The Force of Fluctuations

Analysis and control of extinction in networks

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DISEASES IN THE NEWS...

Novel Coronavirus Continues To Spread

by George Ochoa May 2013

A novel coronavirus (nCoV) outbreak that began in Saudi Arabia has infected 41 people and caused 20 deaths since 2012, according to the World Health Organization (WHO). The outbreak has primarily affected Saudi Arabia, but cases have been reported in Jordan and Qatar as well as in France, Germany and the United Kingdom.

Bird flu: US safe from two new viruses - so far

By Maggie Fox, Senior Writer, NBC News Sun May 12, 2013 9:33 AM EDT

More than 50 travelers just back in the United States from China who had flu-like symptoms have been tested for the H7N9 bird flu virus, federal health officials say. So fa

Now a warni over mumps ollowi mea s outbre

By Claire Duffin, 7:0 M BST 09 May 201 The Telegr h

Public health offic yesterday warne hey fear a ump pidemic in **So Wales** in the wake a growing measl outbreak, er un sity student were diagnosed w the condition.

The number of people infected with measles in South Wales reached 1,039 - 85 of whom have needed hospital treatment.

Europe's Embarrassing Problem

By Kai Kupferschmidt ***Sc nce*** ril 2012

While every country in the Americas, including i oorest, wipe meas s o e map in 2002, Europe has been unable to do so. Ca have quadrupl since 009, a the reemergence has become a threat to other 11, the United States had 222 cases, the highest number since 1996, and most importations come from Europe. Luckily, high vaccination rates in the Americas prevent most imported infections from spreading. Measles' stubborn persistence in Europe would also be a stumbling block in any plan to eradicate the disease globally.

By Laura Shin Published: May 15, 2013

Health Dept: Measles outbreak hits Brooklyn

May 15, 2013

A measles outbreak has hit one Brooklyn community and may be spreading, health officials say.

There have been 21 cases of measles reported in Borough Park and one case in Williamsburg this year, according to the city's Department of Health and Mental Hygiene. Those infected are between 10 months and 32 years old, DNAinfo.com reports.

The first case was imported from London, according to the health department. Measles is a highly contagious disease and can be deadly. Symptoms include rash, fever, cough, runny nose and sore throat.

The disease can be prevented with a vaccine. The new outbreak y have spread between lies who refused vaccinations, ording to DNAinfo.com

lished On: Mon, May 13th, 2013 Outbreak News | By Robert Herriman

ngapore Experi cing D gue Epidemic In The ousands, Bei To 2005 Epidemic

sland city-state of Singap in a real battle with uito borne diseases in 2013, of e cases to date this yea eady 28% more than the whole year of 2012.

oto/CDC-James Gathany

In addition to dengue fever, they are also experiencing a outbreak of chikungunya infections, also transmitted by mosquitoes. To date, there has been 126 cases.

According to the Singapore National Environmental Agency (NEA), as of 4 May 2013, the total number of dengue cases is 5,928. All of 2012 had only 4,632 cases.

This year is already drawing comparisons to 2005 when 14,000 people fell sick and 25 died from the mosquito borne viral disease.

EBOLA

ZIKA

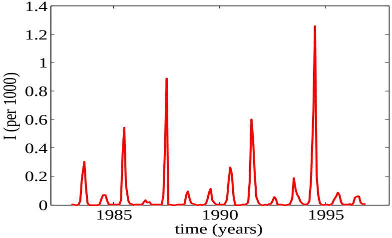
Motivation-Disease extinction

- Control and eradication of infectious diseases are main and important public health goals.

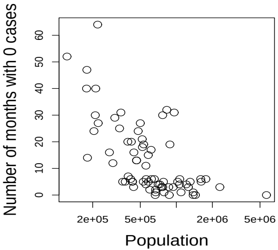
Motivation-Disease extinction

- Control and eradication of infectious diseases are main and important public health goals.
- Extinction is observed in networked populations.
  - Disease extinction occurs when infective population goes to zero.
  - Local extinction in connected patches but reintroduced
  - Global extinction is difficult and a rare event.

Dengue Incidence for
Chiang Mai province (1/72), Thailand.



Measles Incidence by
Thailand province (1980-2001).



***Data provided by Derek Cummings (JHU).*

Outline

- Analyzing fluctuations to extinction
  - All-to-all networked populations
  - Finding out how extinction occurs
  - Predicting extinction times
- Extending fluctuation analysis to networks
  - Homogeneous networks - average degree
  - General theory applied to heterogeneous networks
  - Optimal control on heterogeneous networks
- Conclusion and future work

Analyzing Fluctuations to Extinction

All-to-All Connected Networks

Basic SIS model-All-to-All Coupled Population Network

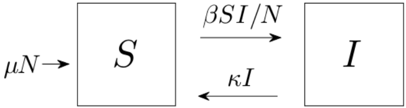
No network structure: all-to-all coupling

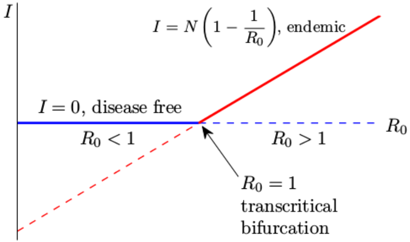
Two state variables:
Susceptibles, S
Infectives, I Total population size, N

$S + I = N$
Assume N is large.

Parameters:
birth and death rates, μ
contact rate, β
recovery rate, κ

Reproductive infection rate
$R_0 = \beta/(\mu + \kappa)$
Distance to the bifurcation point- $R_0 - 1$



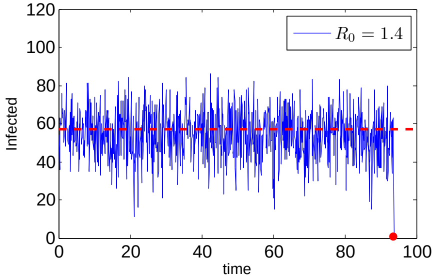


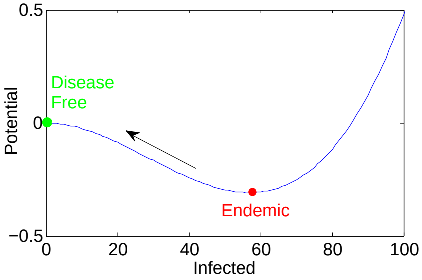
Anderson and May (1991)

Stochastic modeling

There exists random fluctuations, or noise, in the finite N model*

- Markov process
  Internal noise:
  Randomness of the interactions in the system
- Extinction - Analogous to arbitrarily small noise inducing escape of a particle from a potential well.



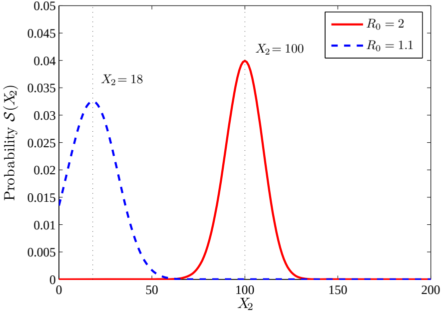


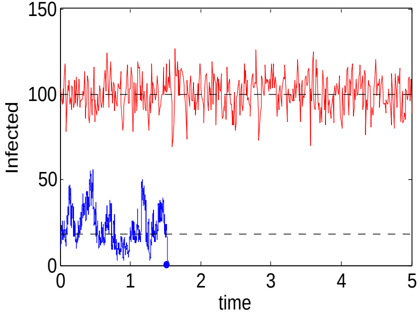
*Schwartz et al J R Soc Interface 8: 1699-1707 (2011)

Characterizing the “almost constant” density

The extinct state ($X_2 = I = 0$) is an absorbing boundary and the system approaches it as $t \to \infty$.

However, if the population size is sufficiently large, the probability density will be Quasi-stationary- $\partial\rho/\partial t \approx 0$.





If $\partial\rho/\partial t \approx 0$, then the value of $\rho(0, t)$ is exponentially small and we define extinction as a rare event.

Master Equation Approach-Modeling the Density

Consider a well-mixed finite population of size N

- Discrete state vector $\mathbf{X} = (S, I, R, ...)$.
- Probability $\rho(\mathbf{X}, t)$ of finding the system in state $\mathbf{X}$ at time t:
- Random state transition rates of increment $\mathbf{r}$: $W(\mathbf{X}, \mathbf{r})$.

