satisfy the condition:

$$\int_0^{\zeta_0} \frac{d\xi}{u(\xi)} + m_a\mathcal{A} + m_b\mathcal{B} = g_0, \tag{13}$$

$$g_0 = \frac{1}{4}\int_{dab} \frac{d\xi}{p^{1/2}(\xi)} + \int_0^{\eta_0} \frac{d\xi}{p^{1/2}(\xi)}, \quad \mathcal{A} = \oint_{\mathbf{a}} \frac{d\xi}{u(\xi)}, \quad \mathcal{B} = \oint_{\mathbf{b}} \frac{d\xi}{u(\xi)}. \tag{14}$$

Solution:

$$\zeta_0 = \mathrm{sn}^2 \frac{ig_0\sqrt{m}}{2},$$

$$m_a = -\frac{\mathrm{Im}\, l_\pm \sqrt{m}}{4\mathbf{K}}, \quad m_b = \frac{\mathrm{Re}\, l_\pm \sqrt{m}}{4\mathbf{K}'},$$

where

$$l_\pm = g_0 \pm \int_0^{\zeta_0} \frac{d\xi}{p^{1/2}(\xi)}, \quad \mathcal{A} = -4i\mathbf{K}/\sqrt{m}, \quad \mathcal{B} = 4\mathbf{K}'/\sqrt{m}.$$

Factorization: $X^+(\xi, v) \cdot [X^-(\xi, v)]^{-1} = G(\xi, v)$, $(\xi, v) \in l_0 \cup l_1$.
Write the inhomogeneous boundary condition as:

$$\frac{\Phi^+(\xi, v)}{X^+(\xi, v)} = \frac{\Phi^-(\xi, v)}{X^-(\xi, v)} + \frac{g(\xi, v)}{X^+(\xi, v)}, \quad (\xi, v) \in l_0 \cup l_1 \subset \mathcal{R}. \tag{15}$$

Solution:

$$\Phi(\zeta, u) = X(\zeta, u)[\Psi(\zeta, u) + \Omega(\zeta, u)], \quad (\zeta, u) \in \mathcal{R}, \tag{16}$$

$$\Psi(\zeta, u) = -\frac{\alpha}{2\pi i} \int_{ab} \frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)}$$

$$+\frac{\pi - \beta}{2\pi i} \int_{da} \frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)} - \frac{\ln(\sigma + 1)}{4\pi} \int_{bcd} \frac{(1 + u/v)d\xi}{X^+(\xi, u)(\xi - \zeta)}, \tag{17}$$

$$\Omega(\zeta, u) = iM_0 \frac{u(\zeta) + u(c)}{\zeta - c} + (M_1 + iM_2) \frac{u(\zeta) + u(\eta_0)}{\zeta - \eta_0} - \tag{18}$$

$$(M_1 - iM_2) \frac{u(\zeta) - \overline{u(\eta_0)}}{\zeta - \bar{\eta}_0},$$

where M_0, M_1, M_2 are real unknowns.

Function $\frac{dw}{d\zeta} = \omega_0(\zeta)$

Introduce a new function

$$\Omega(\zeta, u) = \begin{cases} \omega_0(\zeta) \text{ on } \mathbb{C}_1, \\ \overline{\omega_0(\bar{\zeta})} \text{ on } \mathbb{C}_2. \end{cases} \tag{19}$$

Riemann-Hilbert problem for $\Omega(\zeta, u)$:

$$\Omega^+(\xi, v) = \Omega^-(\xi, v), \quad (\xi, v) \in l_0 \cup l_1, \tag{20}$$

$$\Omega(\zeta, u) = O(\zeta^{-1/2}), \quad \zeta \to \infty, \tag{21}$$

$$\Omega(a, u(a)) = 0, \quad \Omega(c, u(c)) = 0. \tag{22}$$

The solution:

