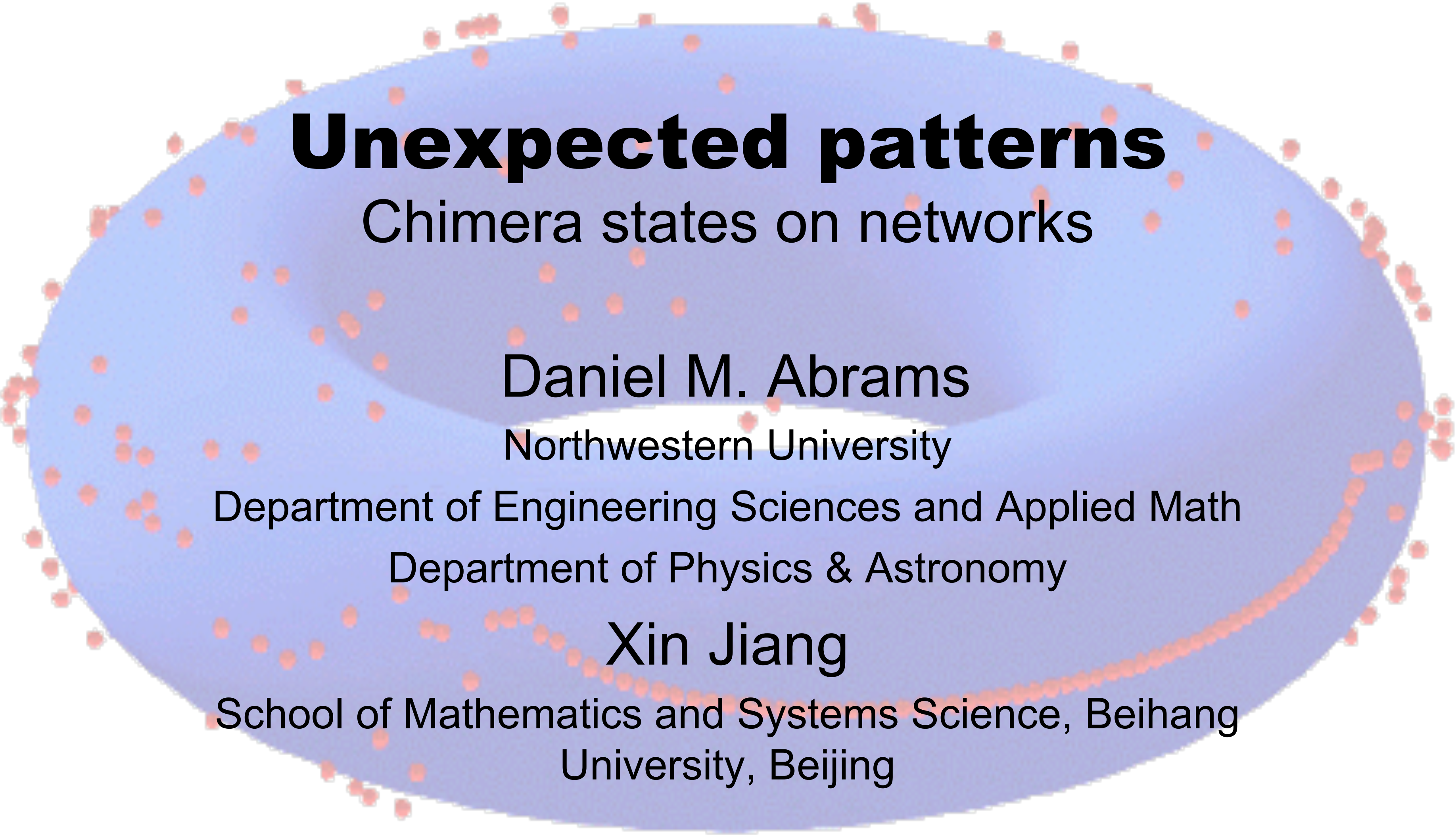
McCormick
Northwestern Engineering



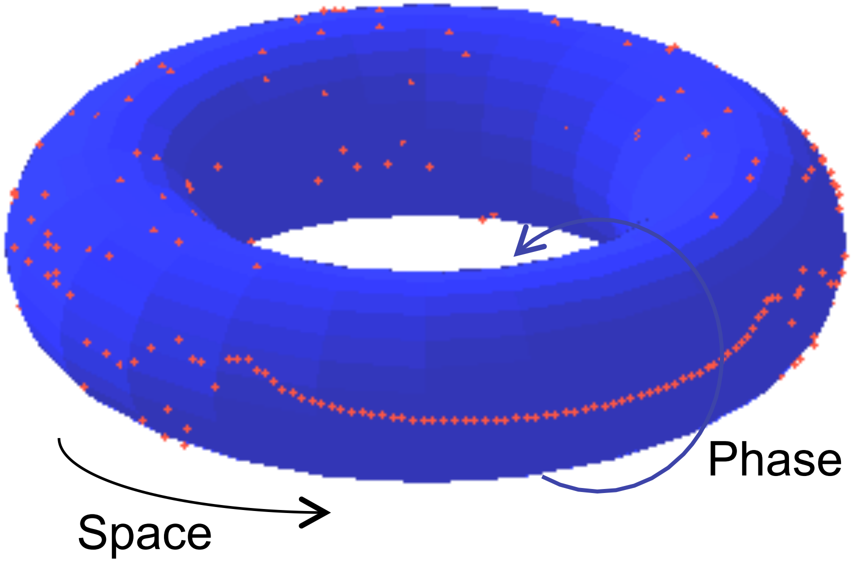
Unexpected patterns

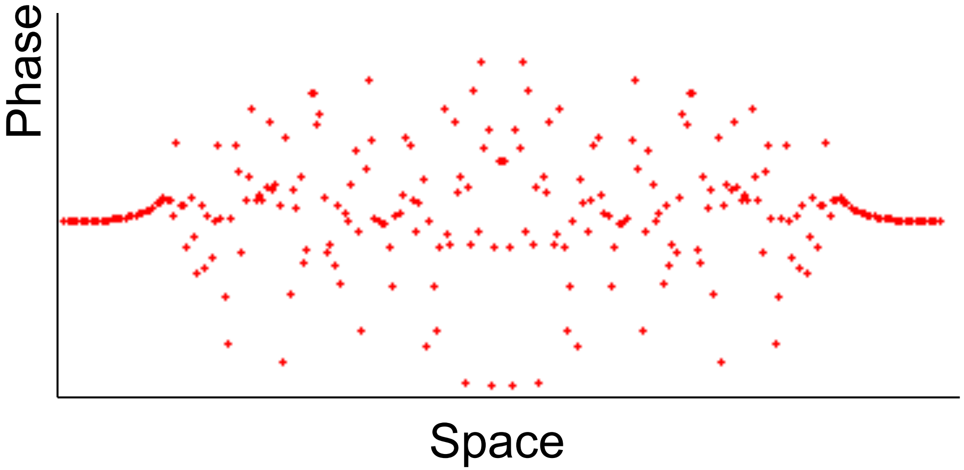
Chimera states on networks

Daniel M. Abrams
Northwestern University
Department of Engineering Sciences and Applied Math
Department of Physics & Astronomy
Xin Jiang
School of Mathematics and Systems Science, Beihang University, Beijing

What is a chimera state?

First example





What is a chimera state?

What is a chimera state?

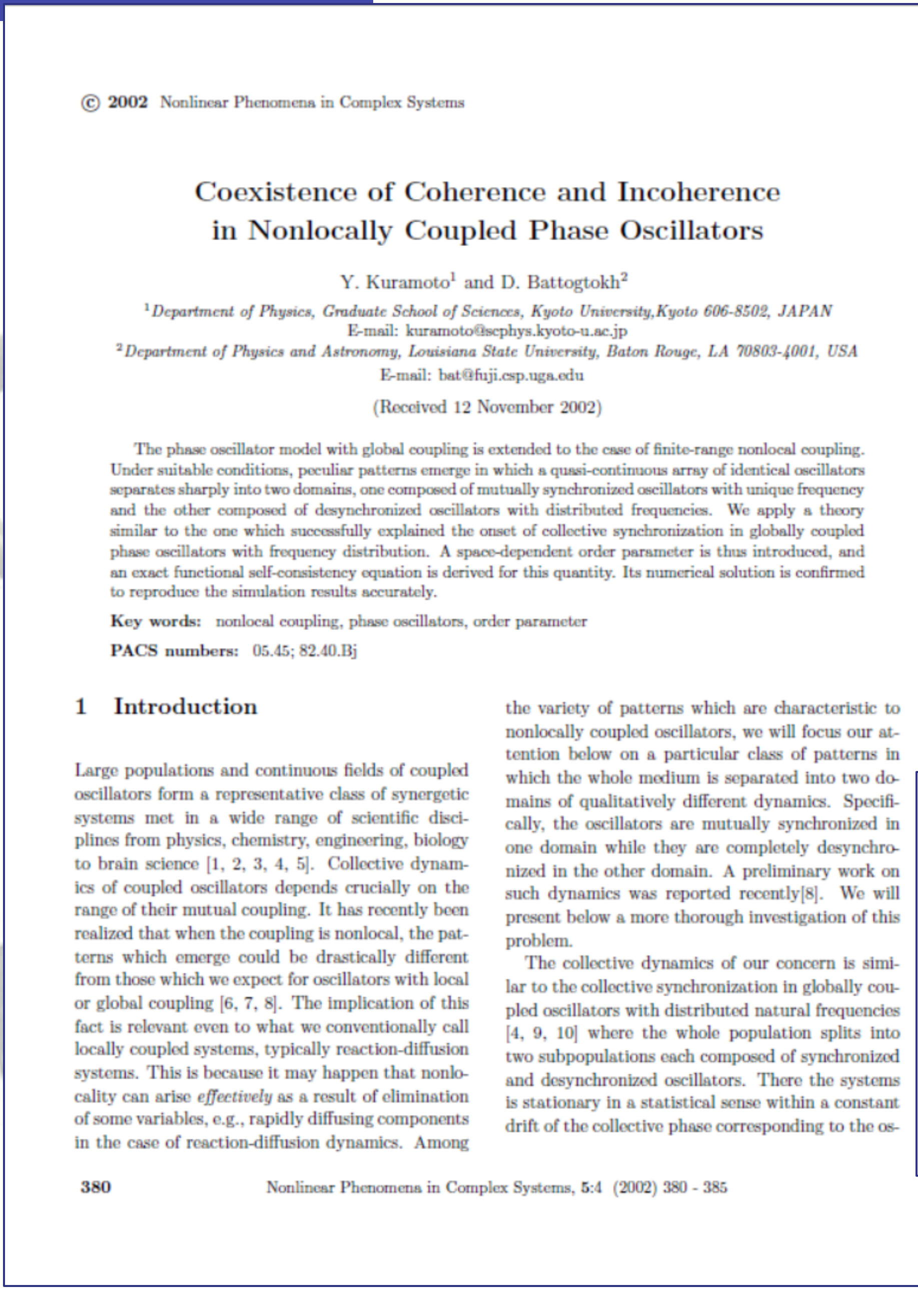
- Surprising patterned state
  - *Identical* oscillators
  - *Identically coupled* to neighbors
  - Symmetry suggests spatially uniform equilibria
  - State has broken symmetry…somehow
  - Result: spatial pattern of partial synchronization

a state?

