Rotation Vectors for Invariant Tori Using Weighted Birkhoff Averages

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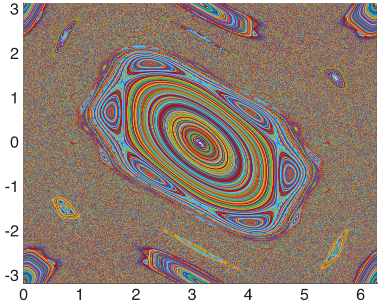
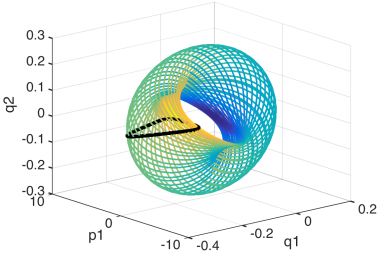
- A quasiperiodic map is a map $F : L \to L$ on a topological circle or torus L such that in some choice of coordinates F is a rigid irrational rotation. That is, for T a canonical torus and a map c

$$c : T \to L, F(c(\theta)) = c(\theta + \rho) \, .$$

This also may occur for some iterate of the map.



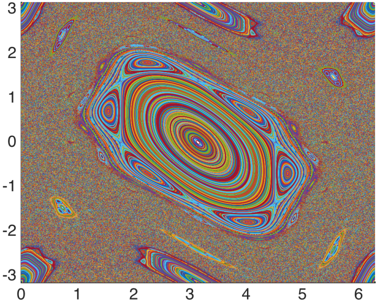


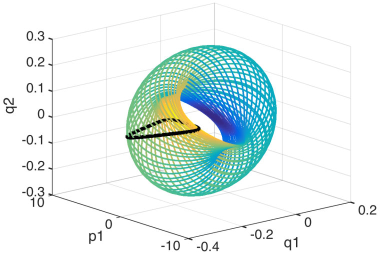
Invariant tori

Das, Saiki, S., Yorke created a weighted Birkhoff average method which can efficiently compute Birkhoff averages for quasiperiodic orbits.

- Compute rotation numbers.
- Distinguish chaos from quasiperiodicity.
- Use rationality of rotation number to distinguish islands.

Current goal is to understand the breakup of invariant KAM tori in volume-preserving systems using these methods.





Birkhoff averages as an integral estimator

Birkhoff ergodic theorem: Time average = space average

- Chaotic or quasiperiodic trajectory

$$x_n = F^n(x_0), n = 0, 1, 2, \dots$$

- Let f be a function along the trajectory
- Time averages are the same as space averages:

$$B_N(f) = \frac{1}{N}\sum_{n=0}^{N-1} f(x_n) \to \int f(x)d\mu(x)$$

  E.g. For rotation number, $f(x_n) = \Delta_n = x_{n+1} - x_n$
  Under mild assumptions including smoothness of f and F, ergodic F, and Lebesgue measure μ.
- Quasiperiodic convergence: $O(1/N)$, lack of smoothness at termination of orbit.
- Chaotic convergence: $O(1/\sqrt{N})$, central limit theorem.

Weighted Birkhoff averages

Slow convergence of the Birkhoff sum is caused by edge effects.

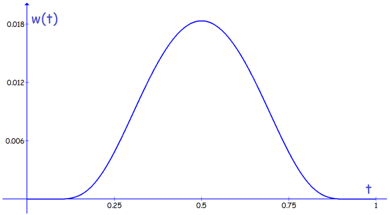
$$B_N(f, x_0) = \sum_{n=0}^{N-1} f(x_n)/N$$

Weighted Birkhoff Average: like windowing methods in signal processing:

$$w_p(t) := \exp(-[t(1-t)]^{-p}),$$

$$\hat{w}_{n,N} = w_p(n/N) / \left[\sum_{j=0}^{N-1} w_p(j/N) \right]$$

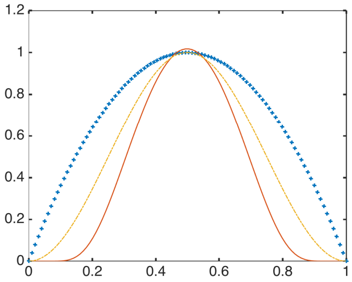
$$WB_{p,N}(f, x) = \sum_{n=0}^{N-1} \hat{w}_{n,N} f(x_n)$$

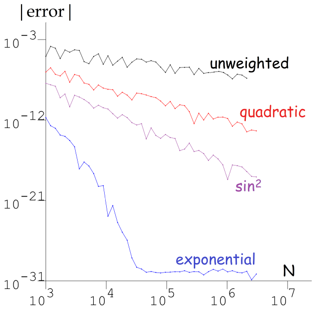


Convergence and smoothness

Convergence determined by smoothness of the weighting function.