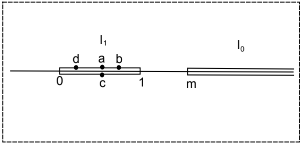
$$\frac{dw}{d\zeta} = N\tilde{\omega}_0(\zeta), \quad \tilde{\omega}_0(\zeta) = \frac{i(\zeta - a)}{p^{1/2}(\zeta)}, \quad a = c, \tag{23}$$

where

$$N = h\left(\mathrm{Im} \int_1^m \tilde{\omega}_0(\xi)d\xi\right)^{-1}. \tag{24}$$

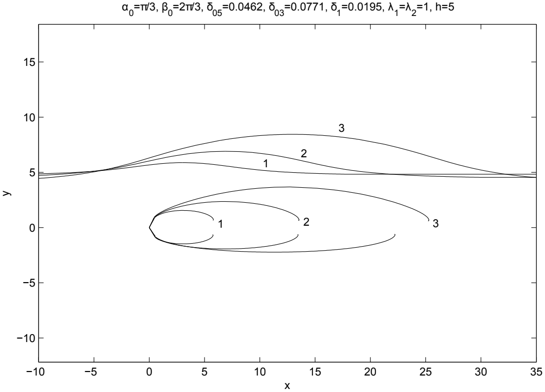
Unknowns and conditions

9 real unknowns need to be fixed: a, b, d, m, δ, N, M_0, M_1, M_2.
9 real linear and transcendental conditions exist.
The resulting system is solved by Newton method.



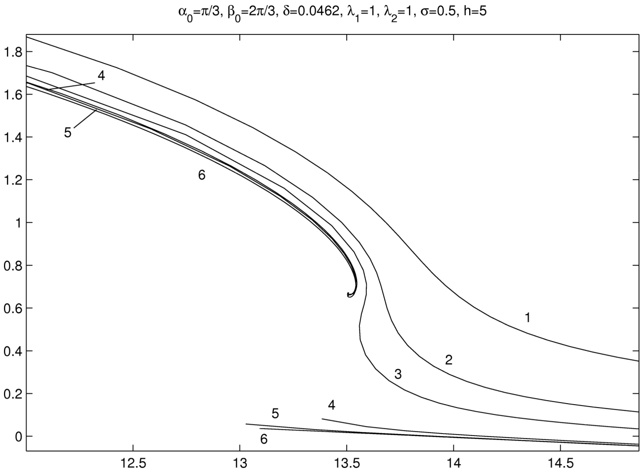
Main difficulties:

- Evaluate the singular and regular integrals with good accuracy and reasonable computational time ($|m - 1| \ll 1$).
- Find good initial approximation to the solution.

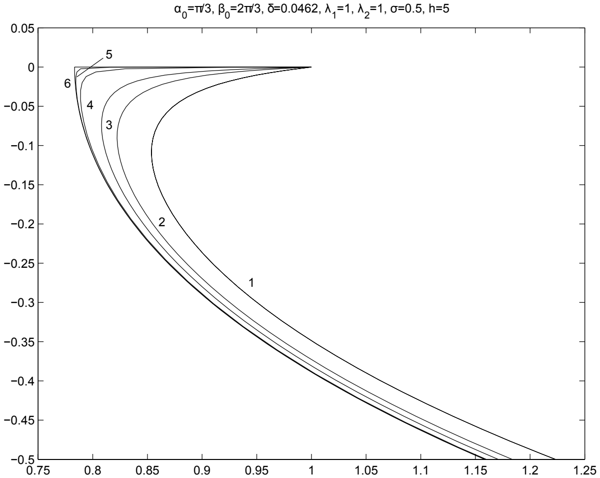


The cavity shape and the free surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $h = 5$ when $\sigma = 1(1)$, $\sigma = 0.5(2)$, $\sigma = 0.3(3)$.