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The master equation definition

$$\frac{\partial \rho(\mathbf{X}, t)}{\partial t} = \sum_{\mathbf{r}} [\underbrace{W(\mathbf{X}-\mathbf{r};\mathbf{r})\rho(\mathbf{X}-\mathbf{r}, t)}_{\substack{\text{the gain to state } \mathbf{X} \\ \text{from state } \mathbf{X}\text{-}\mathbf{r}}} - \underbrace{W(\mathbf{X};\mathbf{r})\rho(\mathbf{X}, t)}_{\substack{\text{the loss of state } \mathbf{X} \\ \text{to other states}}}].$$

It is the gain-loss equation for the probabilities of the separate states **X**.

Van Kampen, N.G., Stochastic processes in physics and chemistry, Elsevier (1992).

Approximating quasi-stationary solutions

To analyze the master equation, make the ansatz:

$$\rho(\mathbf{X}, t) \approx \exp(-N\mathcal{S}(\mathbf{q})), \text{ for } \mathbf{q} = \mathbf{X}/N.$$

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Large N assumption: Action $\mathcal{S}$ satisfies Hamilton-Jacobi equation:

$$\frac{\partial \mathcal{S}}{\partial t} + H\left(\mathbf{q}, \frac{\partial \mathcal{S}}{\partial \mathbf{q}}\right) = 0,$$

with Hamiltonian

$$H(\mathbf{q}; \mathbf{p}) = \sum_{\mathbf{r}} w(\mathbf{q}; \mathbf{r})[\exp(\mathbf{p} \cdot \mathbf{r}) - 1]$$

where $w(\mathbf{q};\mathbf{r}) = W(\mathbf{q};\mathbf{r})/N$

Conjugate momenta

$\mathbf{p} = \partial \mathcal{S}/\partial \mathbf{q}$.

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Conjugate momenta

$\mathbf{p} = \partial \mathcal{S}/\partial \mathbf{q}$.

Since we assume the distribution is quasi-stationary, $\dfrac{\partial \mathcal{S}}{\partial t} = 0$.

Kubo, et al., J. Stat. Phys. 9 (1973); Gang, PRA, 36 (1987); Dykman, et al., J. Chem Phys, 100 (1994); Elgart, et al., PRE, 70 (2004); and many others.

The Stochastic SIS model - Topology*

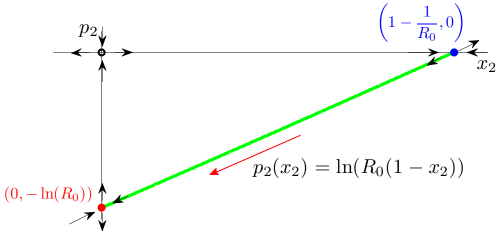
Constrain the population, N: $X_1 + X_2 = N$

Hamiltonian equations of motion- scaled infectives x_2 and momenta p_2:
The Hamiltonian system has three steady states: $\boxed{R_0 = \beta/(\mu+\kappa) > 1}$

- The disease free equilibrium, $(x_2, p_2) = (0, 0)$.
- The endemic state, $(x_2, p_2) = (1 - \frac{1}{R_0}, 0)$.
- The stochastic extinction state, $(x_2, p_2) = (0, -\ln(R_0))$.

Find the action along the path

$$\mathcal{S}_{opt} = \int_{1-\frac{1}{R_0}}^{0} -\ln(R_0(1-x_2))\, dx_2$$
$$= \ln(R_0) - 1 + \frac{1}{R_0}$$

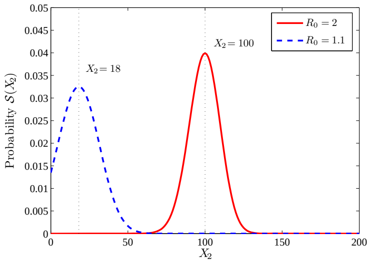


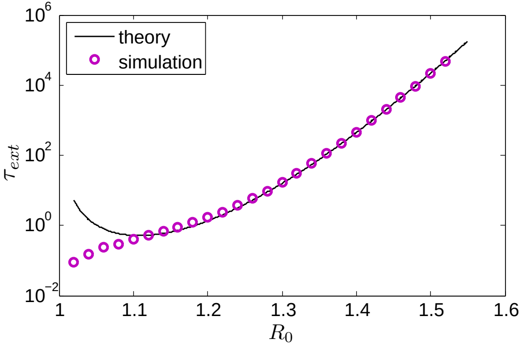
*Forgoston et al Bull Math Bio 73: 495-514 (2011).

The Stochastic SIS model - Mean Time to extinction

To approximate the mean time to extinction $\tau_{ext} \propto 1/\rho_{ext}$:

$$\tau_{ext} = Be^{N\mathcal{S}_{opt}} = \frac{R_0}{(R_0-1)^2}\sqrt{\frac{2\pi}{N}}e^{N(\ln(R_0)-1+\frac{1}{R_0})}$$

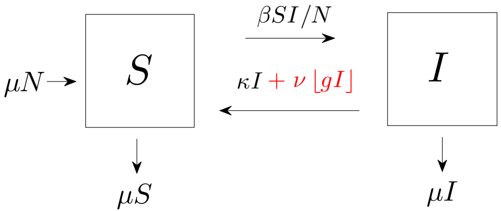




Doering, et al., Multiscale Model. Simul. (2005); Dykman et al, PRL 101 (2008); Schwartz et al, J Stat Mech, P01005 (2009).

Full SIS treatment model (Unconstrained population)

Remove fixed population constraint - N fluctuates



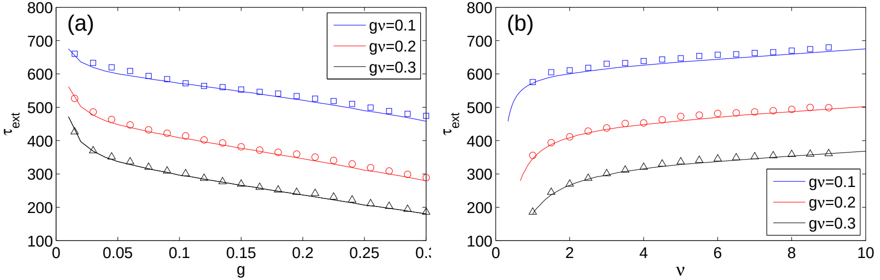
We have two states: susceptible (X_1) or infected (X_2). Use a Poisson based treatment:

- Treat a percentage, g, of infectives
- At a rate of average frequency ν

$$W\left((X_1, X_2); (\lfloor gX_2 \rfloor, -\lfloor gX_2 \rfloor)\right) = \nu, \quad \text{treatment}.$$

Finite resources of SIS treatment - Optimal schedule

Decrease in mean time to extinction as the *g* increases and ν decreases.
Larger fraction treated fewer times per year is most effective.