- Unweighted $= 1$, convergence: $O(1/N)$
- Quadratic $= x(1-x)$, convergence: $O(1/N^2)$
- $\sin^2(\pi x)$ (Laskar) convergence: $O(1/N^3)$
- Exponential $= w$ superconvergence: $O(1/N^m), \forall m$





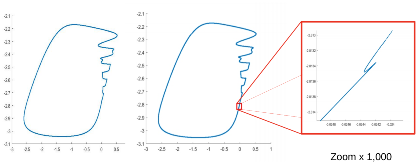
Distinguishing chaos from quasiperiodicity

Chaotic: $O(1/\sqrt{N})$.
Quasiperiodic: $O(1/N^m)\ \forall m$.

- For some function measure WB_N for x_1 and x_{N+1}.
- Matching digits: $zeros_N(f) = -\log_{10}|\text{diff}|$. Small zeros is chaos, large zeros is quasiperiodicity.

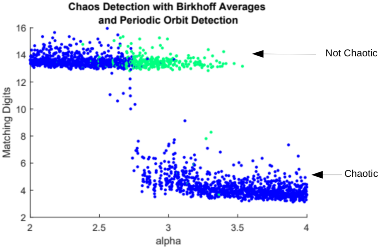
Based on Levnajić & Mezić

Ongoing with Jonathan Jaquette: Chaos in Rulkov map



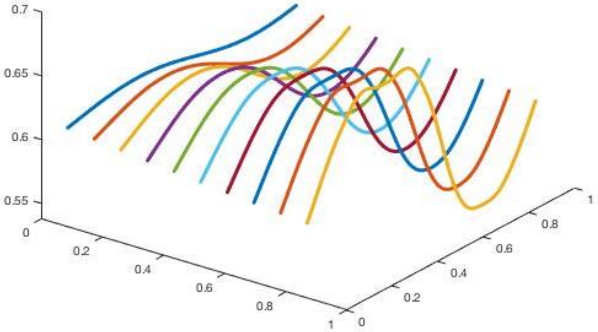
α = 2.728
m.d. = **12**
Not Chaotic

α = 2.739
m.d. = **6**
Chaotic



Persistent invariant tori

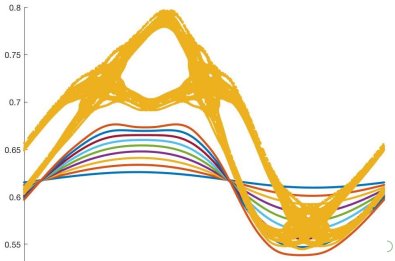
- Integrable Hamiltonian or volume-preserving systems have invariant quasiperiodic tori.
- KAM theory guarantees persistence of tori with Diophantine rotation numbers
- When are the tori destroyed?



Greene's residue criterion, golden and noble numbers

- For volume preserving, symplectic, reversible maps, Greene's residue method: Stable periodic orbits limit to persistent invariant torus.
- Standard map strong conjectures (MacKay, Koch, others)
  - Last rotational circle : $\lambda = 0.971635$, golden mean rotation number, continued fraction of 1's
  - Local max noble rotation number, continued fraction eventually 1's
- Only works well for symmetric reversible case

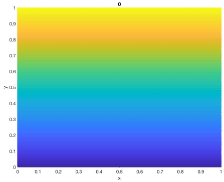
- In higher dimensions, no canonical continued fractions
- Our goal: non-symmetric and higher-dimensional volume-preserving maps

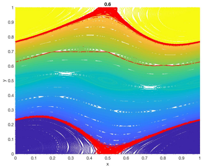


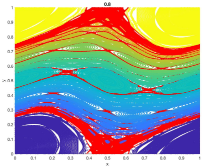
Chirikov's standard map

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \mod\left(x_1 + x_2 - \frac{\lambda}{2\pi}\sin(2\pi x_1), 1\right) \\ \mod\left(x_2 - \frac{\lambda}{2\pi}\sin(2\pi x_1), 1\right) \end{pmatrix}$$