Streamlines at the tail part of the cavity

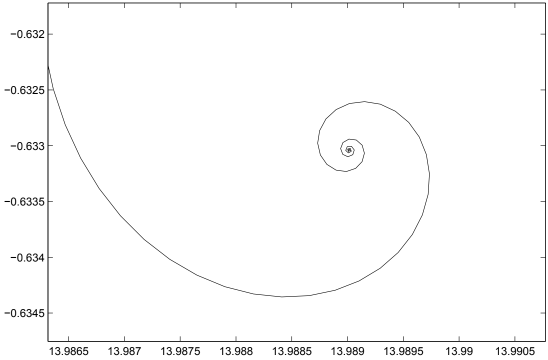


Preimages of the streamlines at the tail part of the cavity

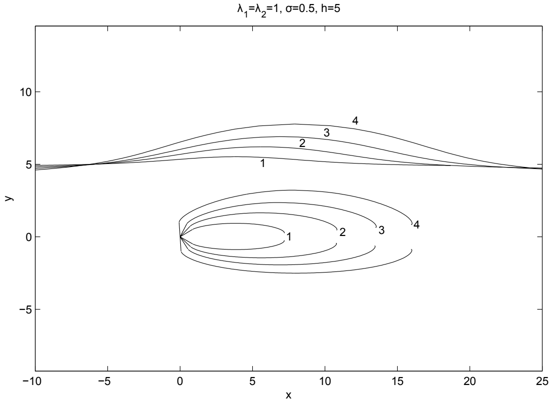


Spiral

Close up of the spiral at the tail part of the cavity:

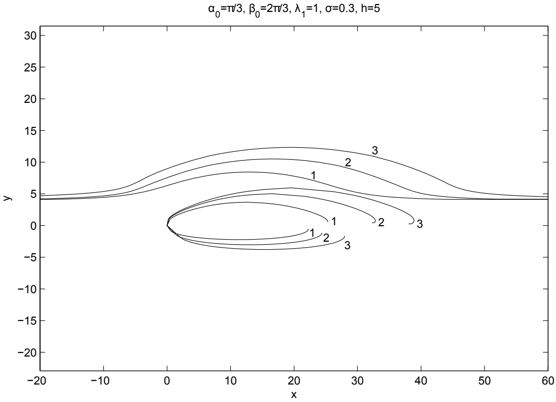


Different angles at the vertex of the wedge



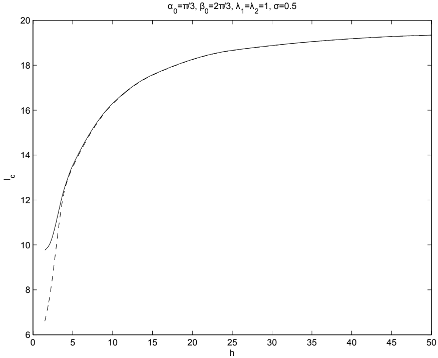
The cavity shape and the free surface for $\lambda_1 = \lambda_2 = 1$, $h = 5$ and $\sigma = 0.5$ for $\alpha_0 = \pi/6$ (1), $\alpha_0 = \pi/4$ (2), $\alpha_0 = \pi/3$ (3) and $\alpha_0 = \pi/2$ (4) ($\beta_0 = \pi - \alpha_0$).

Different side lengths



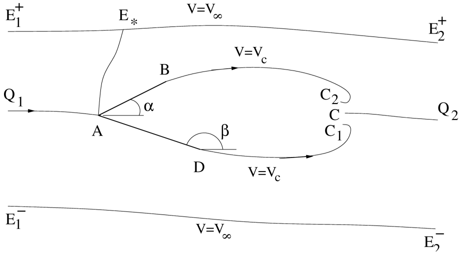
The cavity shape and the free surface for $\lambda_1 = 1$, $h = 5$, $\alpha_0 = \pi - \beta_0 = \pi/3$ and $\sigma = 0.5$ for $\lambda_2 = 1$ (1), $\lambda_2 = 2$ (2) and $\lambda_2 = 3$ (3).

