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SRB Measures for Time-Dependent and Random Attractors

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SRB measures for deterministic, autonomous systems

(no randomness, no drive)

M = finite dim manifold, f = map or f_t = flow

Assume * dissipative, orbits tend to attractors
* chaotic e.g. positive Lyapunov exponents

Conceptually, an SRB measure μ is a prob distribution on M s.t.

(1) (time avg = space avg)

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi \, d\mu \quad \text{Leb-a.e. } x$$

for all cts observables

(2) (characteristic W^u geometry) μ has conditional densities on unstable manifolds

(3) (entropy formula)

$$h_\mu(f) = \int \sum_{\lambda_i > 0} \lambda_i m_i \, d\mu$$

where λ_i = Lyapunov exponents, m_i = multiplicities

(In general, $h \le \int \sum \lambda_i^+ m_i \, d\mu$)

Conceptual properties of SRB measures:
(1) time avg = space avg, (2) characteristic W^u geometry,
(3) entropy formula

Review article : Young, J Phys A 2013

Rigorous results :

Axiom A attractors : (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3)

(Sinai, Ruelle, Bowen 1970s)

General diffeomorphisms and arbitrary inv measures:

(2) $\Longrightarrow$ (1) if no zero Lyap exp and ergodic (Pugh-Shub 1990)

(2) $\Longleftrightarrow$ (3) finite dim diffeo (Ledrappier-Strelcyn, L, L-Young 1980-88)

inf dim - dissipative PDEs (Li-Shu, Blumenthal-Young 2014,15)

Interpretation of (3) : For arbitrary inv meas μ,

(a) $h = \sum \lambda_i^+ \delta_i m_i$, $0 \leq \delta_i \leq 1$ (Ledr-Young 1985)

where $\delta_i m_i$ = dim of μ in directions E_i

(b) Under certain assumptions, $\sum \lambda_i^+ m_i - h$ ~ *escape rate*

Gap in entropy formula ~ measure of dissipation

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Gap in entropy formula/fractal dim ~ degree of dissipation

Generalizations to random / nonautonomous frameworks

(A) Periodic forcing

$$\frac{dx}{dt} = X(x) + p(x,t) \ , \quad p(x, t+T) = p(x,t)$$

Many *rigorous* examples of strange attractors w/ SRB measures are related to *shear-induced chaos*

Idea : unforced system has nonchaotic dynamics; forcing magnifies underlying shear to produce ``folds" --- and strange attractors.

Examples : periodic kicking of limit cycles (Wang-Young 2003)

In ODE as well as PDEs, e.g. periodically forced *Brusselator* (autocatalytic chemical reaction) near Hopf bifurcation

$$\begin{cases} u_t = d_1 \Delta u + a - (b+1)u + u^2 v \\ v_t = d_2 \Delta v + bu - u^2 v \end{cases} \quad + \text{ periodic forcing}$$

(Lu-Wang-Young 2013)

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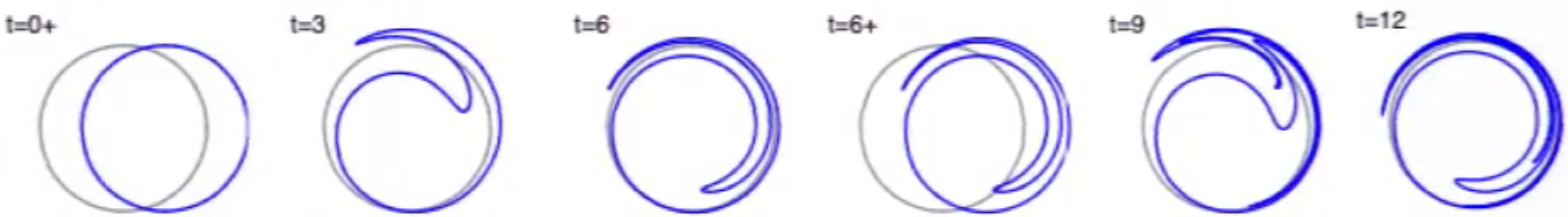