Rotational circles, librational circles (islands), chaos movie

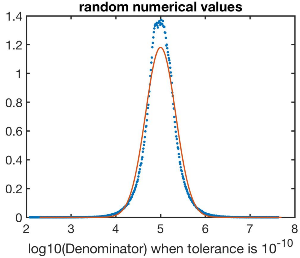


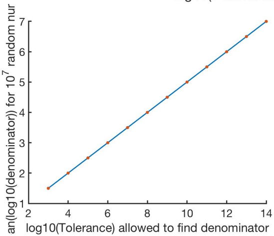


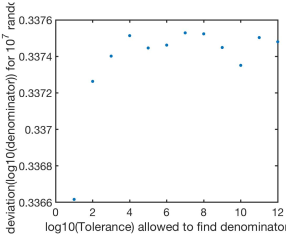


Distinguishing rational numbers in floating point arithmetic

A floating point rational number: within ε of p/q where q is unexpectedly small.

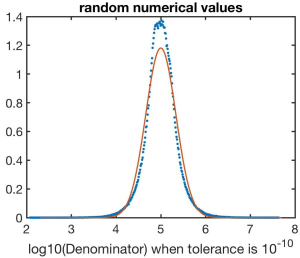


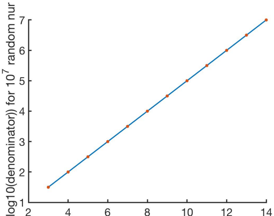


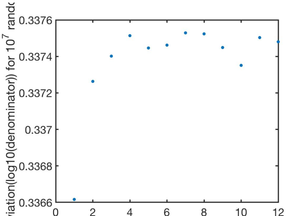


Distinguishing rational numbers in floating point arithmetic

- Bottom left: Farey sequences, Euler totient function
- Mystery 1: What is the distribution? Why symmetric?
- Mystery 2: Lower right: Fixed standard deviation?

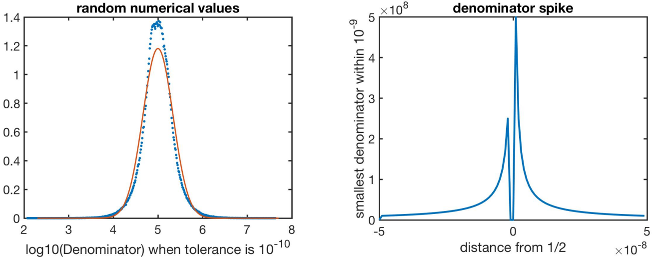






Denominator spike

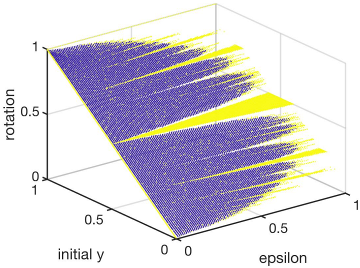
Near rationals, unexpectedly small denominators are accompanied by denominator spikes

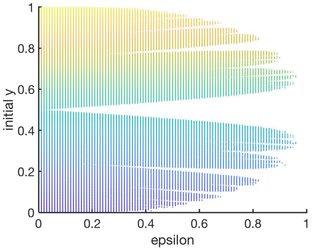


Distinguishing islands using weighted Birkhoff averages

A proof of concept for Chirikov's standard map

- Starting with the movie data
- Remove chaotic orbits
- Islands rational rotation (yellow, tongues)
- Computed final rotational circle parameter: 0.972351 (overshoot of 0.0007)





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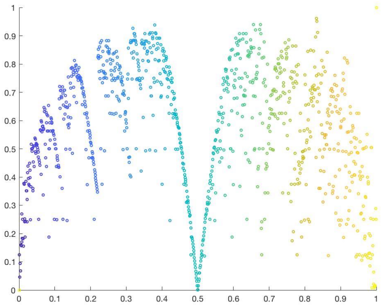
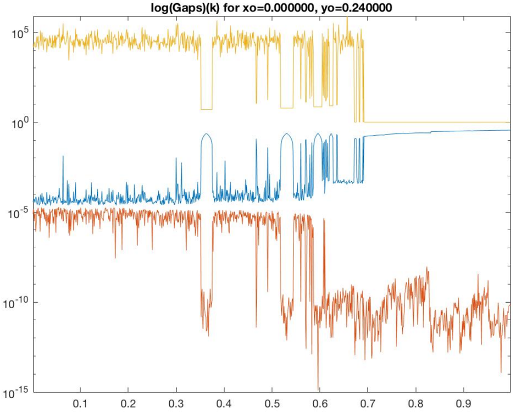