Cavity length



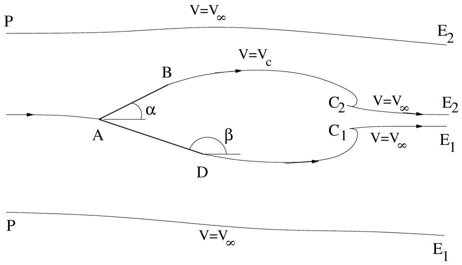
The length of the upper (solid line) and the lower (dash line) boundaries of the cavity vs the depth when $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $\sigma = 0.5$.

A nonsymmetric wedge in a jet: Comparison of the single- and the double-spiral-vortex models



SINGLE-SPIRAL-VORTEX MODEL: 10 real unknowns, 10 real conditions.
DOUBLE-SPIRAL-VORTEX MODEL: 6 real unknowns, 6 real conditions.

Double spiral vortex model

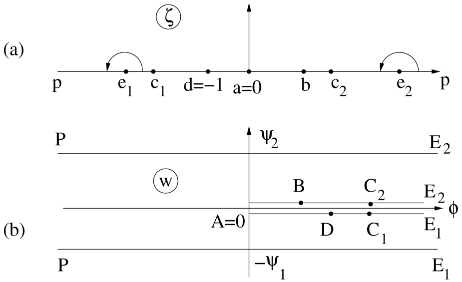


Find an analytic function $w(z) = \phi(z) + i\psi(z)$ from the conditions

$$\operatorname{Im} w(z) = \begin{cases} \psi_0, & z \in ABC_2E_2 \cup ADC_1E_1, \\ -\psi_1, & z \in PE_1, \\ \psi_2, & z \in PE_2, \end{cases} \tag{25}$$

$$\arg \frac{dw}{dz} = \begin{cases} -\alpha, & z \in AB, \\ \pi - \beta, & z \in AD, \end{cases} \tag{26}$$

$$\left|\frac{dw}{dz}\right| = \begin{cases} V_c, & z \in BC_2 \cup DC_1, \\ V_\infty, & z \in C_2E_2 \cup C_1E_1 \cup PE_1 \cup PE_2. \end{cases} \tag{27}$$



The complex potential has the same value at the points C_1 and C_2:

$$\mathrm{Re}\, w(C_1) = \mathrm{Re}\, w(C_2). \tag{28}$$

Choose $a = 0$, $d = -1$ and $p = \infty$.

Function $\omega_0(\zeta)$ and $\omega_1(\zeta)$

Using Schwarz-Christoffel formula and conservation of mass law:

$$\omega_0(\zeta) = -\frac{hV_\infty \zeta}{\pi[\zeta - (1 - h/h_1)e_2](\zeta - e_2)} \tag{29}$$

Boundary conditions for $\omega_1(\zeta)$:

$$\mathrm{Re}\,\omega_1(\xi) = \begin{cases} 0, & \xi \in pc_1 \cup c_2p, \\ \frac{1}{2}\ln(1+\sigma), & \xi \in c_1d \cup bc_2, \end{cases} \tag{30}$$

$$\mathrm{Im}\,\omega_1(\xi) = \begin{cases} -\alpha, & \xi \in ab, \\ \pi - \beta, & \xi \in da, \end{cases} \tag{31}$$

Introduce an auxiliary function:

$$\Phi(\zeta) = \begin{cases} -i\omega_1(\zeta), & \mathrm{Im}\,\zeta > 0, \\ \overline{i\omega_1(\bar{\zeta})}, & \mathrm{Im}\,\zeta < 0. \end{cases} \tag{32}$$

Riemann-Hilbert problem

Find an analytic function vanishing at infinity and satisfying the condition

$$\Phi^+(\xi) = G(\xi)\Phi^-(\xi) + g(\xi), \quad -\infty < \xi < +\infty, \tag{33}$$

$$G(\xi) = \begin{cases} 1, & \xi \in pd \cup bp, \\ -1, & \xi \in db, \end{cases} \tag{34}$$

$$g(\xi) = \begin{cases} 0, & \xi \in pc_1 \cup c_2 p, \\ -i\ln(1+\sigma), & \xi \in c_1 d \cup bc_2, \\ -2\alpha, & \xi \in ab, \\ 2(\pi - \beta), & \xi \in da. \end{cases} \tag{35}$$