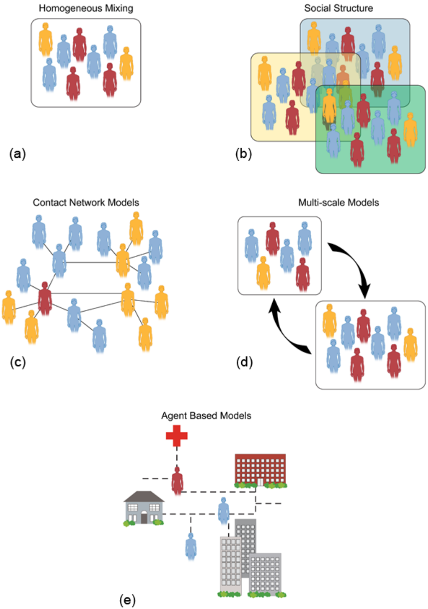


Results: average of 10,000 simulations
Master equation theory (solid)
MC simulation (symbols)

Parameters: $g\nu =$ constant
$\beta = 105$ year^{-1}
$N = 8000$ people

Billings et al PLOS ONE 8 (8), e70211 (2013)

Fluctuation Analysis on Networks

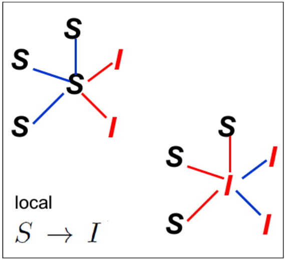


- Extend fluctuation analysis and control to stochastic networks

Rewiring for adaptation, IB Schwartz and LB Shaw Physics 3 (17) (2010)

SIS on Networks[1]

Structure of the network consists of numbers of nodes AND links:
$\mathbf{X} = [N_S, N_I, N_{SS}, N_{SI}, N_{II}]$
N nodes, K links, $2K/N$ mean degree



Three state transitions (assume no births/deaths):

- $S \to I$ along the network (local)
- $S \to I$ Global transmission
- $I \to S$ Recovery

Changing a node means links change; e.g.,
$S \to I$ means $SS \to SI$

[1] Brandon Lindley, Leah Shaw, Ira B. Schwartz EPL 108 58008(2014)

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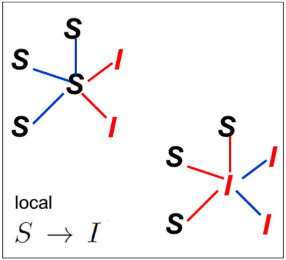

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Link numbers may be large when they change

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Continuation for Networks

- Optimal path is known numerically for SIS with all to all coupling (lower dimensional mean field)

- In a globally coupled network, links scale quadratically with nodes
  - For an all-to-all connected graph, $N_{AB} \propto N_A N_B$

- Constructive approach - Perturb from population with global coupling to population on a network and track the optimal path
  - ϵ is a homotopy parameter
  - $\epsilon = 0$ corresponds to all to all coupling No structure
  - $\epsilon = 1$ corresponds to local network coupling Network structure

Modeling the Transitions for a Homogeneous Network

- $\epsilon = 0$ corresponds to all to all coupling
- $\epsilon = 1$ corresponds to local network coupling

- p is infection rate
- r is a recovery rate
- $\mathbf{X} = [N_S, N_I, N_{SS}, N_{SI}, N_{II}]$

[2] Tim Rogers et al J. Stat. Mech. Theory and Experiment, PO8018 2012

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Transition rates[2]

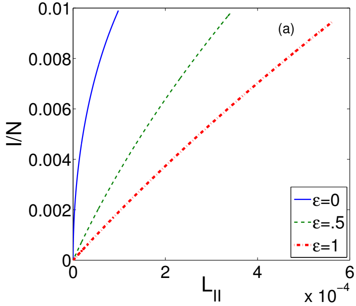
$W(\boldsymbol{X}, \boldsymbol{\nu}_1) = \epsilon p N_{SI}$	$S \rightarrow I$	Local
$W(\boldsymbol{X}, \boldsymbol{\nu}_2) = (1-\epsilon) p \dfrac{2K}{N} \dfrac{N_S N_I}{N}$	$S \rightarrow I$	Global
$W(\boldsymbol{X}, \boldsymbol{\nu}_3) = r N_I,$	$I \rightarrow S$	Recovery

[2] Tim Rogers et al J. Stat. Mech. Theory and Experiment, PO8018 2012

Optimal paths for a Stochastic Network

Computed paths from theory

Lindley et al, Physica D 255, 22-30 (2013)



Infective fraction vs II links

$p = 1.03x10^{-4}$, $r = 0.002$,$N = 10^4$,$K = 10^5$

- $\epsilon = 0$ - all to all coupling
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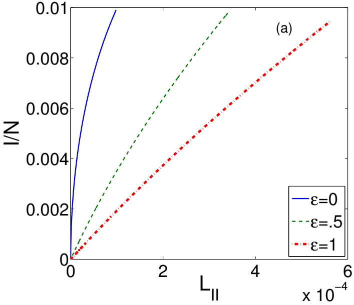

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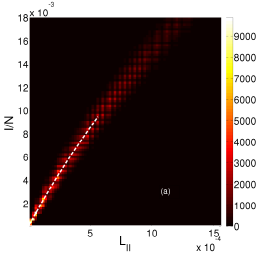
Compared to Monte Carlo PDF ($\epsilon = 1$)



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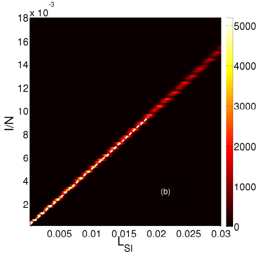
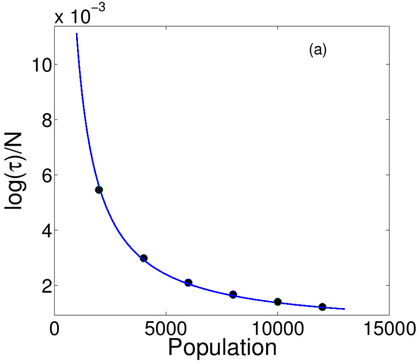


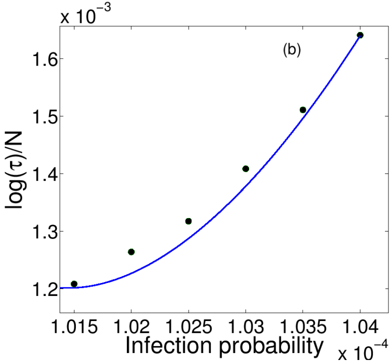
SI Links

Extinction Times for a Stochastic Network

As a function of population size



As a function of infection probability



$$log(\tau)/N \approx S + log(B)/N$$

Pre-factor at $\epsilon = 0$: $B = \frac{\sqrt{2\pi \frac{R_0^{eff}}{N}}}{r(R_0^{eff} - 1)^2}$:

No fitting parameters

Extinction in heterogeneous networks-General Theory

Consider SIS transitions on network having degree distribution g_k

- Assume adjacency matrix follows : $A_{ij} \approx k_i k_j / (N \langle k \rangle)$.
- Bin infected nodes of degree k, I_k
- Transition rates

  Infection rate $w_k^{inf}(\mathbf{I}) = \beta k(N_k - I_k) \sum_{k'} k' I_{k'} / (N \langle k \rangle)$ with $I_k \to I_k + 1$

  Recovery rate $w_k^{rec}(\mathbf{I}) = \alpha I_k$ with $I_k \to I_k - 1$.

  $N_K \equiv g_k N$

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Master Equation

$$\frac{\partial \rho}{\partial t}(\mathbf{I}, t) = \sum_k w_k^{inf}(\mathbf{I} - \mathbf{1}_k)\rho(\mathbf{I} - \mathbf{1}_k, t) - w_k^{inf}(\mathbf{I})\rho(\mathbf{I}, t)$$

$$+ \sum_k w_k^{rec}(\mathbf{I} + \mathbf{1}_k)\rho(\mathbf{I} + \mathbf{1}_k, t) - w_k^{rec}(\mathbf{I})\rho(\mathbf{I}, t),$$

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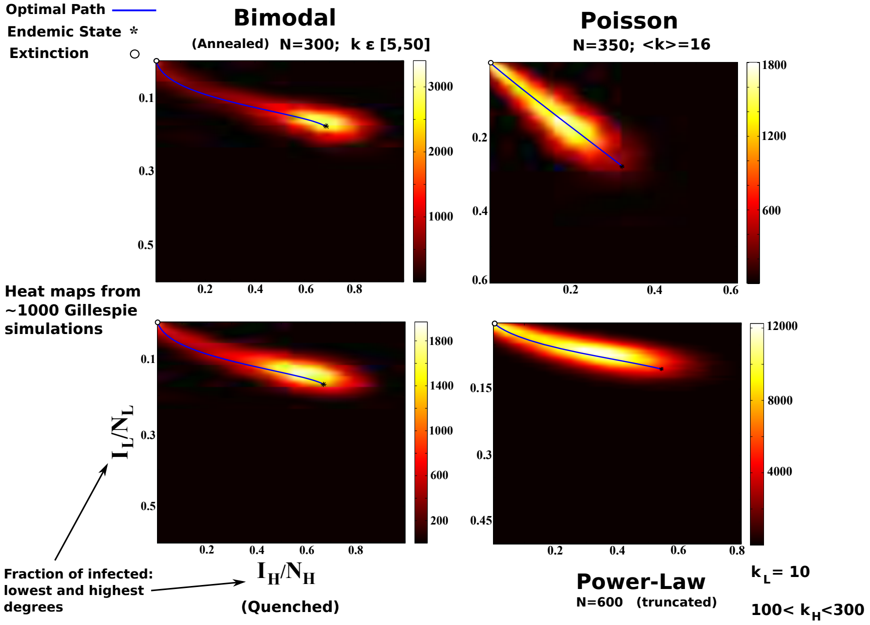
$$+ \sum_k w_k^{rec}(\mathbf{I} + \mathbf{1}_k)\rho(\mathbf{I} + \mathbf{1}_k, t) - w_k^{rec}(\mathbf{I})\rho(\mathbf{I}, t),$$