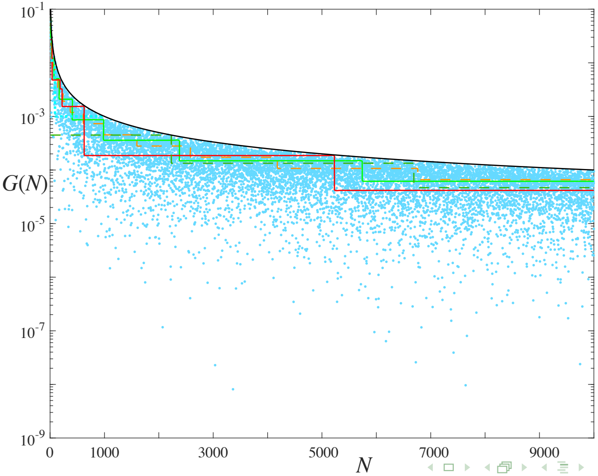
Distinguishing islands with gap size

- Gaps between islands – large max gaps
- Compression of non-embedding – small minimum gaps
- Preliminary results: more accurate
- Relates to Slater's method



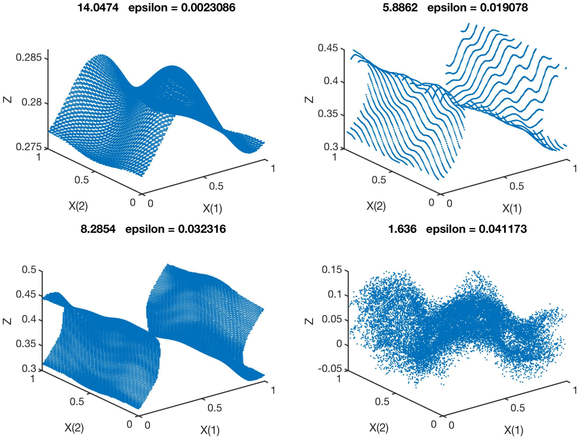
Typical gap size

For rigid rotation, the minimum gap size has a supremum of $1/N$ where N is the number of iterates.
Lines: Golden mean, $\sqrt{2}$, and dashed: rationals.



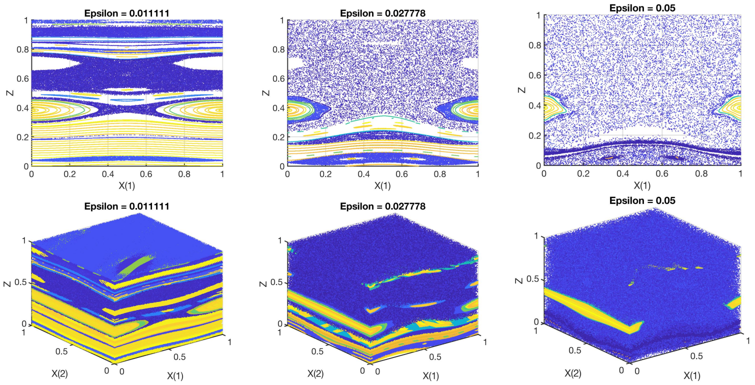
Action-angle map in three dimensions

Similarly in three dimensions, we get rotational tori, circles, islands, and chaos in a system previously studied by Fox and Meiss.



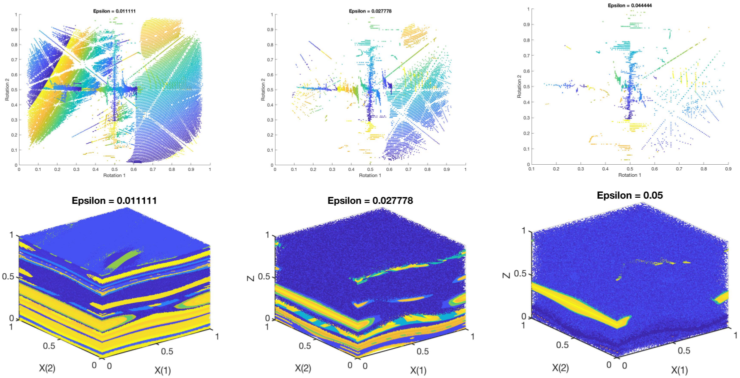
Action-angle map in three dimensions

Chaos still be distinguished with weighted Birkhoff method.



Action-angle map in three dimensions

Kim-Ostlund coordinates



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