Solution to the problem:

$$\Phi(\xi) = \frac{\chi(\zeta)}{2\pi i} \int_{c_1}^{c_2} \frac{g(\xi)d\xi}{\chi^+(\xi)(\xi - \zeta)}, \tag{36}$$

$$\chi(\zeta) = \sqrt{(\zeta - b)(\zeta + 1)}. \tag{37}$$

Unknowns and conditions

6 real unknowns: b, c_1, c_2, e_1, e_2, δ.
Additional conditions:

$$e_1 = -\frac{h - h_1}{h_1} e_2, \tag{38}$$

$$\ln(1+\sigma) \ln \frac{-2\chi^+(c_1) - 2c_1 + b - 1}{2\chi^+(c_2) + 2c_2 - b + 1} + 2\alpha \left(\frac{\pi}{2} - \arcsin \frac{1-b}{1+b} \right) \tag{39}$$

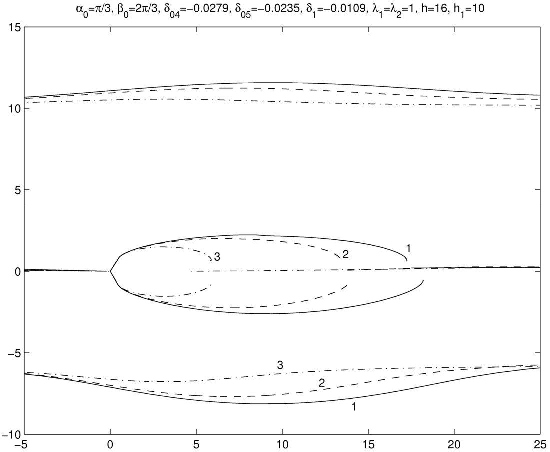
$$-2(\pi - \beta) \left(\frac{\pi}{2} + \arcsin \frac{1-b}{1+b} \right) = 0,$$

$$\ln \frac{c_1 - e_1}{c_2 - e_1} = \frac{e_2}{e_1} \ln \frac{e_2 - c_1}{e_2 - c_2}, \tag{40}$$

$$\mathrm{Im}\, \omega_1(e_1) = \mathrm{Im}\, \omega_1(e_2), \tag{41}$$

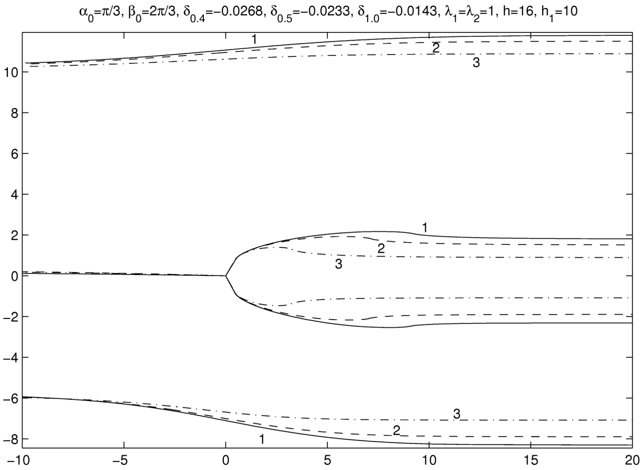
$$\int_0^b \left| \frac{df}{d\zeta} \right| |d\zeta| = \lambda_1, \quad \int_{-1}^0 \left| \frac{df}{d\zeta} \right| |d\zeta| = \lambda_2. \tag{42}$$