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(B) Random dynamical systems

$$dx_t = a(x_t)dt + \sum_{i=1}^{n} b_i(x_t) \circ dW_t^i$$ W_t^i = Brownian motion

Solution has representation as stochastic flow of diffeomorphisms

$\cdots f_{\omega_3} \circ f_{\omega_2} \circ f_{\omega_1},$ $i.i.d.$ or $\cdots f_{\omega_1} \circ f_{\omega_0} \circ f_{\omega_{-1}} \cdots$

(averaged) stationary measure $\mu = \int (f_\omega)_* \mu \, \mathbb{P}(d\omega)$

Distributions at time 0 given history are given by

$$\mu_{\omega^-} := \mu | \{\omega_i, i \le 0\} = \lim_{n\to\infty} (f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n+1}} \circ f_{\omega_{-n}})_* \mu$$

THEOREM (a) If $\lambda_{\max} < 0$, μ_{ω^-} are random sinks (Le Jan 1987)

(b) If $\lambda_{\max} > 0$, μ_{ω^-} are *random SRB measures* in terms of

(i) characteristic geometry and (ii) entropy formula

(Ledrappier-Young 1988)

Rmk : *randomness leads to simpler picture*

For deterministic maps, SRB measures hard to prove w/out invariant cones

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Recall dim formula for single maps : $h_\mu(f) = \sum (\delta_i m_i)\lambda_i^+, \quad 0 \le \delta_i \le 1$

THEOREM. With sufficient randomness, dim of μ_ω – satisfies

$$(\delta_1, \cdots, \delta_r) = (1, 1, \cdots, 1, *, 0, \cdots, 0)$$

(Ledrappier-Young 1988)

Interpretation : effective dim ~ # positive Lyap exp

Application to biological & engineered systems : *reliability*

$I_\omega(t)$ → fluctuating input → (large) dynam sys → $R(t)$ → response

Say a system is *reliable* if same $I_\omega(t)$ elicits same $R(t)$ following transient

Mathematically : reliable iff random sinks iff $\lambda_{\max} < 0$

unreliable iff random SRB measure iff $\lambda_{\max} > 0$

Example: coupled oscillators at $t = 50, 500, 2000$

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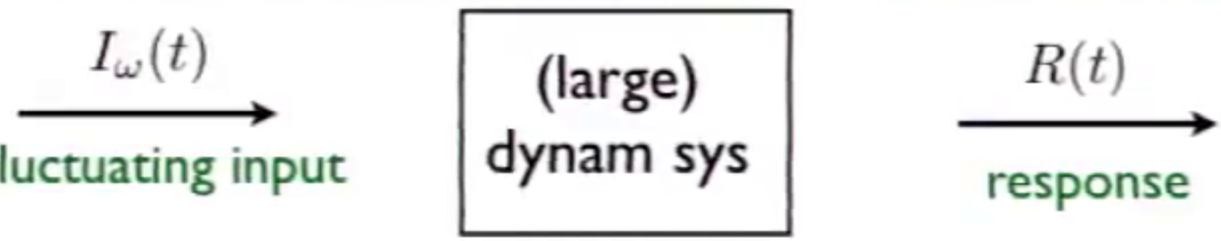

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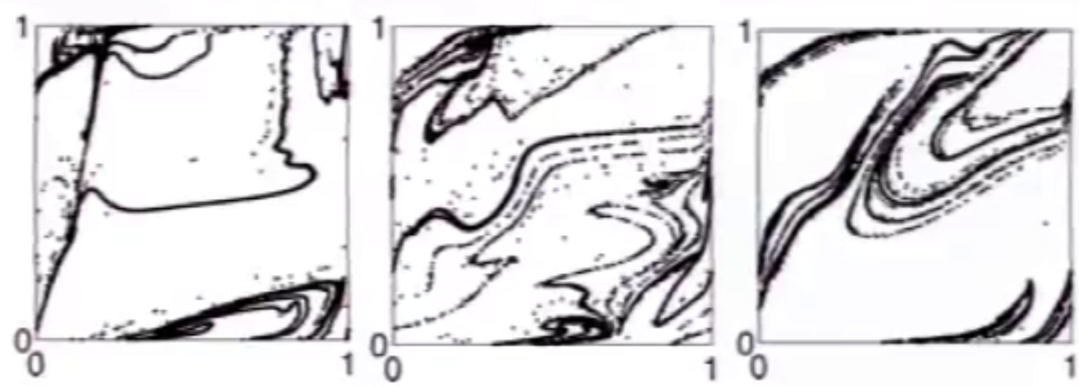