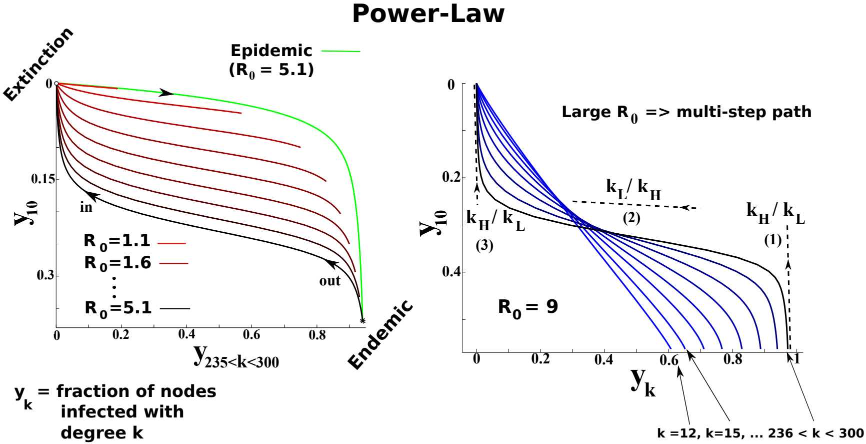
Hamiltonian from WKB ansatz ($\mathbf{x} = \mathbf{I}/N$):

$$H(\mathbf{x}, \mathbf{p}) = \sum_k \left[\beta k \left(g_k - x_k \right) \left(e^{p_k} - 1 \right) \sum_{k'} \frac{k' x_{k'}}{\langle k \rangle} + \alpha x_k \left(e^{-p_k} - 1 \right) \right].$$

Extinction paths in hetergeneous networks



Extinction in a Power Law Network

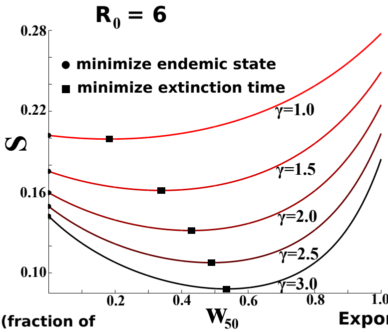


y_k = fraction of nodes infected with degree k

Action and extinction times in a bi-modal network

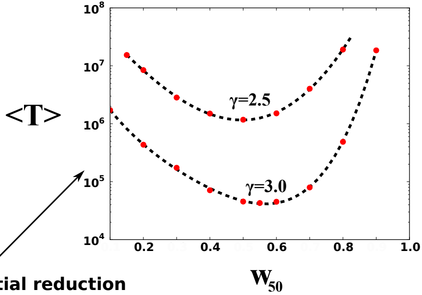
Bimodal distribution: %90 with k=5 and 10% with k=50

Treatment for nodes with degree k: (recovery) + $w_k \gamma$; $\Sigma w_k = 1$



(fraction of high-degree)

Quenched Network: N=200

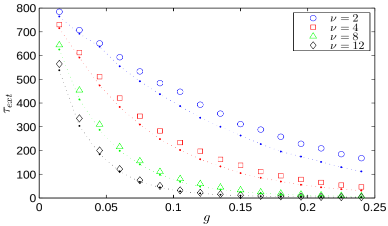


Exponential reduction in extinction times

Conclusions

- A general formulation of extinction for a disease in a finite population is developed.
- We can quantify the effect of treatment programs on extinction rates.
- For limited resources, larger treatment pulses less often are most effective.
- Used optimal paths to predict extinction times in terms of bifurcation parameters.
- Can choose topology of the network to optimally control extinction times.

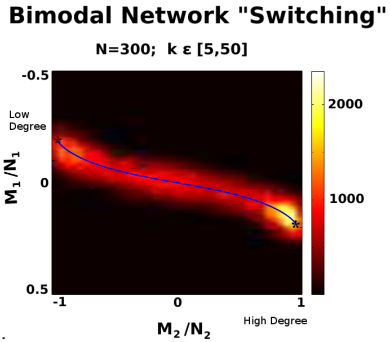
Periodic and random vaccination schedules



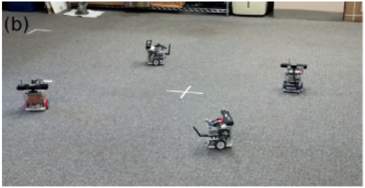
Future Directions and Things Not Discussed

- How does complex network structure affect route to extinction?
  - Topolgy, deterministic time dependent contacts, etc..
  - Beyond pairwise approximation
  - Non-Markovian assumptions
- Extend theory to other networks
  - Switching and adaptive networks
  - Networks with delays
  - Noise...

Dynamics Ising-J. Hindes, in prep



Noise-delay interaction in swarms
K. Szwaykowska,Phys. Rev. E 93, 032307 2016.



Related papers

On Extinction

- Jason Hindes and Ira B. Schwartz, "Epidemic Extinction and Control in Heterogeneous Networks", Physical Review Letters, 117, 028302 (2016).
- Klementyna Szwaykowska, Ira B. Schwartz, Luis Mier-y-Teran Romero et al, "Collective motion patterns of swarms with delay coupling: Theory and experiment," Phys. Rev. E 93, 032307 (2016).
- Brandon S. Lindley, Leah B. Shaw, Ira B. Schwartz, “Rare Event Extinction on Stochastic Networks,” arXiv:1411.0017 (2014), and EPL 108 58008(2014)
- Lora Billings, Luis Mier-y-Teran-Romero, Brandon Lindley, Ira B. Schwartz, “Intervention-Based Stochastic Disease Eradication," PLOS ONE 8 (8), e70211 (2013).
- Brandon S. Lindleyand Ira B. Schwartz, “An iterative action minimizing method for computing optimal paths in stochastic dynamical systems,” Physica D 255, 22-30 (2013).
- Brandon S. Lindley, Luis Mier-y-Teran-Romero, and Ira B. Schwartz, “Noise induced pattern switching in randomly distributed delayed swarms,” American Control Conference (ACC), 2013, 4587-45 (2013).
- Max S. Shkarayev, Ira B. Schwartz, Leah B. Shaw, “Recruitment dynamics in adaptive social networks,” Journal of Physics A: Mathematical and Theoretical 46 (24), 245003 (2013)
- Ira B. Schwartz, Eric Forgoston, Simone Bianco, Leah B. Shaw “Converging towards the optimal path to extinction,” J R Soc Interface 8: 1699-1707 (2011).
- Eric Forgoston, Simone Bianco, Leah B. Shaw, Ira B. Schwartz “Maximal sensitive dependence and the optimal path to epidemic extinction," Bull Math Bio 73: 495-514 (2011).
- LB Shaw, IB Schwartz,” Enhanced vaccine control of epidemics in adaptive networks,” Physical Review E 81 (4), 046120 (2010)
- IB Schwartz, LB Shaw, “Rewiring for adaptation,” Physics 3 (17) (2010)