-

equilibria

ow



© 2002 Nonlinear Phenomena in Complex Systems

Coexistence of Coherence and Incoherence in Nonlocally Coupled Phase Oscillators

Y. Kuramoto[1] and D. Battogtokh[2]
[1] *Department of Physics, Graduate School of Sciences, Kyoto University, Kyoto 606-8502, JAPAN*
E-mail: kuramoto@scphys.kyoto-u.ac.jp
[2] *Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803-4001, USA*
E-mail: bat@fuji.csp.uga.edu

(Received 12 November 2002)

The phase oscillator model with global coupling is extended to the case of finite-range nonlocal coupling. Under suitable conditions, peculiar patterns emerge in which a quasi-continuous array of identical oscillators separates sharply into two domains, one composed of mutually synchronized oscillators with unique frequency and the other composed of desynchronized oscillators with distributed frequencies. We apply a theory similar to the one which successfully explained the onset of collective synchronization in globally coupled phase oscillators with frequency distribution. A space-dependent order parameter is thus introduced, and an exact functional self-consistency equation is derived for this quantity. Its numerical solution is confirmed to reproduce the simulation results accurately.

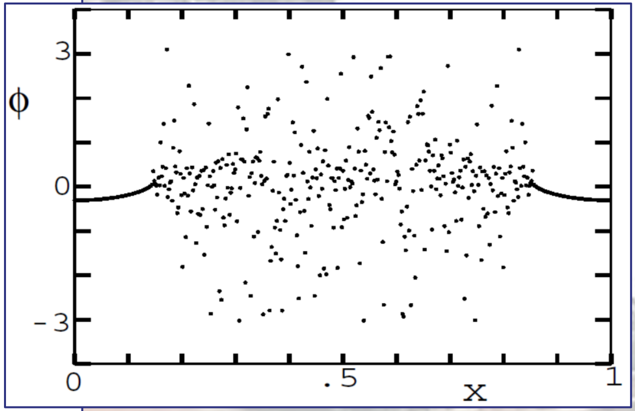
Key words: nonlocal coupling, phase oscillators, order parameter

PACS numbers: 05.45; 82.40.Bj

1 Introduction

Large populations and continuous fields of coupled oscillators form a representative class of synergetic systems met in a wide range of scientific disciplines from physics, chemistry, engineering, biology to brain science [1, 2, 3, 4, 5]. Collective dynamics of coupled oscillators depends crucially on the range of their mutual coupling. It has recently been realized that when the coupling is nonlocal, the patterns which emerge could be drastically different from those which we expect for oscillators with local or global coupling [6, 7, 8]. The implication of this fact is relevant even to what we conventionally call locally coupled systems, typically reaction-diffusion systems. This is because it may happen that nonlocality can arise *effectively* as a result of elimination of some variables, e.g., rapidly diffusing components in the case of reaction-diffusion dynamics. Among the variety of patterns which are characteristic to nonlocally coupled oscillators, we will focus our attention below on a particular class of patterns in which the whole medium is separated into two domains of qualitatively different dynamics. Specifically, the oscillators are mutually synchronized in one domain while they are completely desynchronized in the other domain. A preliminary work on such dynamics was reported recently[8]. We will present below a more thorough investigation of this problem.

The collective dynamics of our concern is similar to the collective synchronization in globally coupled oscillators with distributed natural frequencies [4, 9, 10] where the whole population splits into two subpopulations each composed of synchronized and desynchronized oscillators. There the systems is stationary in a statistical sense within a constant drift of the collective phase corresponding to the os-



First example

- Phase oscillators $(\dot{\theta}_i = d\theta_i/dt = \omega)$
- Ring geometry
- Coupling strength decays with distance
  - Not global
  - Not local
- "Phase lag" in coupling:

Necessary??

$$\dot{\theta}_i = \omega + \sum_{j=1}^{N} G_{ij} \sin(\theta_j - \theta_i - \alpha)$$

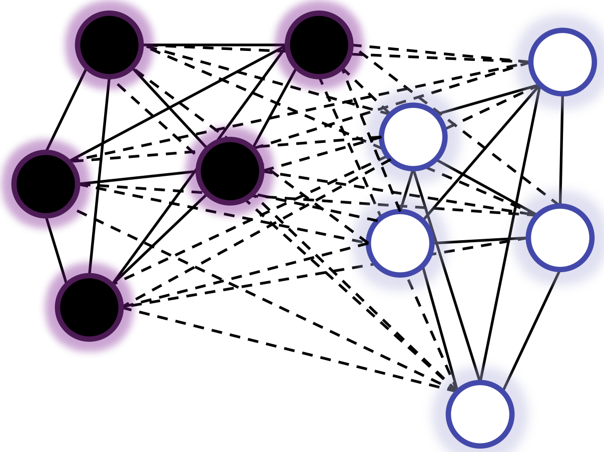
Phase lag

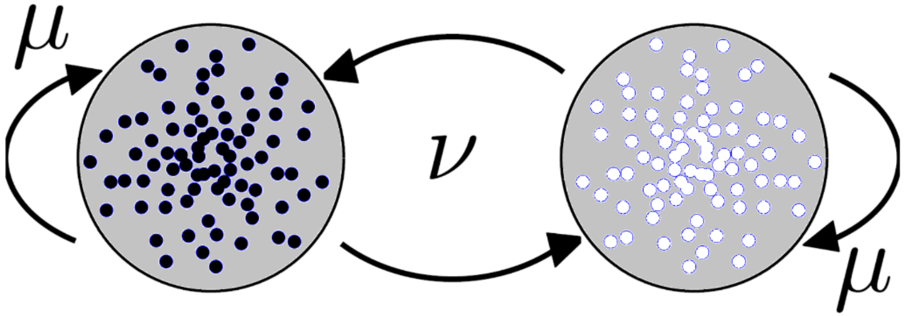
Natural frequency ω for all i

Coupling matrix: for ring e.g. decaying with distance: $G_{ij} \propto \exp(-d_{ij})$

Second example

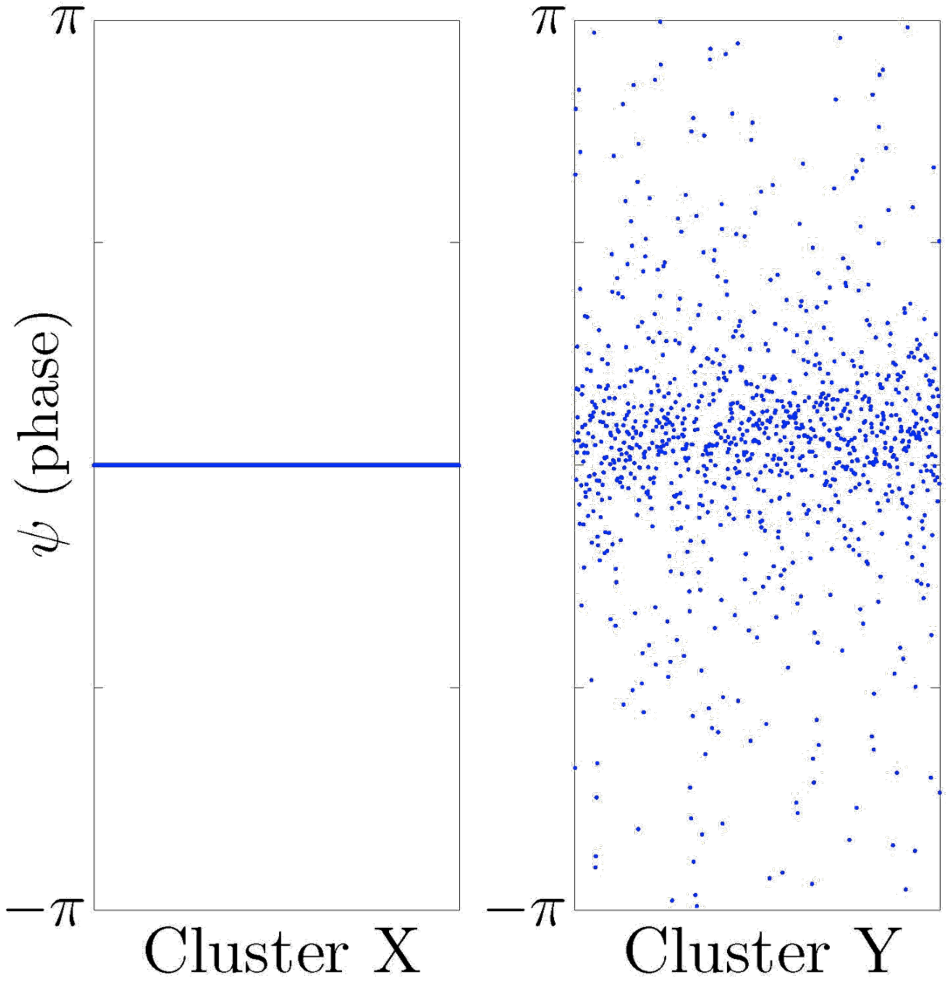
- Simpler math with different coupling: two-cluster
  - Stronger in-group coupling μ
  - Weaker out-group coupling ν