(C) Dynamical systems driven by other dyn sys or stoch processes : math framework unifying (A) and (B)

$\Omega =$ Diff(M), Z = map or flow or stoch process on Ω

Skew product representation :

e.g. Z = $(f_1, f_2, f_3, \cdots)$
Diff(M)-valued stationary process

Equivalently $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ inv prob ν

Consider

$$F^+ : ((f_n), x) \mapsto (\sigma(f_n), f_0(x))$$

Taking inverse limit : get $F : \Omega^{\mathbb{Z}} \times M \to \Omega^{\mathbb{Z}} \times M$
with invariant measure projecting to stationary measure of Z.

Idea of SRB measures on M-fibers *describing state of system at time 0 given history* can make sense

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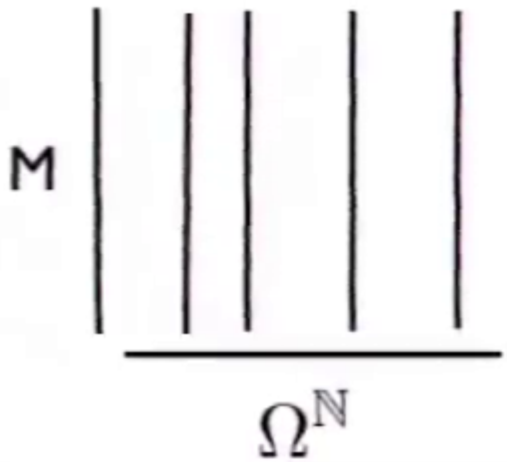

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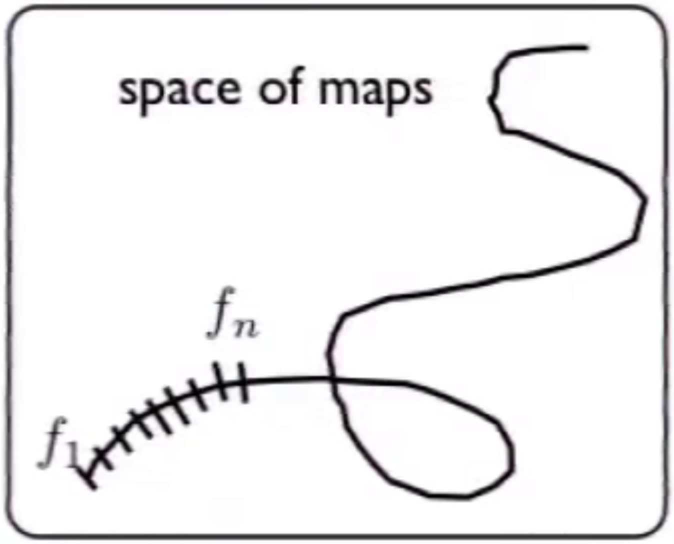
(D) Time-dependent dynamical systems *(no stationarity)*



Consider $\cdots \circ f_3 \circ f_2 \circ f_1$

along arbitrary path in space of maps

Conceptually, expect :

if f_i changes slowly enough, then

$(f_n \circ \cdots \circ f_1)_*$ (init distr) $\longrightarrow \mu_n$

where $\mu_n \approx$ SRB measure of f_n

I propose:

"adiabatic dynamics" = systems with slowly drifting parameters

and *time-dependent SRB measures*

Illustrating example: billiards with slowly moving scatterers

THEOREM. For μ, ν in large class of suitable initial distr,

$|(f_n \circ \cdots \circ f_1)_*\mu - (f_n \circ \cdots \circ f_1)_*\nu| \to 0$

exponentially fast as $n \to \infty$.