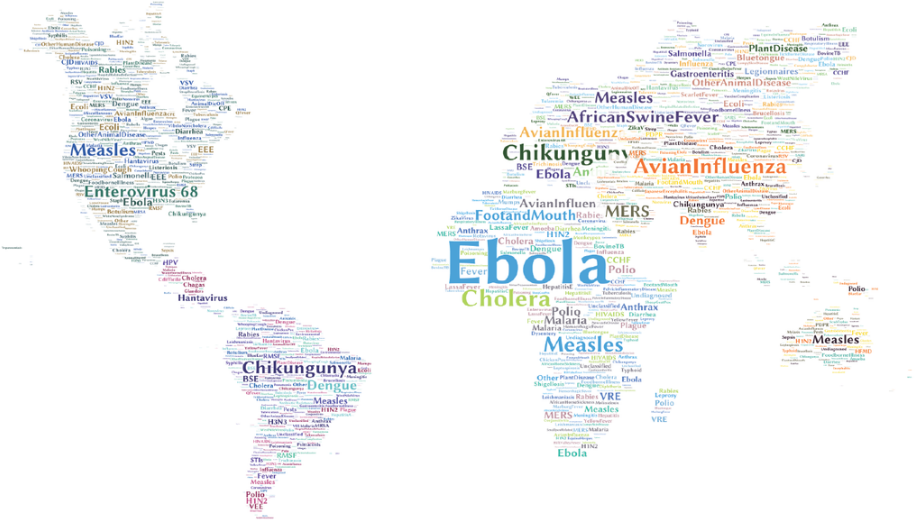
Related noise papers

On Noise-Induced Phenomena

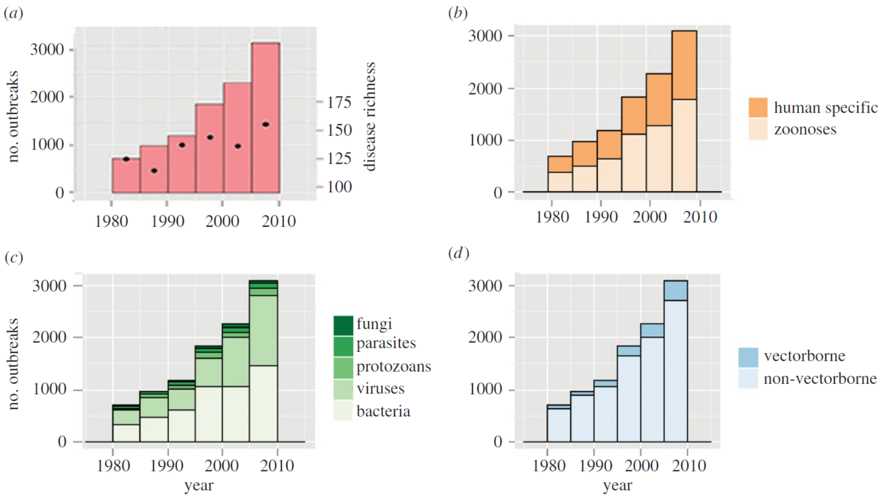
- Christoffer R. Heckman, M. Ani Hsieh, and Ira B. Schwartz, “Going With the Flow: Enhancing Stochastic Switching Rates in Multigyre Systems,” J. Dynamic Systems, Measurement, and Control 137, 031006-1 (2014).
- Christoffer R. Heckman and Ira B. Schwartz, “Stochastic switching in slow-fast systems: A large-fluctuation approach’,” PHYSICAL REVIEW E 89, 022919 (2014)
- Lora Billings, Mark I. Dykman, Marie McCrary, A. N. Korotkov, and Ira B. Schwartz, “Switching barrier scaling near bifurcation points for non-Gaussian noise," Physical Review Letters 104(14), 140601 (2010).
- Eric Forgoston, Lora Billings, and Ira B. Schwartz, “Accurate Time Series Prediction in Reduced Stochastic Epidemic Models," Chaos 19, 043110 (2009).
- Lora Billings, Ira B. Schwartz, and Mark I. Dykman, “Thermally activated switching in the presence of non-Gaussian noise," Physical Review E 78 (2008) 051122.

EXTRA SLIDES

Spatial View of Epidemic Outbreaks



Temporal Increase of Epidemic Outbreaks



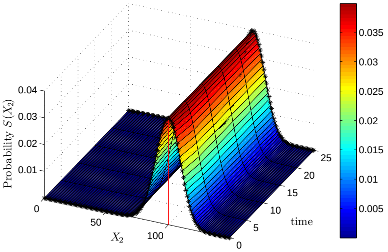
Smith KF, 2014 J. R. Soc. Interface 11: 20140950. http://dx.doi.org/10.1098/rsif.2014.0950

The drift of probability distributions in time

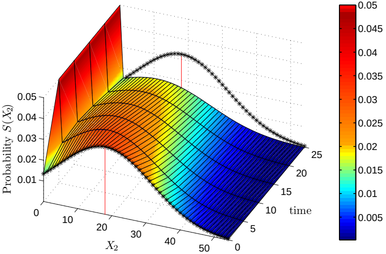
The system always decays to $X_2 = 0$ as $t \to \infty$ (absorbing boundary).

The drift to $X_2 = 0$ is slow for quasi-stationary systems $\dfrac{\partial \rho}{\partial t} \approx 0$, and fast for systems that are not quasi-stationary.

$R_0 = 2$, quasi-stationary



$R_0 = 1.1$, not quasi-stationary



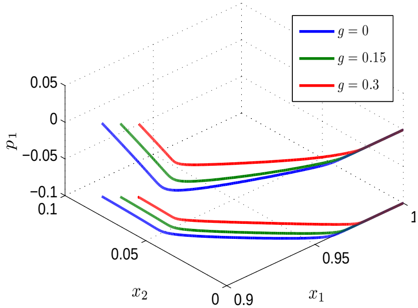
Solution for $\rho(X_2, t)$ from the master equation.

Full SIS treatment model - Optimal path

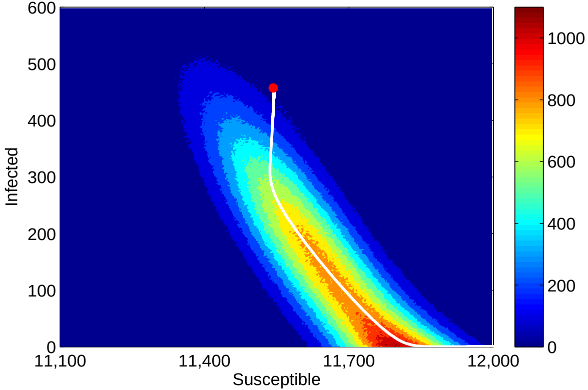
The Hamiltonian in normalized coordinates ($x_1 = X_1/N$, $x_2 = X_2/N$) is

$$H(\mathbf{x},\mathbf{p}) = \mu(e^{p_1}-1) + \beta x_1 x_2 (e^{-p_1+p_2}-1) + \kappa x_2 (e^{p_1-p_2}-1) \\ +\mu x_1 (e^{-p_1}-1) + \mu x_2 (e^{-p_2}-1) + \frac{\nu}{N}(e^{g x_2 N p_1 - g x_2 N p_2}-1).$$

Form the auxiliary Hamiltonian system and identify the extinction path.



Full SIS treatment model - Extinction realizations



Parameters:

$\beta = 105$ year^{-1}

$R_0 = 1.0479$

$g = 0.1$

$\nu = 8$ year^{-1}

$N = 12,000$ people

Probability density of extinction prehistory. Spatial frequency for the last five years of data from 200,000 Monte Carlo extinction realizations.

The optimal path (white curve) connects the endemic state to the extinct state. Notice that it lies on the peak of the probability density of extinction prehistory.

The SIR Model

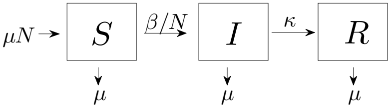
SIR model (Deterministic)

Captures dynamics of most common childhood diseases that confer long-lasting immunity: chickenpox, measles, mumps, rubella, etc.

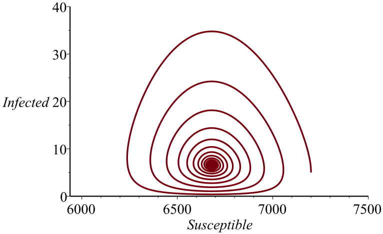
Population of individuals: susceptible (S), infected (I) or recovered (R). Total population: $N = S + I + R$.

