The Bayesian Formulation and Well-Posedness of Fractional Elliptic Inverse Problems

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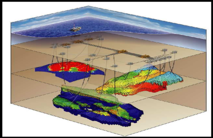
BROWN

Background

Elliptic Inverse Problem. Dashti, Stuart



MATHEMATICAL MODEL

$$-\nabla \cdot (e^{u(x)} \nabla p(x)) = f \quad \text{in } D,$$
$$p = h \quad \text{on } \partial D.$$

DATA

$$y_j = p(x_j) + \eta_j, \quad x_j \in D.$$

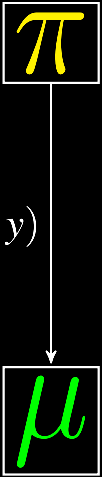
Uncertainty in input $u \in \mathcal{H}$.
Treat it as random.

Inverse Problems

u —(F)PDE— p — $y = \mathcal{O}(p) + \eta$

$u \sim \pi$ —BIP→ $u|y \sim \mu$

Goal



$$\frac{d\mu}{d\pi}(u) \propto g(u; y)$$

Bayesian Inverse Problem

g involves an FPDE.
Formulation, well-posedness?

Setting

Basic Framework

$$L_A := -\nabla \cdot \big(A(x)\nabla_x\big),\, s \in (0,1).$$

$$\begin{cases} L_A^s p &= f, \quad \text{in } \ D, \\ \partial_A p &= 0, \quad \text{on } \ \partial D. \end{cases}$$

Set-up: Uncertainty on A and s. f known.

Aim: Learn $A(x)$ and s from partial and noisy observations of p

$$y = \mathcal{O}(p) + \eta \in \mathbb{R}^m, \quad \eta \sim N(0, \Gamma).$$

Why Fractional Order Bayesian Inverse Problems?

- Order uncertainty e.g. in viscoelastic materials (Zayernouri).
- Integer order PDE $\leftrightarrow$ fractional order in boundary.

Bayesian Formulation

Forward Map $u := (s, A) \mapsto p$ $\qquad \begin{cases} L_A^s p &= f, \quad \text{in } D, \\ \partial_A p &= 0, \quad \text{on } \partial D, \end{cases}$

Data $\qquad y = \mathcal{O}(p) + \eta =: \mathcal{G}\big(u\big) + \eta.$

NON-SMOOTH CASE

- $f \in L^2_{\text{avg}}(D)$. (Right-hand side)
- $X := (0,1) \times L^\infty(D)$. (Prior space)

SMOOTH CASE

- $f \in L^p_{\text{avg}}(D)$ for some $p \geq 2$. (Right-hand side)
- $X := (\frac{d}{2p}, 1] \cap C^1(\overline{D})$. (Prior space)

Results

Main results

Theorem 1

If $\pi(X) = 1$, *the BIP is well formulated:*

$$\frac{d\mu^y}{d\pi}(u) = \frac{1}{Z}\exp\Big(-\frac{1}{2}|y - \mathcal{G}(u)|_\Gamma^2\Big).$$

Theorem 2

There is $C = C(r)$ *such that, for all* $y_1, y_2 \in \mathbb{R}^m$ *with* $|y_1|, |y_2| \leq r$,

$$D_{\text{Hell}}(\mu^{y_1}, \mu^{y_2}) \leq C|y_1 - y_2|.$$

Two Comments

Fractional Laplacian

Orthonormal basis of eigenfunctions of L_A, $\psi_k \in H^1_{\text{avg}}(D)$,
Eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ For $p(x) = \sum_{k=1}^{\infty} p_k \psi_k(x)$,

$$L_A^s p(x) := \sum_{k=1}^{\infty} \lambda_k^s p_k \psi_k(x).$$

Domain of L_A^s: functions $p \in L^2_{\text{avg}}(D)$ with

$$\|p\|_{H^s}^2 := \sum_{k=1}^{\infty} \lambda_k^s p_k^2 < \infty.$$

Extension Problem. Caffarelli, Silvestre

Let $a := 1 - 2s$,

$$B(x) := \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}.$$

We denote by $P_{s,A} : D \times (0, \infty) \to \mathbb{R}$ the solution to

$$\begin{cases} \nabla \cdot (\xi^a B \nabla P_{s,A}) = 0, & \text{in } D \times (0, \infty), \\ \partial_A P_{s,A} = 0, & \text{on } \partial D \times [0, \infty), \\ P_{s,A}(x, 0) = p_{s,A}(x), & \text{on } D. \end{cases}$$

Weak form: for $\phi \in H^1(D \times (0, \infty), \xi^a d\xi dx)$

$$\int_D \int_0^\infty \langle B \nabla P_{s,A}, \nabla \phi \rangle \xi^a d\xi dx = c_s \int_D L_A^s p_{s,A} \phi(x, 0) dx.$$

Literature and Current Work

Literature

- BAYESIAN INVERSE PROBLEMS

  Kaipio-Somersalo 2006, Marzouk-Xiu 2009, Stuart 2010

- REGULARITY THEORY OF FPDEs

  Caffarelli-Silvestre 2007, Stinga 2010, Caffarelli-Stinga 2016

- NUMERICAL SOLUTION OF FPDEs
  Nochetto-Otarola-Salgado 2015, Zayernouri-Karniadakis 2015

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Brown: Ainsworth, Darbon, Guzmán, Karniadakis,...

Current Directions

- Numerical aspects: work in progress with C. Glusa (Brown).
  - Numerical methods for FPDEs via extension problem.
  - Scalable MCMC algorithms: extended pCN.
- Other Laplacians.
- Variable order forward maps, priors, and noise models.

N. Garcia Trillos, D. Sanz-Alonso, *The Bayesian formulation and well-posedness of fractional elliptic inverse problems.* Inverse Problems, 2017.