Second example

Typical stable chimera state



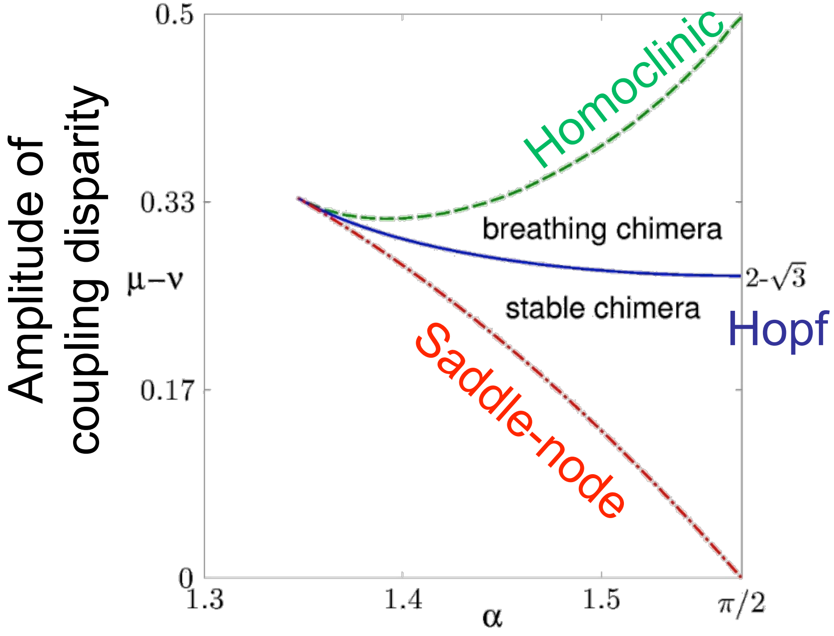
Second example

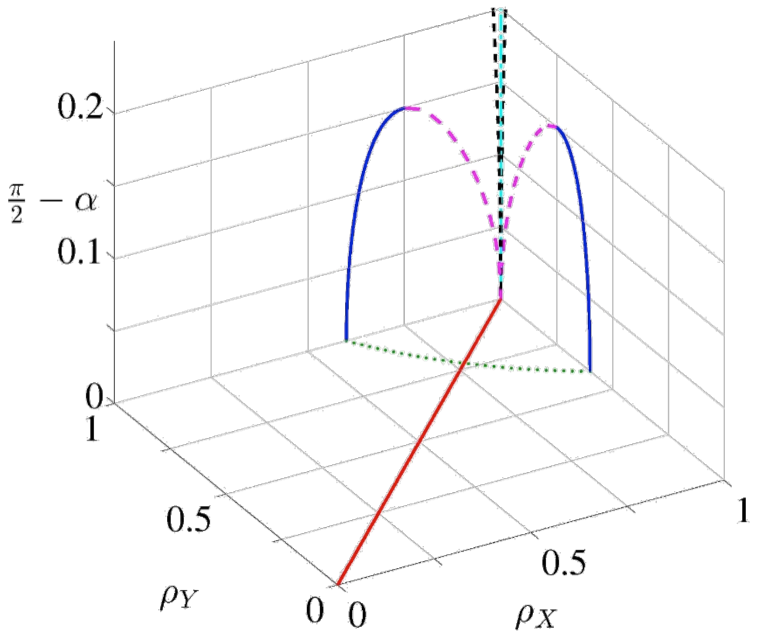
- Big idea: solve for *distribution* of phases
  - Can apply continuity equation $\frac{\partial p}{\partial t} + \frac{\partial}{\partial \theta}(pv) = 0$
  - N ODEs → single integro-differential equation for $p(\theta, t)$
- Big idea: reexpress using "order parameter"
  - Same as in soln. to classical Kuramoto model
- Big idea: Ott-Antonsen manifold (2008)
  - Assume Fourier series for p has particularly simple form

$$2\pi p(\theta, t) = 1 + \sum_{n=1}^{\infty} \left\{ \underbrace{\left[a_\sigma(t) e^{i\theta}\right]^n}_{\text{Fourier coefficients are geom. sequence}} + \left[a_\sigma^*(t) e^{-i\theta}\right]^n \right\}$$

Second example

- End up with low-dim (4D) ODE system
- Chimera state: one cluster sync's, other not
- Behaves qualitatively like ring - but solvable!





Big questions

- This is more than 10 years old…
- *How can chimera states exist?*
  - In a sense already known
- Intuitively, where do they come from?
  - Connection to resonance?
  - Bridge between synchrony and incoherence?
- Perhaps gain insight from limiting cases

Interesting limits

- Already many limits of interest:
  - Phase lag α: small (near zero) or large (near $\pi/2$)
  - Coupling strength: weak or strong
  - Coupling range: local or global
  - System size: small or large N
- Some have been explored, some not
- Some results dimension-dependent
- I won’t talk about any of them now…

Connection to “classical” Kuramoto model?

re interesting lim

Known chimera states

Let’s revisit initial assumptions.

- Identical oscillators
  - Relax: distribution of oscillator frequencies $g(\omega)$
    - g → delta function
    - g → finite variance distribution
    - g → divergent variance distribution
- Identically coupled (perfectly regular n work)
  - Relax: arbitrary coupling network
    - “small” amount of rewiring (perturbation from lattice)
    - “large” deviation from regular lattice

Numerical evidence for persistence of chimera states.

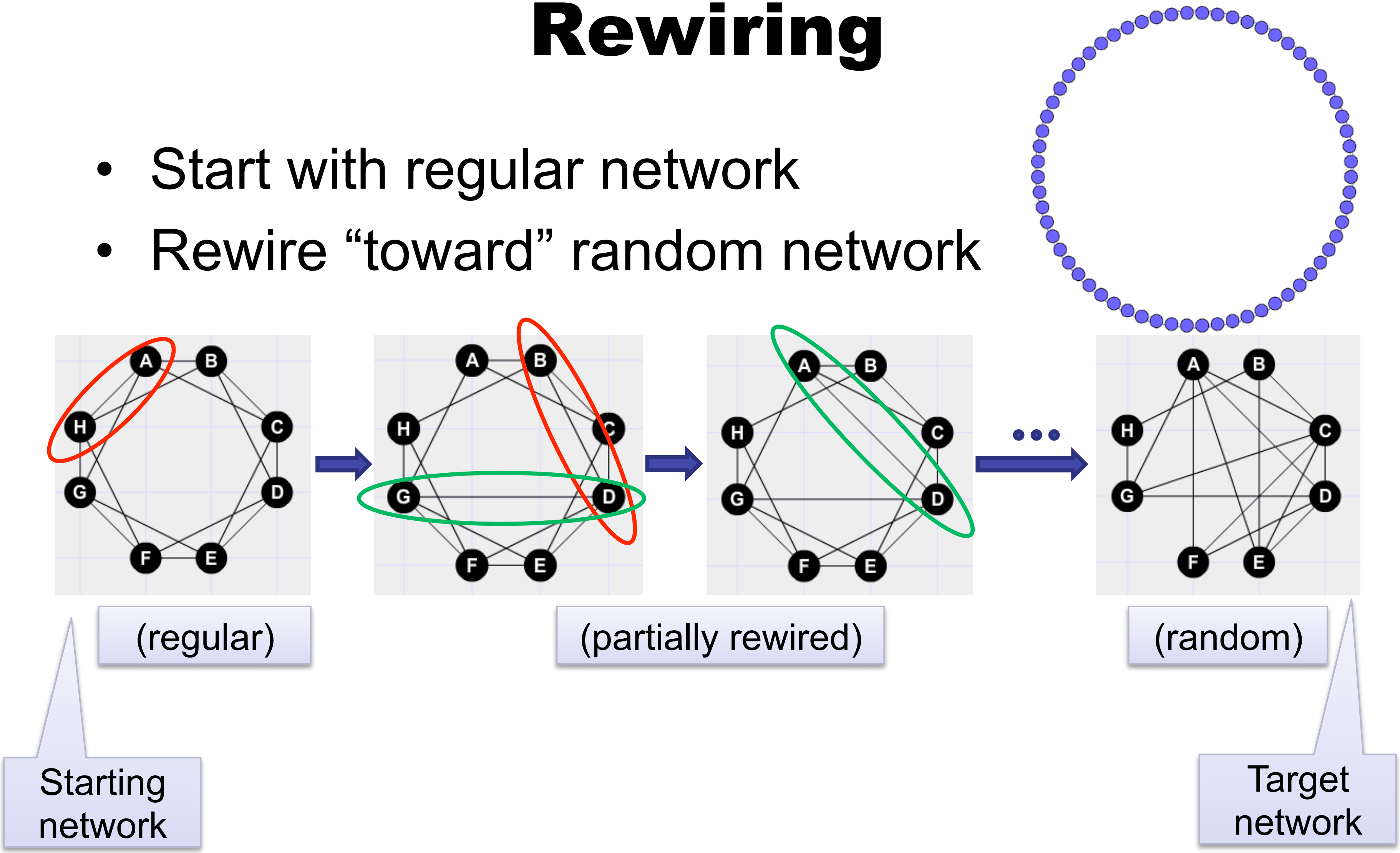
?

Numerical evidence for persistence of chimera states.

?