Mean field equations:

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \frac{\beta}{N} SI - \mu S \\ \frac{dI}{dt} &= \frac{\beta}{N} SI - \kappa I - \mu I \\ \frac{dR}{dt} &= \kappa I - \mu R \end{aligned}$$



Since $R = N - S - I$, consider the (S, I) system



Basic reproduction number: $R_0 = \frac{\beta}{\mu+\kappa}$

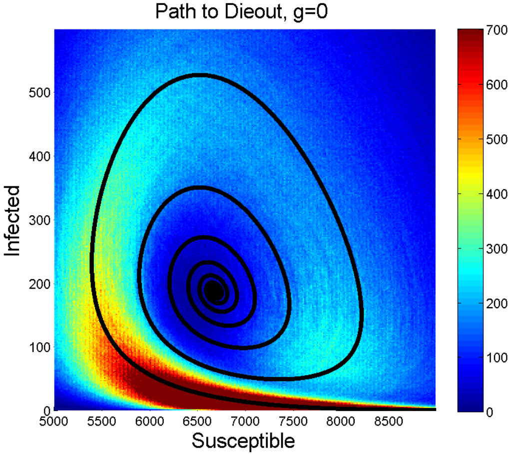
Steady states:

- disease free, $(S, I) = (N, 0)$
- endemic, $(S, I) = (\frac{N}{R_0}, \frac{\mu N}{\beta}(R_0 - 1))$

  $R_0 > 1 \rightarrow$ endemic stable

The Stochastic SIR model

Master equation approach: Optimal path lies along PDF local maxima



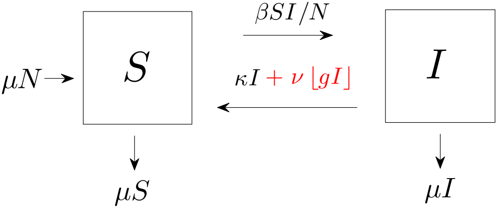
Probability density of extinction prehistory over 40,000 simulations.

Parameters:
$\beta = 1500$ year^{-1}
$R_0 = 14.97$
$N = 100,000$ people

[Schwartz, et al., J R Soc Interface, 2011]

The Stochastic SIS treatment model

Consider the SIS treatment model ($X_2 = I$) with transitions:



$$
\begin{array}{ll}
W\big(X_2; -1\big) = \kappa X_2, & \text{recovery} \\
W\big(X_2; -1\big) = \mu X_2, & \text{death} \\
W\big(X_2; 1\big) = \beta X_2(N - X_2)/N, & \text{infection} \\
W\big(X_2; -\lfloor gX_2 \rfloor\big) = \nu, & \text{treatment}
\end{array}
$$

Infectives receive treatment, which removes them from the infective group. We remove a fraction of infectives (g) at a mean frequency (ν) per year: Poisson treatment

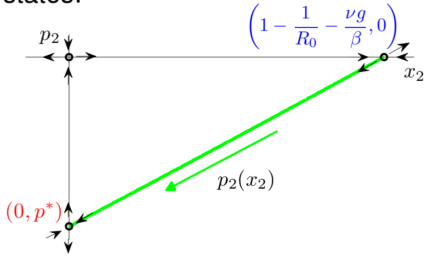
The Stochastic SIS treatment model - Topology

The equations of motion are

$$\dot{x}_2 = \beta x_2 (1 - x_2) e^{p_2} - (\mu + \kappa) x_2 e^{-p_2} - \nu g x_2 e^{-gN x_2 p_2},$$
$$\dot{p}_2 = -\beta (1 - 2x_2)(e^{p_2} - 1) - (\mu + \kappa)(e^{-p_2} - 1) + \nu g p_2 e^{-gN x_2 p_2}.$$

The Hamiltonian system has three steady states:

- The disease free equilibrium, $(x_2, p_2) = (0, 0)$.
- The endemic state, $(x_2, p_2) = (1 - \frac{1}{R_0} - \frac{\nu g}{\beta}, 0)$.
- The stochastic extinction state, $(x_2, p_2) = (0, p^*)$, $\nu g p^* = \beta(e^{p^*} - 1) + (\mu + \kappa)(e^{-p^*} - 1)$.



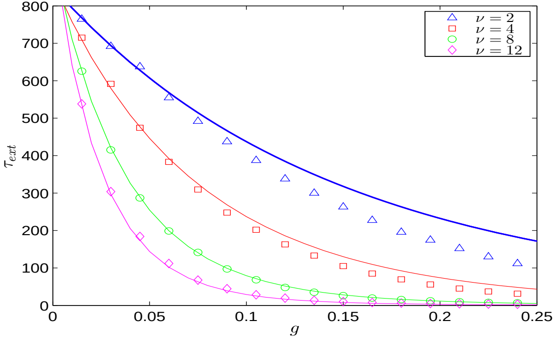
Find the action along the path of an extinction event ($\mathcal{S}_{opt}$).

For $g > 0$, $\mathcal{S}_{opt}$ will have to be approximated.

The Stochastic SIS treatment model - Mean Time to Extinction

To approximate the mean time to extinction: $\tau_{ext} = Be^{N\mathcal{S}_{opt}}$ (years)

Compare the result to Monte Carlo (Gillespie) simulations.



Results:

Theory (solid)

MC simulation (symbols)

- avg 2000 simulations

Parameters:

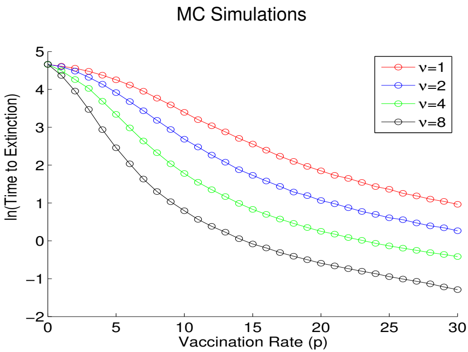
$\beta = 105$ year^{-1}

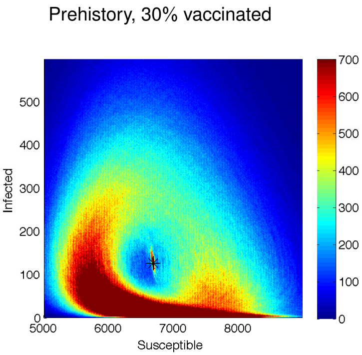
$R_0 = 1.0479$

$N = 8000$ people

Notice the decrease in mean time to extinction as g and ν increases.

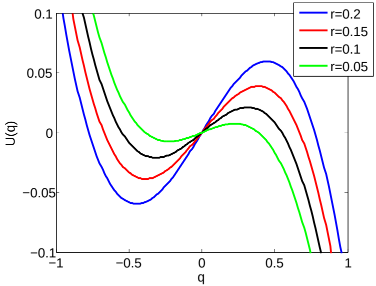
The Stochastic SIR with Vaccinations





Other types of noise induced behavior-Bifurcations

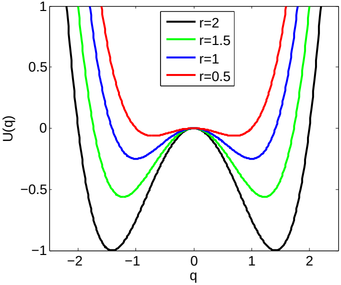
Escape from a single well potential



$$U(q) = -q^3/3 + rq$$

Saddle node bifurcation at $r = 0$

Switching in a double well potential



$$U(q) = q^4/4 - rq^2/2$$

Pitchfork bifurcation at $r = 0$

[Billings, et al., PRL (2010) 140601; Billings,et al., PRE 78 (2008) 051122.]

Other Optimal Path Experiments

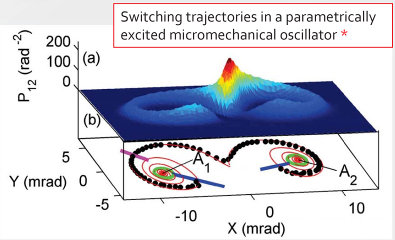
- Microscopic and mesoscopic systems:

  Josephson junctions, mechanical nanoresonators, nanomagnets.

  Fluctuations usually due to thermal or externally applied Gaussian noise.