Numerical Results: Single-spiral-vortex model



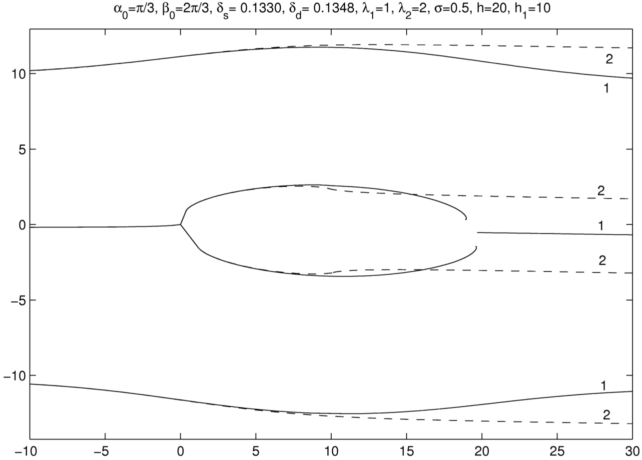
The cavity shape and the jet surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $h = 16$, $h_1 = 10$, when $\sigma = 0.4(1)$, $\sigma = 0.5(2)$, $\sigma = 1(3)$.

Numerical Results: Double-spiral-vortex model



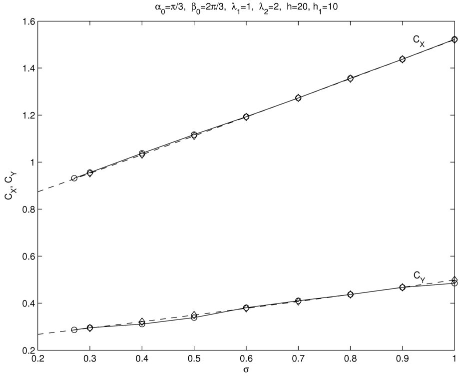
The cavity shape and the jet surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = \lambda_2 = 1$, $h = 16$, $h_1 = 10$, when $\sigma = 0.4(1)$, $\sigma = 0.5(2)$, $\sigma = 1(3)$.

Comparison of the results



The cavity shape and the jet surface for $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 2$, $h = 20$, $h_1 = 10$ for the single- (1) and double-spiral-vortex(2) models.

Drag and lift coefficients



The drag and lift coefficients, C_X and C_Y, when $\alpha_0 = \pi - \beta_0 = \frac{\pi}{3}$, $\lambda_1 = 1$, $\lambda_2 = 2$, $h = 20$, $h_1 = 10$ vs the parameter σ: the single-spiral-vortex model (-) and the double-spiral-vortex model (–).

The angle of the rotation of the wedge δ

σ	Single-spiral-vortex model	Double-spiral-vortex model
0.4	-0.0279	-0.0268
0.5	-0.0235	-0.0233
1.0	-0.0109	-0.0143

Table 1. The angle of the rotation of the wedge δ of the wedge for $\alpha_0 = \pi - \beta_0 = \pi/3$, $\lambda_1 = 1$, $\lambda_2 = 1$, $h = 16$, $h_1 = 10$.

The angle of the deflection of the jet

σ	Single-spiral-vortex model	Double-spiral-vortex model
0.3	-0.01524	-0.01661
0.5	-0.02294	-0.01811
0.7	-0.02745	-0.01954
1.0	-0.03392	-0.02160

Table 2. The angle of deflection ϵ of the jet at infinity for $\alpha_0 = \pi - \beta_0 = \pi/3$, $\lambda_1 = 1$, $\lambda_2 = 2$, $h = 20$, $h_1 = 10$.

Comparison of the results

The parameters determined mostly by the flow in the front part of the cavity such as

- streamlines in the front part of the flow,
- drag and lift coefficients,
- yaw angle,

do not change significantly.

The parameters determined mostly by the flow at the tail part of the cavity, such as

- cavity length,
- streamlines at the tail part of the cavity,
- angle of the deflection of the jet,

depend strongly on the chosen model.

Possible future work

- Numerical scheme for two or more wedges in the presence of free surfaces of liquid.
- Supercavitating flows around curvilinear, polygonal bodies, including flexible curvilinear shells.
- Transient supercavitating flows (changing velocity at infinity).
- Supercavitating flows in the presence of a surface tension and gravity.

Thank You!