Rewiring

- Start with regular network
- Rewire “toward” random network



(regular)

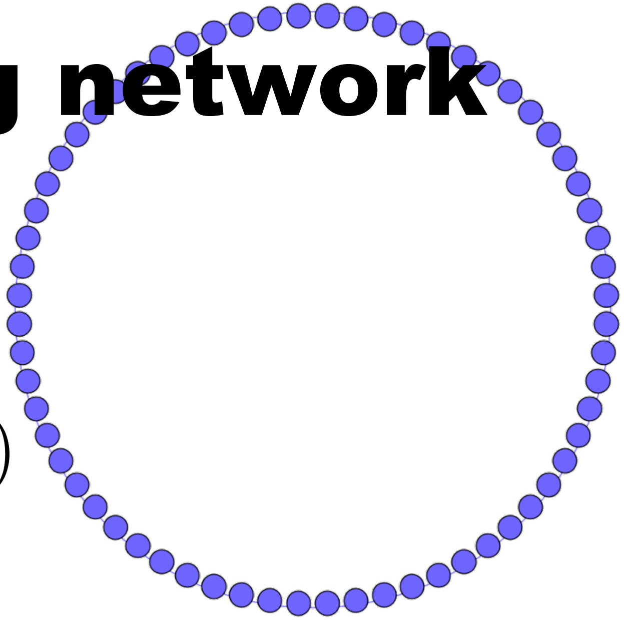
(partially rewired)

(random)

Starting network

Target network

Rewiring: starting network

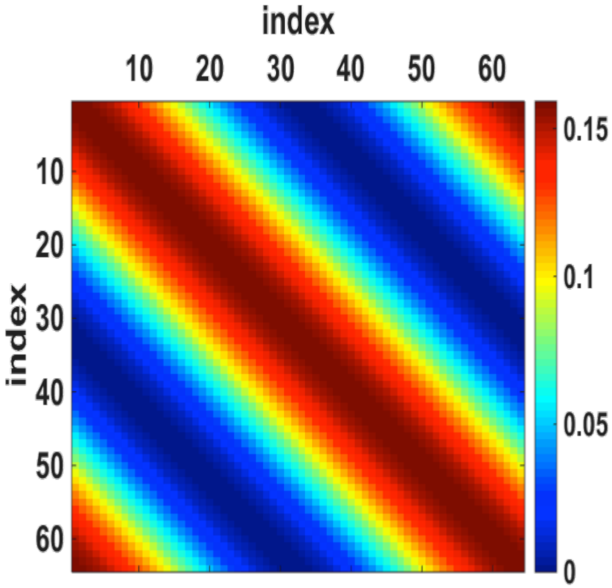


- Assume coupling strength decays with distance:
  $G(x,x') \propto 1 + A\cos\left(|x-x'|\right)$

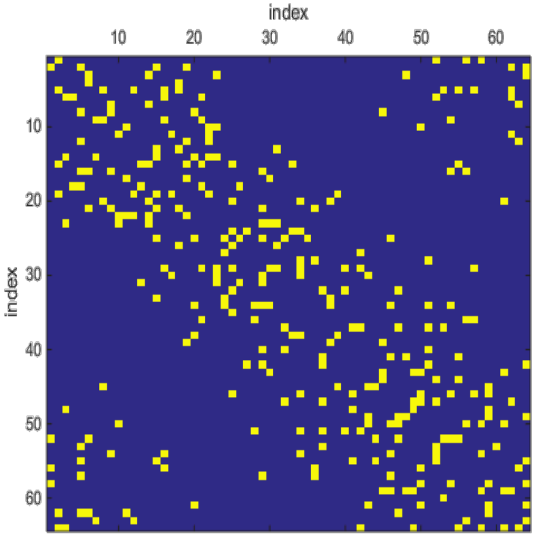
- Generate starting network with link probability proportional to coupling strength

Rewiring: starting network

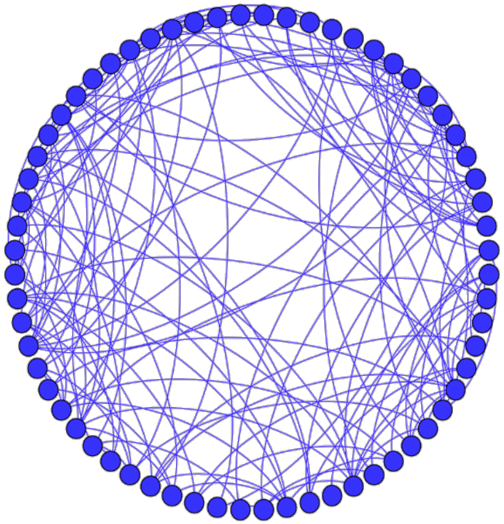
$c = 0.5$



Link probability



Adjacency matrix

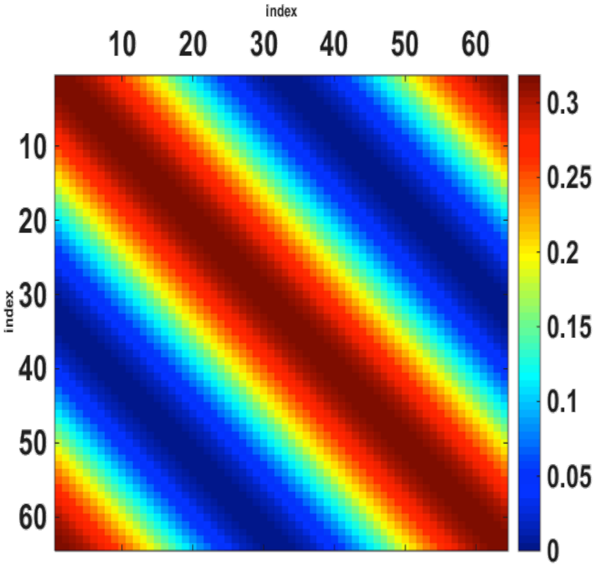


Network

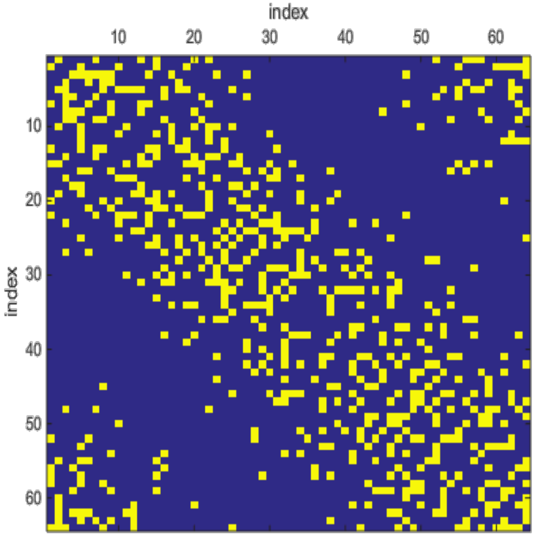
$$p_{ij} = \frac{c}{2\pi}\left[1 + A\cos(|x_i - x_j|)\right]$$

Rewiring: starting network

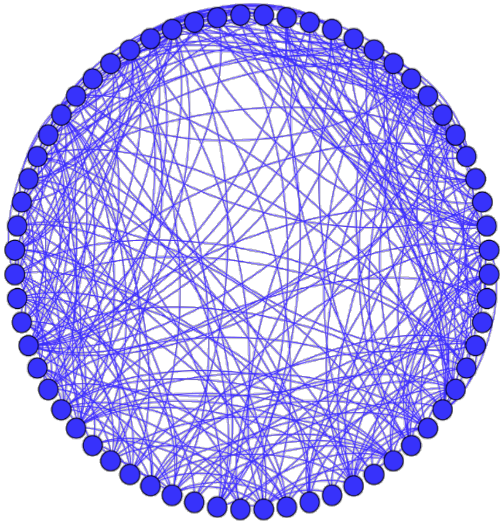
$c = 1.0$



Link probability



Adjacency matrix

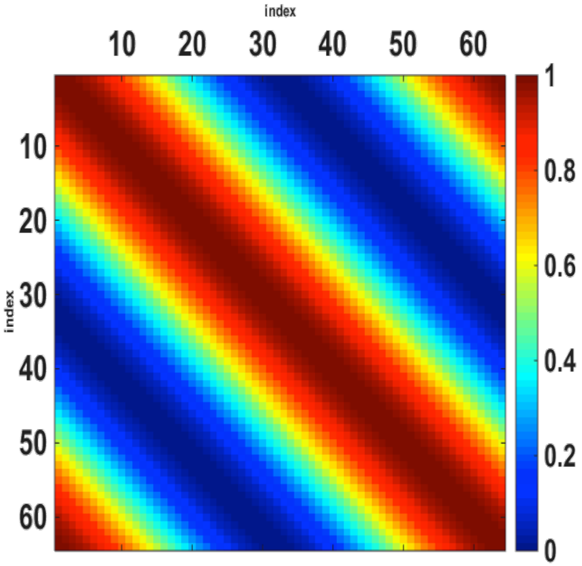


Network

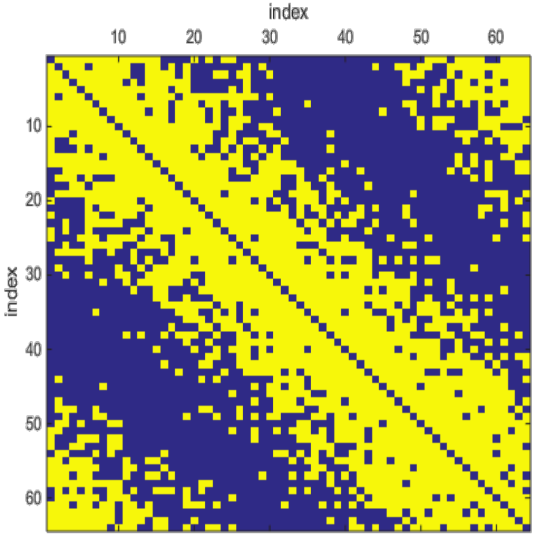
$$p_{ij} = \frac{c}{2\pi}\left[1 + A\cos(|x_i - x_j|)\right]$$