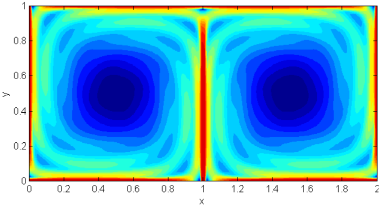
Trajectories form a narrow tube centered at the most probable switching path.

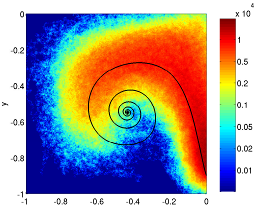
Result: Efficient control of switching rates.



[Chan, et al. PRL 100 (2008) 130602; Chan, et al. PRE 78 (2008) 051109]

- Autonomous robot escape from gyre flows





Going With the Flow: Enhancing Stochastic Switching Rates in Multigyre Systems C Heckman, MA Hsieh, IB Schwartz J. Dynamic Systems, Measurement, and Control 137, 031006-1 2014

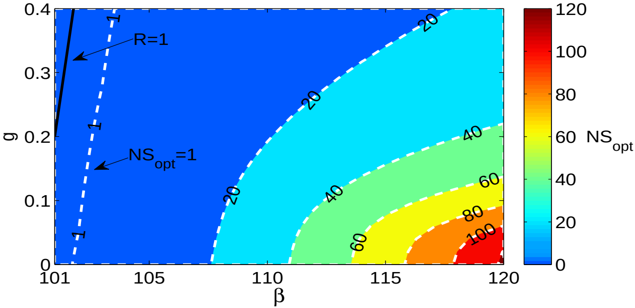
Stochastic SIS treatment model - Quasi-stationarity

When is ρ Quasi-stationary?

$\rho(0) = Ae^{-N\mathcal{S}_{opt}}$

$0 < Ae^{-N\mathcal{S}_{opt}} \ll 1$

$N\mathcal{S}_{opt} \gg 1$



Numerical approximation of the optimal path:

- Iterative action minimizing method.

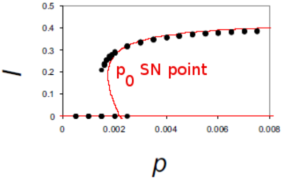
  [Lindley and Schwartz, *Physica D* (2013).]

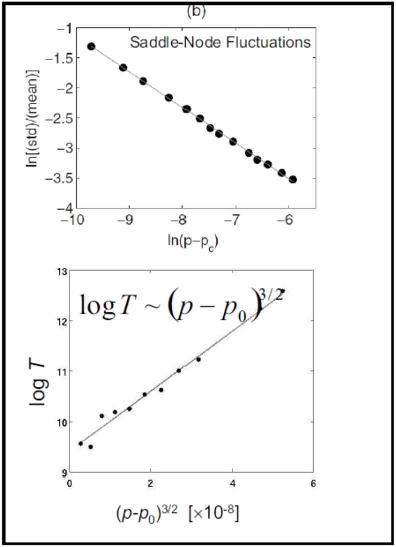
- Other methods: Shooting, String method, Minimum action method
  [Keller(1976); E, Ren, and Vanden-Eijnden (2002) and (2004).]

$\mu = 0.2$ year^{-1}
$\kappa = 100$ year^{-1}
$\nu = 4$ year^{-1}
$N = 8000$ people

Fluctuations and Lifetimes of Endemic State

- Lifetime is defined as the time to extinction of I nodes
- Fluctuations increase near the Saddle-Node point p_0 (Bistable state)
  - Scaling of fluctuations explained by noise-induced dynamics near a saddle-node point
- Mean lifetime T of the endemic state becomes shorter near the bifurcation point
- Lifetime scaling is consistent with a saddle-node bifurcation





q=0.0016, r=0.002, w=0.04

Quasi-stationary solutions in a dynamical systems framework

A physically meaningful distribution must satisfy correct boundary conditions.

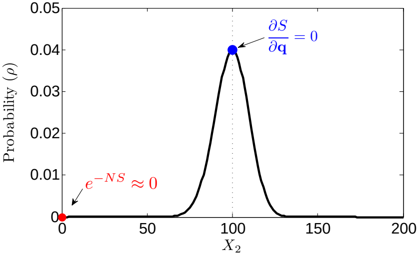
Recall $\rho(\mathbf{q}, t) \approx e^{-N\mathcal{S}(\mathbf{q})}$, so

$$\frac{\partial \rho(\mathbf{q}, t)}{\partial \mathbf{q}} \approx -N e^{-N\mathcal{S}} \frac{\partial \mathcal{S}}{\partial \mathbf{q}}$$

Since $\mathbf{p} = \partial \mathcal{S} / \partial \mathbf{q}$, then
$p = 0$ at the endemic state
$p \neq 0$ at the extinction state.

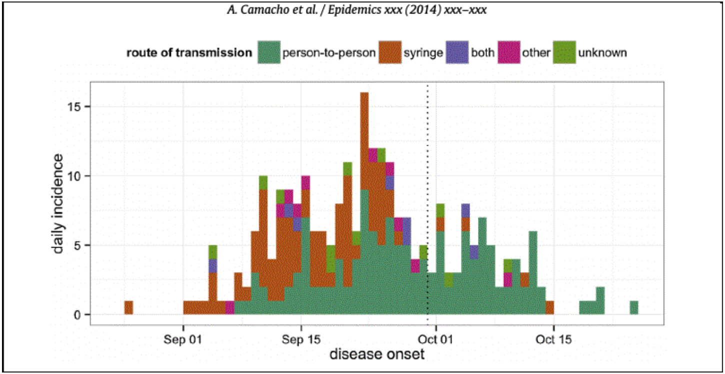
The stationary states:

- endemic state
- stochastic extinction state



Human Behavior Modifies Disease Fade Out

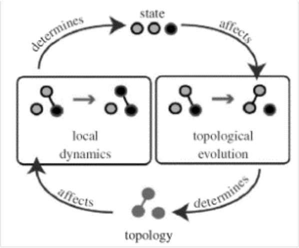
- Strong evidence hospital and person-to-person transmission declined over the course of the outbreak.
- Epidemiological reports the community stopped coming to the outpatient department as they associated the epidemic with the hospital, which eventually was closed on 30th September.
- The population became very suspicious and did not touch the corpses anymore, not even to bury them.

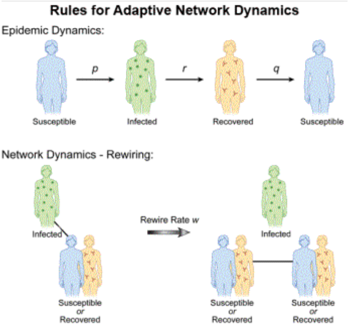


EPIDEMICS, v. 9, pp. 70-78, 2014, Zaire, 1976 Ebola

Dynamics of Stochastic Adaptive Networks

- In real networks nodes and links change in time-Dynamic networks
- Node dynamics affects network geometry
  Network geometry affects node dynamics
- Feedback loop interaction
- Adaptive networks have many applications
  - Human social networks
  - Fads, terrorist networks
  - Self healing networks
  - Swarming of autonomous agents
  - Immune system networks
  - Biological networks (e.g., food webs)





Epidemics on Adaptive Social Networks

Epidemic dynamics: Avoidance Behavior

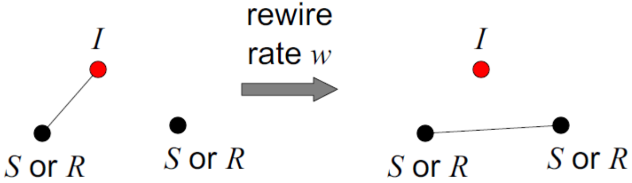
$$S \xrightarrow{p} I \xrightarrow{r} R \xrightarrow{q} S$$

S: susceptible
I: infected
R: recovered

N_{AB}: AB links

p: infection rate
r: recovery rate
q: resusceptibility rate
w: rewiring rate

Network dynamics—rewiring:



Run Monte Carlo simulation for $N=10^4$ nodes, $K=10^5$ links

(Shaw and Schwartz PRE 77: 066101, 2008)

Mean Field Approximation

$$\boldsymbol{S} \xrightarrow{p} \boldsymbol{I} \xrightarrow{r} \boldsymbol{R} \xrightarrow{q} \boldsymbol{S}$$