(Stenlund-Young-Zhang 2013)

Note *time-dep "limits"*

(a)

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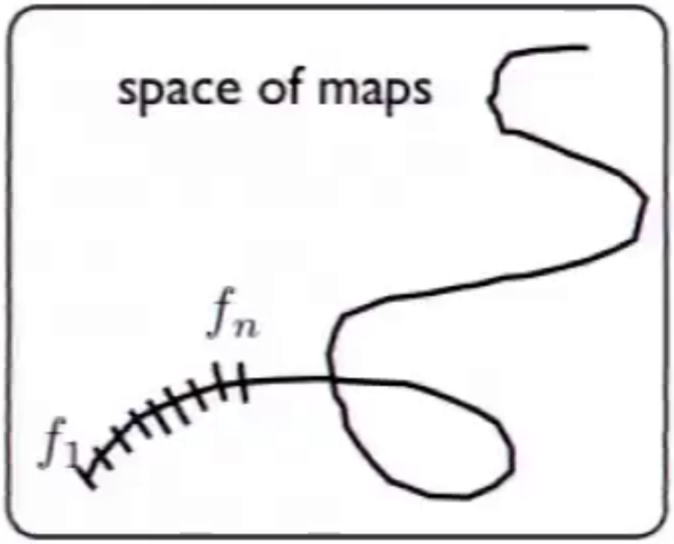

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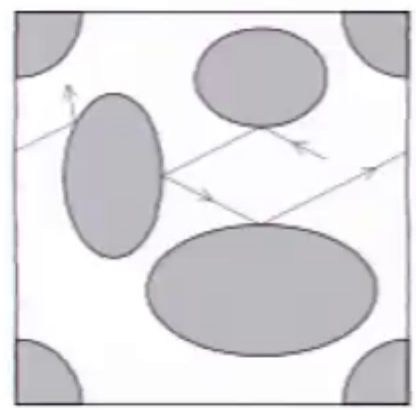
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time-dep "limits" w/ SRB geometry

(E) Leaky dynamical systems (phase space with holes)

open set $H \subset M$ ``hole" : *orbit "lost forever" once it enters hole*

Equiv : *not fully invariant domains, i.e.* $U \subset M, \; f(U) \not\subset U$

Questions: *escape rates*, *surviving distributions*, *hole dependence etc.*

e.g. μ_0 = reasonable init distr, $\mu_n = f_*(\mu_{n-1})|_{M \setminus H}$ normalized

$\mu_n \to \mu_\infty$ as $n \to \infty$ (if limit exists)

Illustrating example: periodic Lorentz gas with holes

(Demers-Wright-Young 2010, 2012)

THEOREM (1) escape rate $\lambda > 0$ well defined

(2) μ_∞ well defined, has SRB geometry, and satisfies

$$f_*(\mu_\infty)|_{M \setminus H} = e^{-\lambda} \mu_\infty$$

conditionally invariant SRB measures

(3) assoc with μ_∞ is an inv meas μ characterized by

$$h_\mu(f) - \int \sum \lambda_i^+ m_i \, d\mu = -\lambda$$

$\longleftarrow$ cf entropy formula

(4) as hole size goes to 0, μ_∞ tends to SRB measure

Above extended to nonuniformly hyperbolic systems admitting Markov towers in the sense of (Young 1998)

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Conclusions

- *SRB measures* are known to be *the* natural physical measures in chaotic dissipative autonomous dynamical systems

- Many real-world systems have *stochastic* components; they are often *driven*, *time-dependent*, *leaky*, etc.

Main message of this talk :

- *Ideas surrounding SRB measures can be adapted to these more realistic settings, both conceptually and rigorously, and they are equally relevant in these settings.*

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