Rewiring: starting network

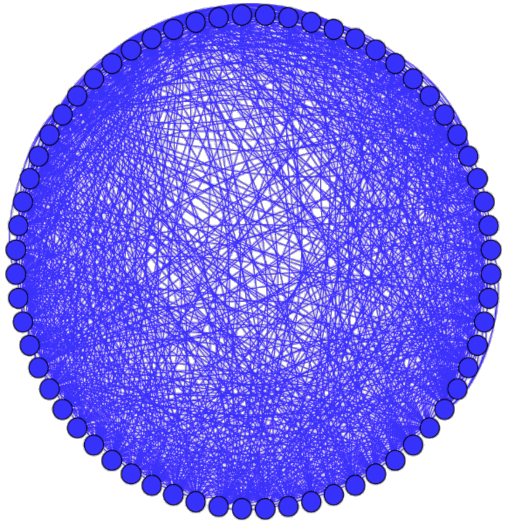
$c = \pi$



Link probability



Adjacency matrix



Network

$$p_{ij} = \frac{c}{2\pi}\left[1 + A\cos(|x_i - x_j|)\right]$$

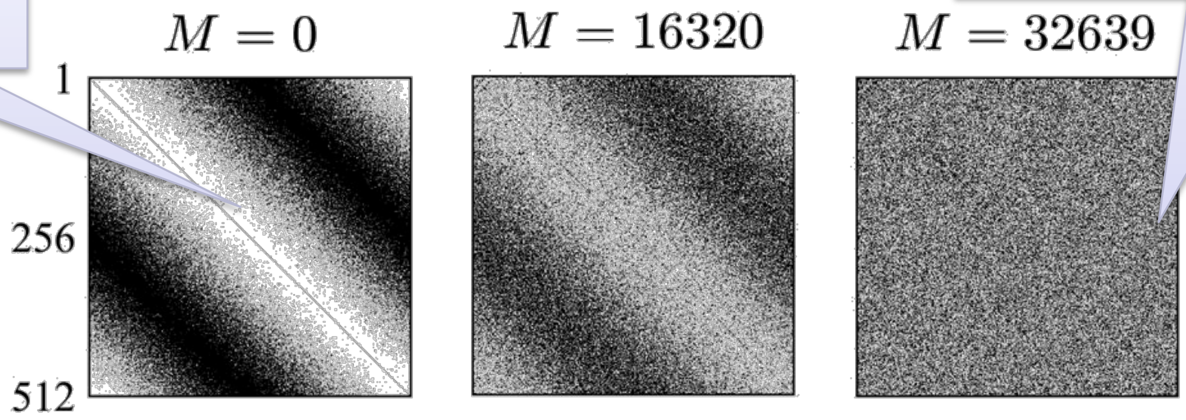
Rewiring: experiment

Starting network (structured)

More rewiring →

Target network (ER random)

With known structure:



Rewiring: experiment

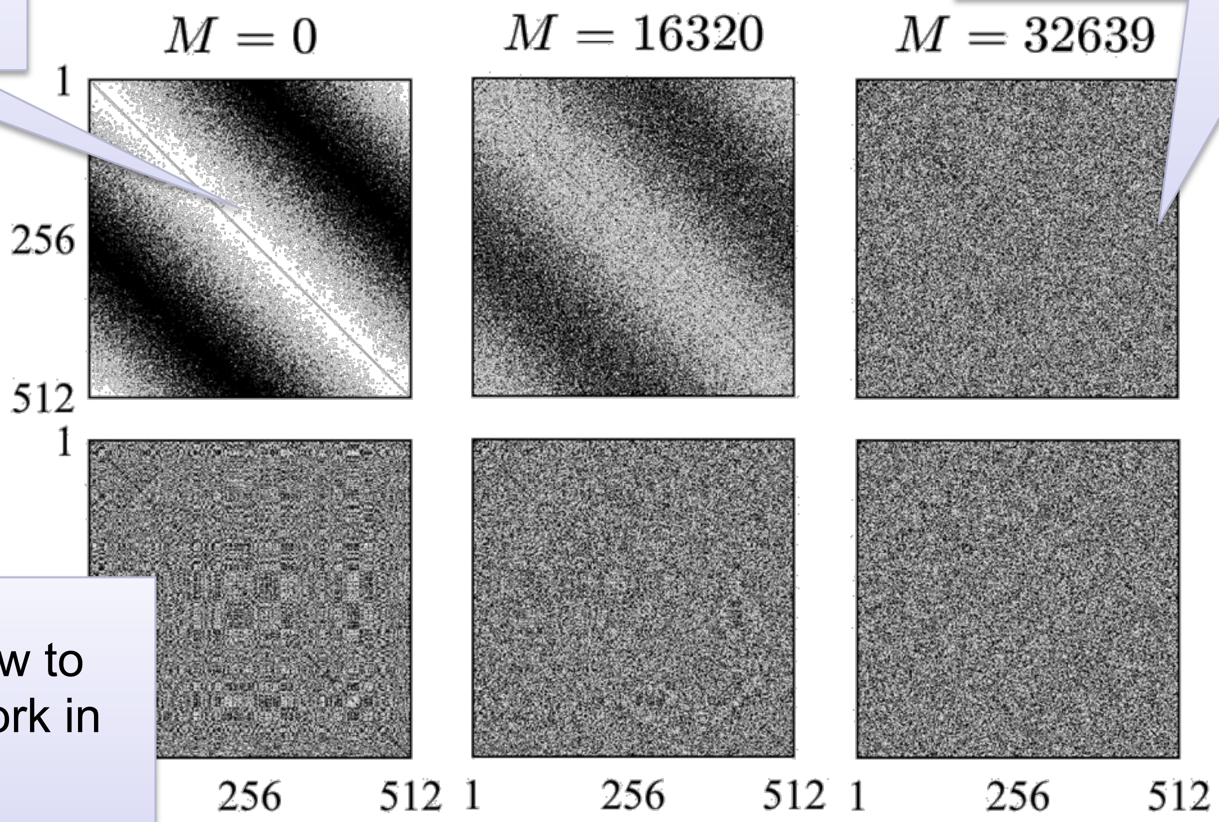
Starting network (structured)

More rewiring →

Target network (ER random)

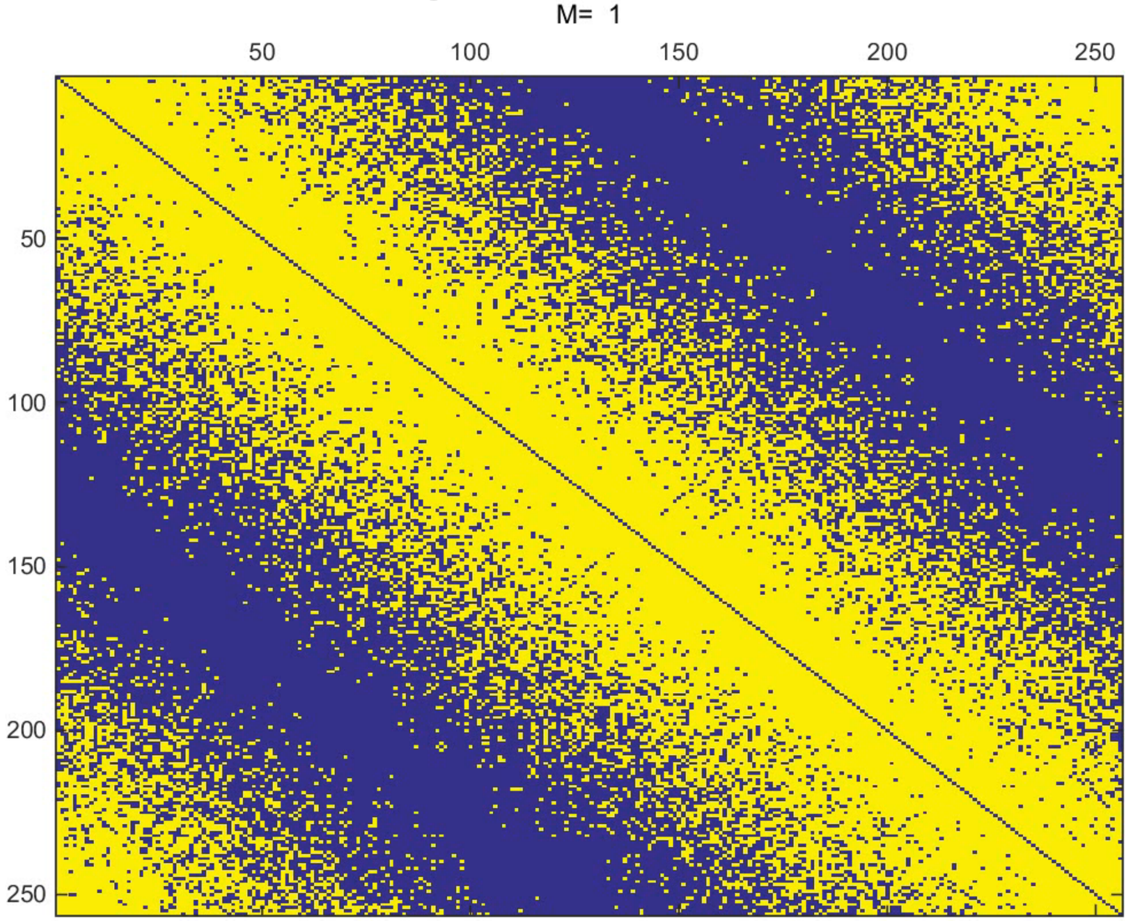
With known structure:

If structure unknown:

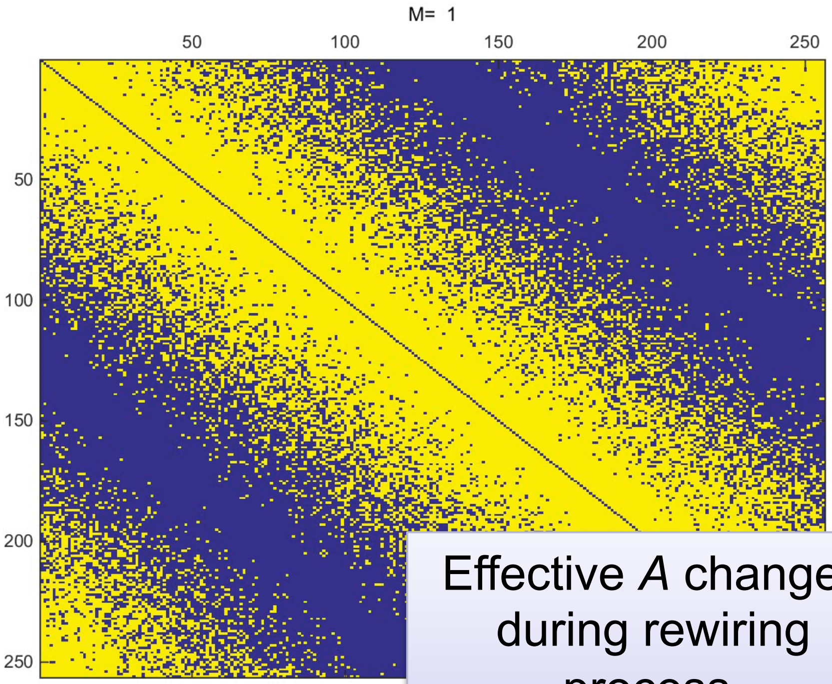


Deeper problem: how to embed a given network in space?

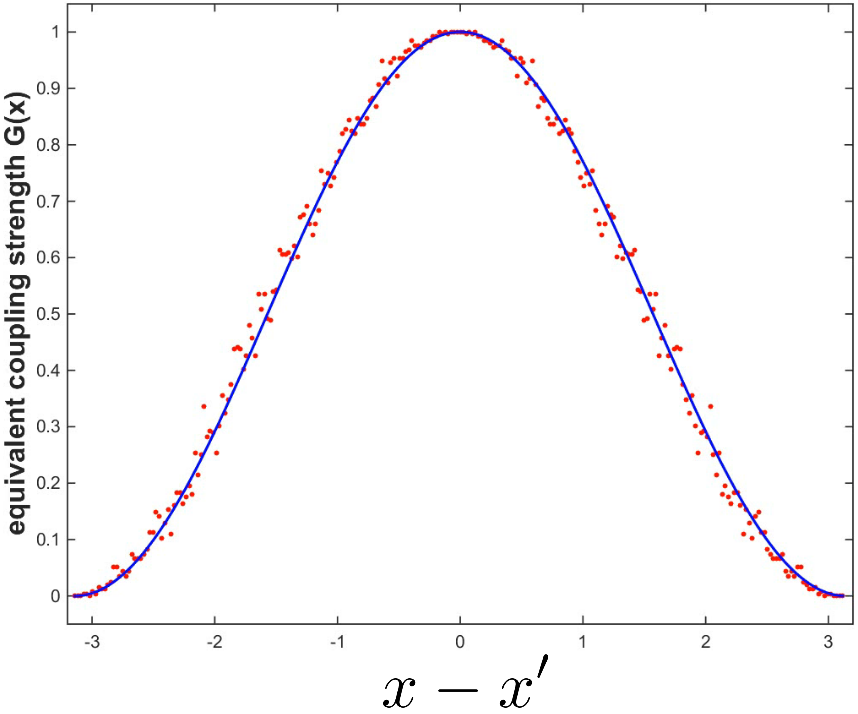
Rewiring: experiment



Rewiring: experiment



$$G(x - x') \propto 1 + A\cos(x - x')$$

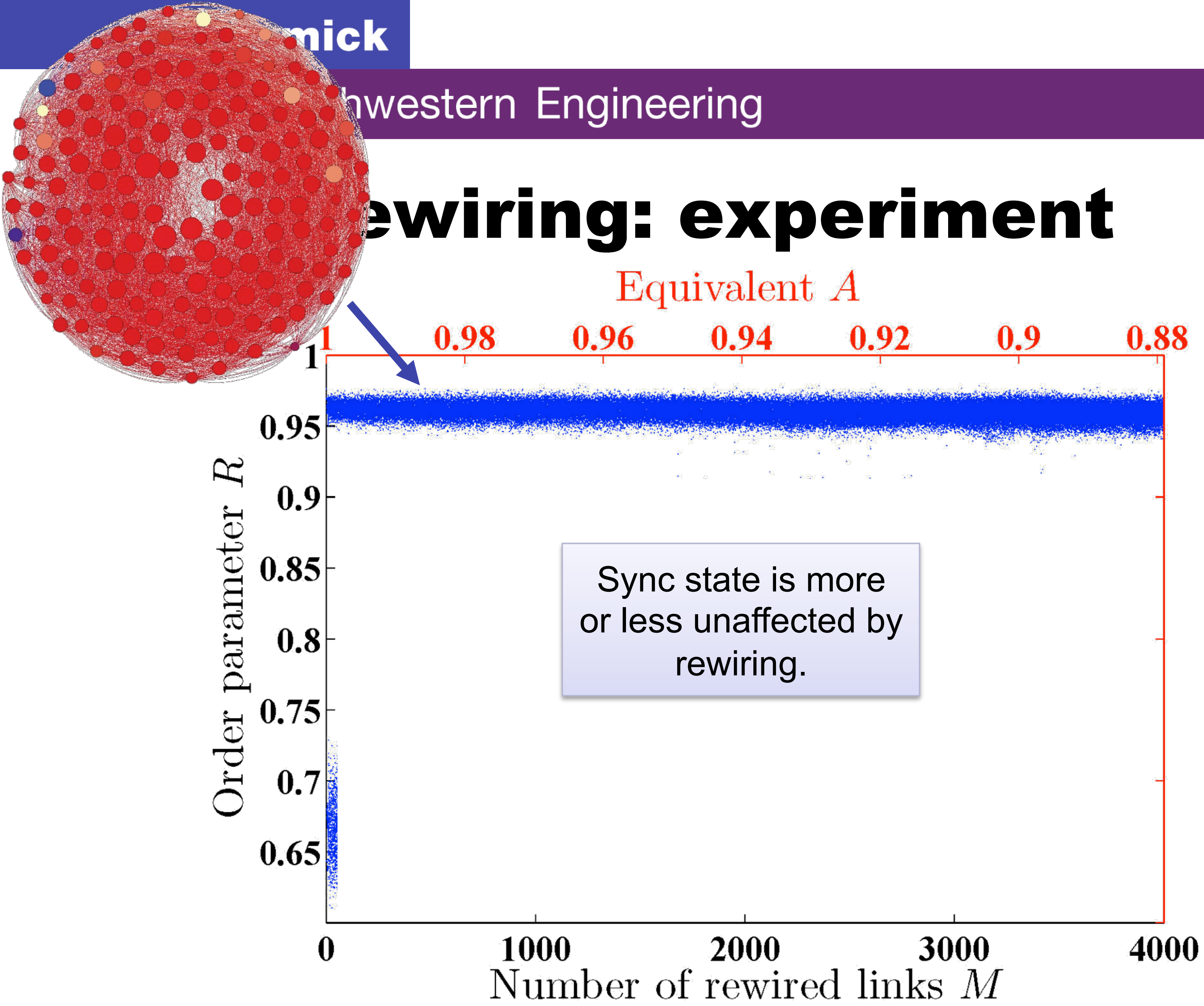


Effective *A* changes during rewiring process.

Rewiring: experiment

- Run Kuramoto-Sakaguchi model on network
- Rewire and equilibrate after each step
- We see:

ewiring: experiment

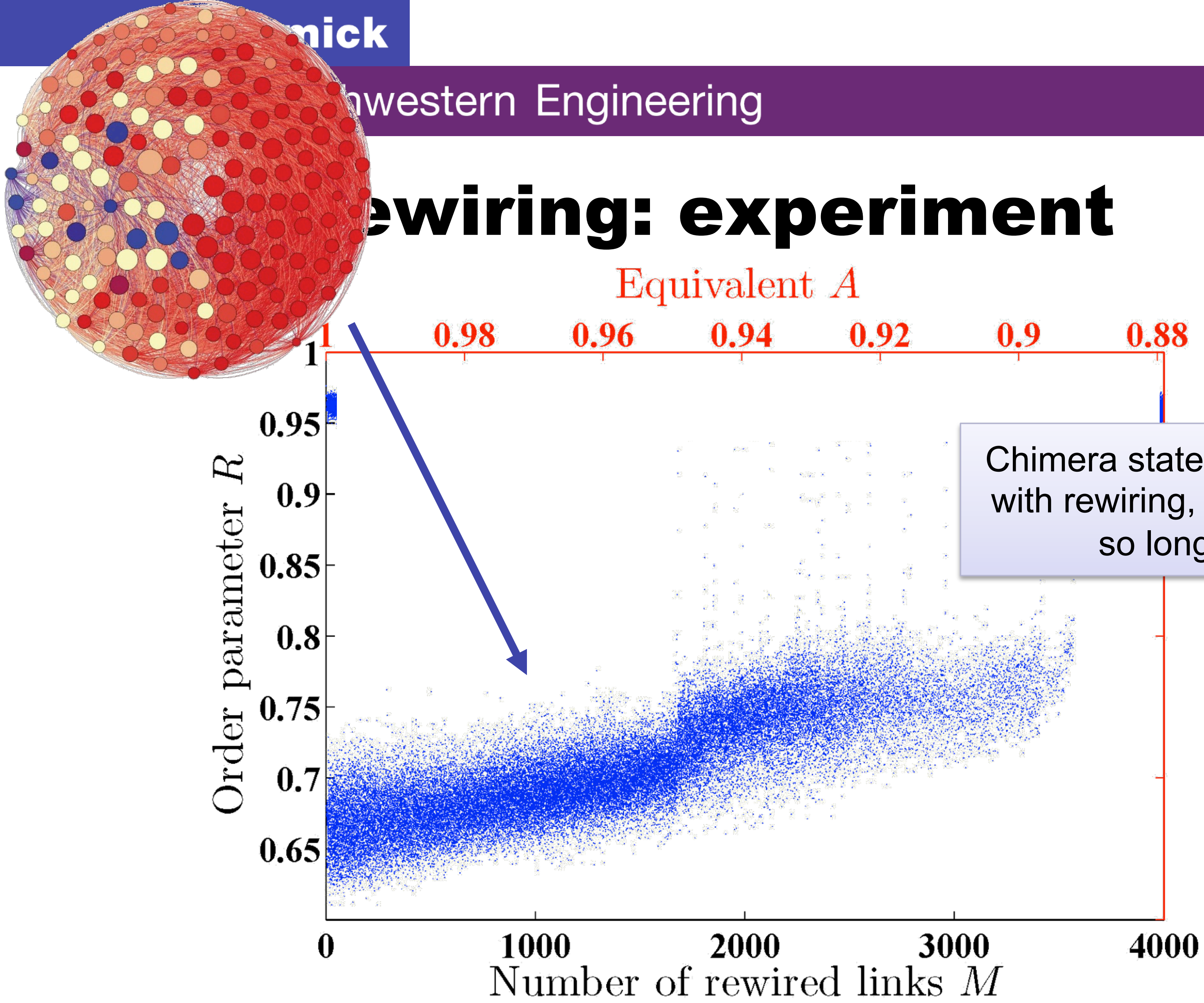


Sync state is more or less unaffected by rewiring.

Rewiring: experiment

- What if we had started from the “chimera” state instead?

ewiring: experiment

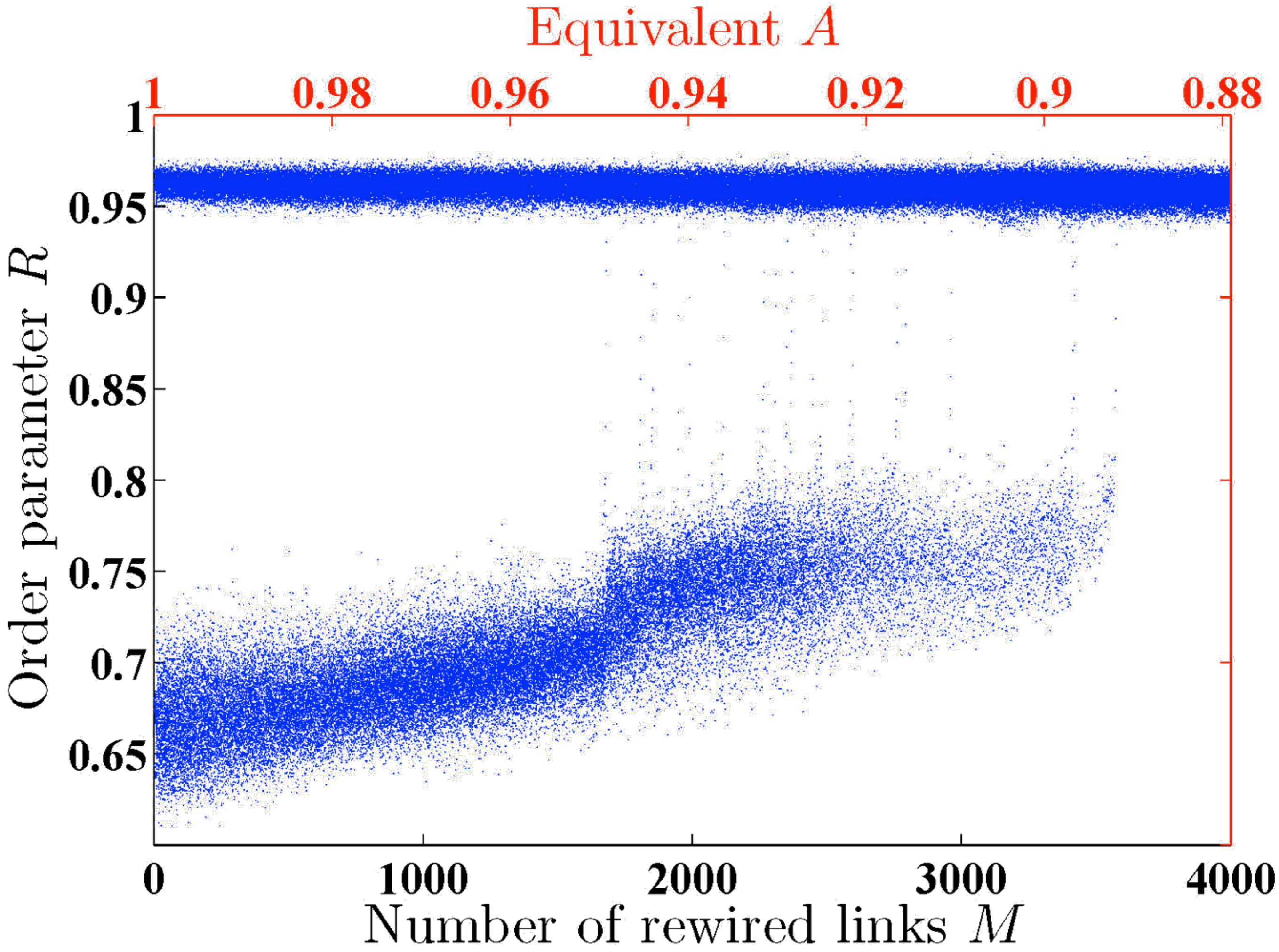


Chimera state persists with rewiring, but only so long.

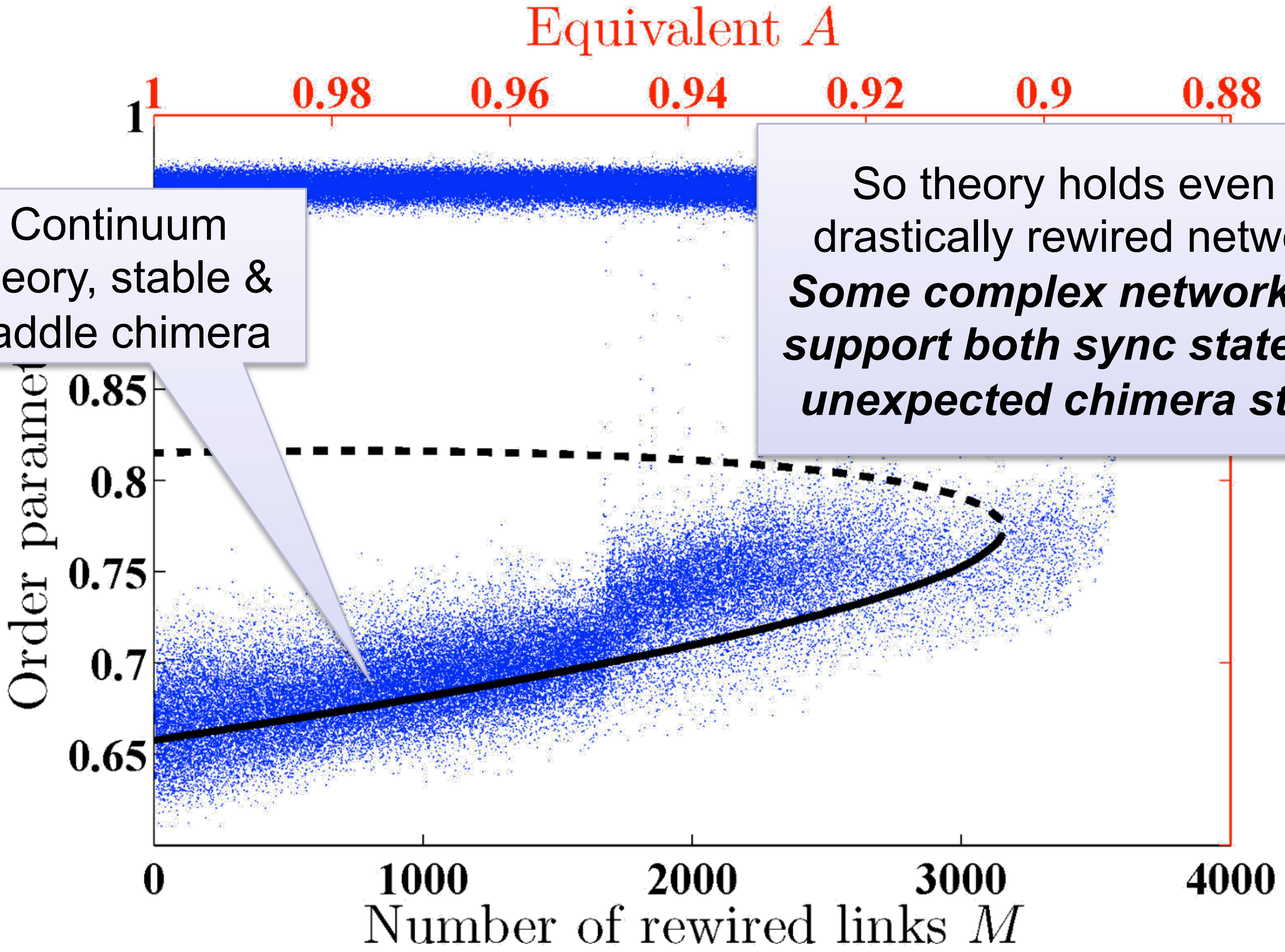
Rewiring: experiment

- Can continuum theory for “true” chimera states help with understanding?