- Node dynamics—depends on node pairs (links)

$$\boxed{\begin{aligned} \dot{P}_S &= qP_R - p\frac{K}{N}P_{SI}, \\ \dot{P}_I &= p\frac{K}{N}P_{SI} - rP_I, \\ \dot{P}_R &= rP_I - qP_R. \end{aligned}}$$

N_{AB}: AB links

$\boldsymbol{p}$: infection rate
$\boldsymbol{r}$: recovery rate
$\boldsymbol{q}$: resusceptibility rate
$\boldsymbol{w}$: rewiring rate

Mean Field Approximation

$$S \xrightarrow{p} I \xrightarrow{r} R \xrightarrow{q} S$$

- Node dynamics—depends on node pairs (links)

$$\dot{P}_S = qP_R - p\tfrac{K}{N}P_{SI},$$

$$\dot{P}_I = p\tfrac{K}{N}P_{SI} - rP_I,$$

$$\dot{P}_R = rP_I - qP_R.$$

N_{AB}: AB links

$\boldsymbol{p}$: infection rate
$\boldsymbol{r}$: recovery rate
$\boldsymbol{q}$: resusceptibility rate
$\boldsymbol{w}$: rewiring rate

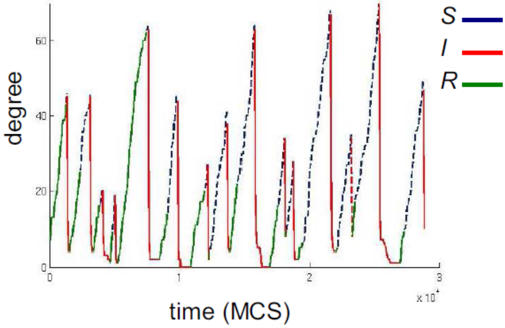
- Link dynamics—depends on triples

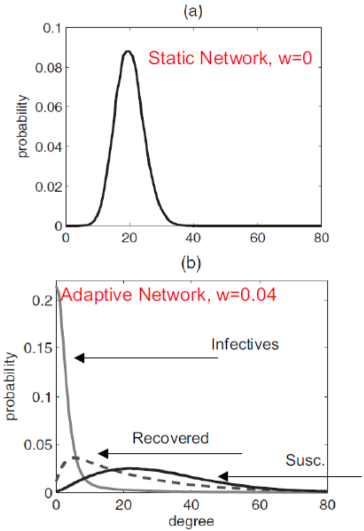
$$\dot{P}_{SI} = 2p\frac{K}{N}\frac{P_{SS}P_{SI}}{P_S} + qP_{IR} - rP_{SI} - \boxed{wP_{SI}} - p\left(P_{SI} + \frac{K}{N}\frac{P_{SI}^2}{P_S}\right)$$

Network structure analysis-Degree distribution

q=0.0016, r=0.002

Time series for degree of a single node:





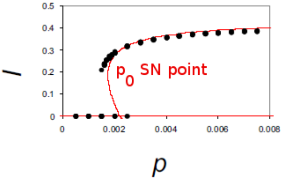
Node degree cycles in time

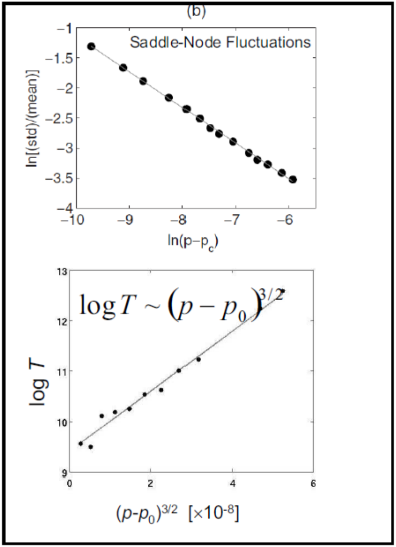
I loses links

R and S gain links

Fluctuations and Lifetimes of Endemic State

- Lifetime is defined as the time to extinction of I nodes
- Fluctuations increase near the Saddle-Node point p_0 (Bistable state)
  - Scaling of fluctuations explained by noise-induced dynamics near a saddle-node point
- Mean lifetime T of the endemic state becomes shorter near the bifurcation point
- Lifetime scaling is consistent with a saddle-node bifurcation





q=0.0016, r=0.002, w=0.04

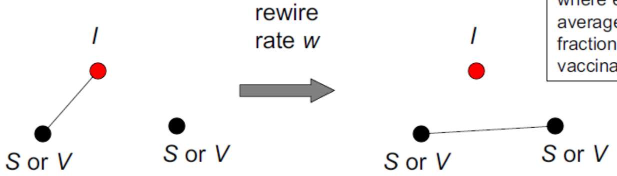
Extinction and Control in Adaptive Networks

Adaptive network with vaccinations

Epidemic dynamics:

$$\begin{tikzcd} S \arrow[r, shift left, "p"] \arrow[d, shift right, "\eta(t)"'] & I \arrow[l, shift left, "r"] \\ V \arrow[u, shift right, "q"'] & \end{tikzcd}$$

Network dynamics—rewiring:



S: susceptible
I: infected
V: vaccinated

p: infection rate
r: recovery rate
$\boldsymbol{\eta(t)}$: vaccination rate
q: resusceptibility rate
w: rewiring rate

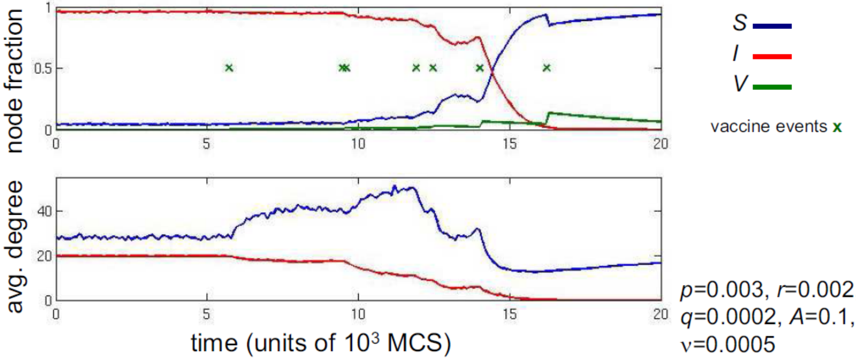
Vaccination rate is Poisson where events happen with average frequency ν and a fraction A of susceptibles are vaccinated in each event.

Run Monte Carlo simulation for $\boldsymbol{N}=10^4$ nodes, $\boldsymbol{K}=10^5$ links

L. B. Shaw and I. B. Schwartz, Phys. Rev. E (2010).

Effect of vaccination and rewiring on degree

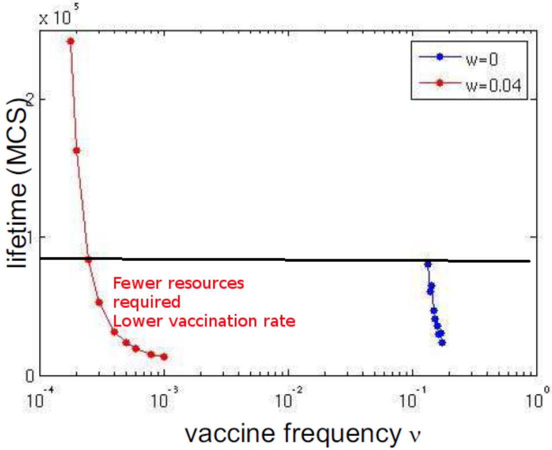
- Vaccination occurs on susceptible nodes
- In the adaptive network, susceptible nodes have higher degree due to rewiring
- Vaccination of high degree nodes provides better protection (e.g., Pastor-Satorras and Vespignani PRE 65: 036104, 2002)
- In the static network, high degree nodes tend to be infected and are not vaccinated



p=0.003, r=0.002, q=0.0002, A=0.1, ν=0.0005

Adaptive network with vaccinations

- Poisson-distributed pulse vaccine control
- Compute lifetime of the infected state
- Average over 100 runs
- Rewiring in combination with vaccination significantly shortens the disease lifetime



p=0.003, r=0.002, q=0.0002, A=0.1