Rewiring: experiment



Rewiring: experiment/theory



Continuum theory, stable & saddle chimera

So theory holds even for drastically rewired networks: ***Some complex networks can support both sync states and unexpected chimera states!***

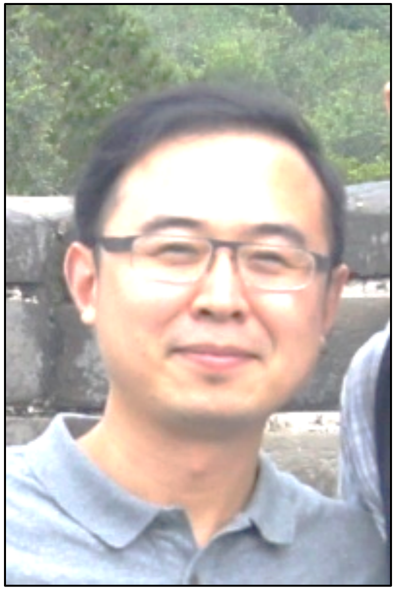
Some questions

- Why sensitivity to density of links?
  - More stable on larger, denser networks
- Is there a way to know “correct” embedding in space?
  - (would help predict/understand these states)
- Small network limit?
- Spontaneous alternation for small *N*?
  - *Maybe extremely common!*

Conclusions/Questions

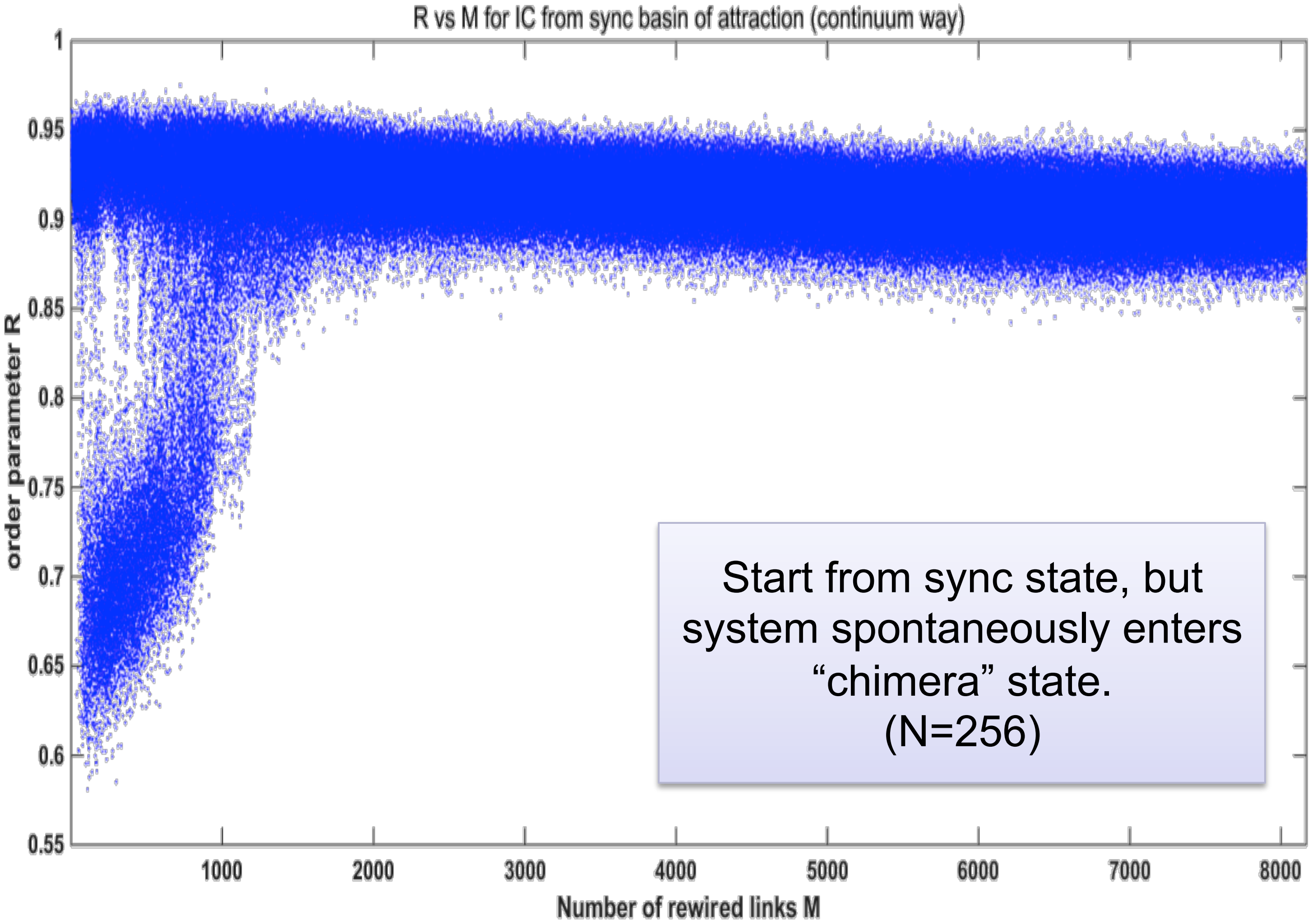
- Chimera state analogs do exist on networks
    - Likely even networks very close to random!
- Patterned state might be unexpected if network were not created this way
    - Could be dangerous, e.g. power grid desync
- They *do not* reflect exact network symmetries
- How to reconcile this with exact symmetry methods?
    - (see, e.g., Pecora et al., Nature Comm., 2014)

Thanks!



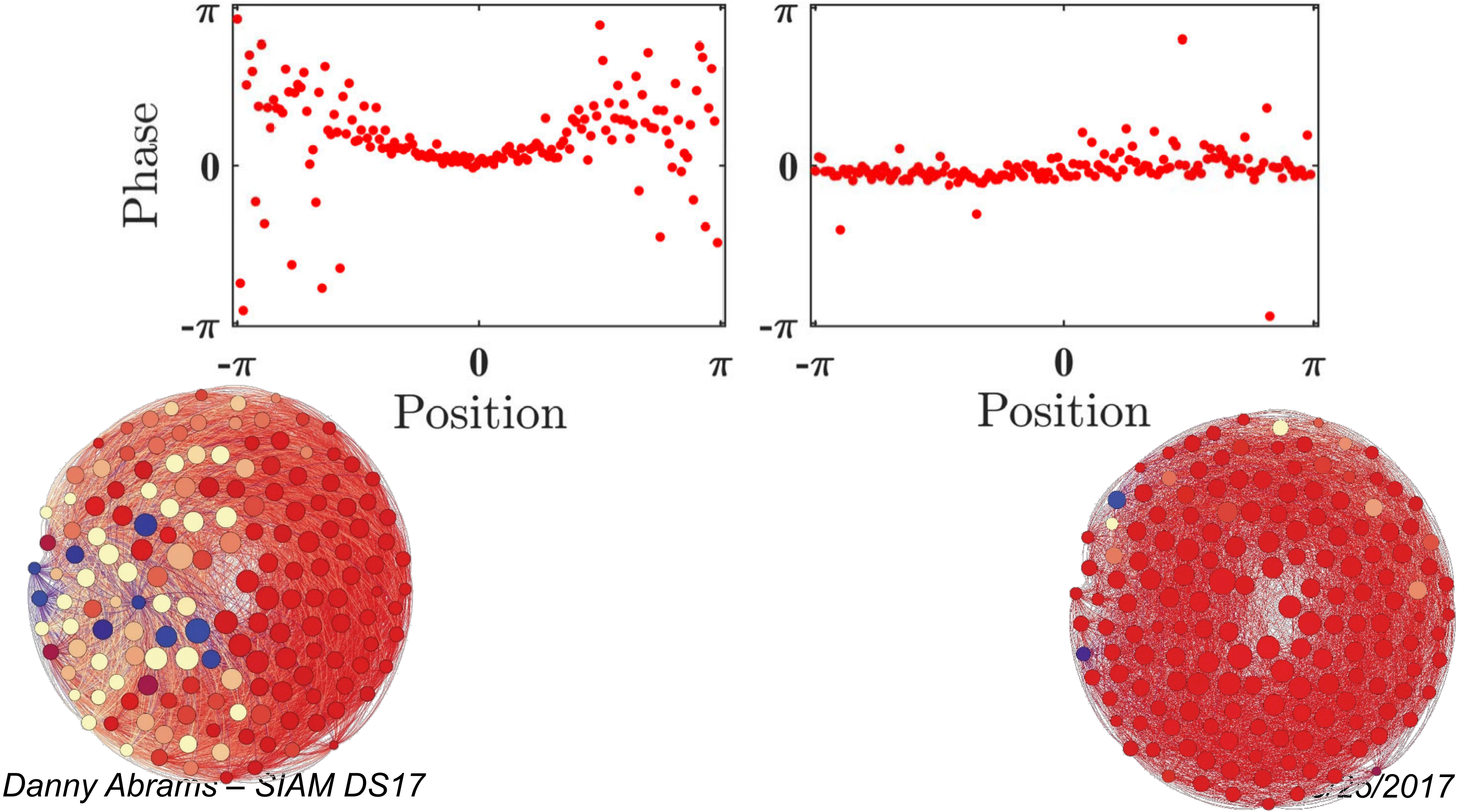
- Joint work with Xin Jiang
- Useful conversations with many others (special thanks to Oleh Omel’chenko and Matthias Wolfrum)
- For more, see Jiang & Abrams, PRE 93, 052202 (2016).

Extra: spontaneous switching



Start from sync state, but system spontaneously enters “chimera” state.
(N=256)

Extra: snapshots (N=160)



Extra: open questions

- Many big open questions remain:
  - Intuition (resonance? Connection between sync and inc?)
  - Existence in alternate geometries/topologies
  - Stability
  - Theory for non-constant amplitude oscillators
  - Theory for inertial oscillators
  - Theory for iterated maps
  - Noise
  - Connection to classical Kuramoto model? (co-dimension 3 bifurcation in (σ, α, A)?)

Extra: thoughts

- Experiments show chimera states are real
- Many possible applications
  - Biological – e.g., ventricular fibrillation in heart, suprachiasmatic nucleus in brain;
  - Chemical – e.g., electrodissolution, BZ reaction;
  - Physical – e.g., power grid, metronomes;
- Much more work to be done!