Wavelet Frame Based Piecewise Smooth Image Model and It's Relation to Mumford-Shah Functional

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Outline

- Review of sparsity based image restoration models
- Wavelet frames based image restoration and relation to variational and PDE models
- Piecewise smooth image model by wavelet frames
- Asymptotic analysis and relation to Mumford-Shah functional
- Numerical experiments
- Concluding remarks

Jian-Feng Cai, B. Dong and Zuowei Shen, *Image restorations: a wavelet frame based model for piecewise smooth functions and beyond*, **Applied and Computational Harmonic Analysis**, 2016.

Image Restoration Model

- Image Restoration Problems

$$f = Au + \eta$$

  - Denoising, when A is identity operator
  - Deblurring, when A is some blurring operator
  - Inpainting, when A is some restriction operator
  - CT/MR Imaging, when A is partial Radon/Fourier transform
- Challenges: large-scale & ill-posed

How to Obtain a Good Recovery

- Variational and Optimization Models

$$\min_u \lambda R(u) + \|Au - f\|^2$$

  - Total variation (TV) and generalizations: $R(u) = \|\nabla u\|_1$ or $\|\boldsymbol{D}u\|_1$
  - Wavelet frame based: $R(u) = \|Wu\|_1$ or $\|Wu\|_0$
  - 1-norm v.s. 0-norm:

    [Zhang, Dong and Lu, Math Comput. 2013] & [Dong and Zhang, JSC, 2013]
  - Others: total generalized variation, low rank, NLM, BM3D, dictionary learning, etc.
- PDEs and Iterative Algorithms
  - Perona-Malik equation, shock-filtering (Rudin & Osher), etc

$$u_t = \sum_{\ell=1}^{L} \frac{\partial^{\boldsymbol{\alpha}_\ell}}{\partial x^{\boldsymbol{\alpha}_\ell}} \Phi_\ell(\boldsymbol{D}u, u) - A^*(Au - f), \qquad \text{with } \boldsymbol{D} = \left(\frac{\partial^{\boldsymbol{\beta}_1}}{\partial x^{\boldsymbol{\beta}_1}}, \ldots, \frac{\partial^{\boldsymbol{\beta}_L}}{\partial x^{\boldsymbol{\beta}_L}}\right)$$

  - Iterative shrinkage algorithm

$$\boldsymbol{u}^k = \widetilde{\boldsymbol{W}}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) - \boldsymbol{A}^\top(\boldsymbol{A}\boldsymbol{u}^{k-1} - \boldsymbol{f}), \quad k = 1, 2, \cdots$$

- What do they have in common?

Shrinkage in **sparse** domain under **transformation**!

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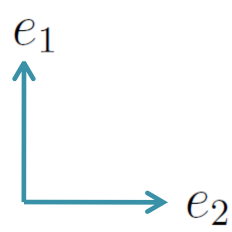
"Dong and Shen, *Image restoration: a data-driven perspective,* Proceedings of the International Congress of Industrial and Applied Mathematics (ICIAM), 2015"

Tight Frames in $\mathbb{R}^2$

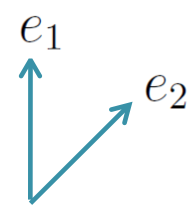
- Orthonormal basis

$$e_1 = (0,1)^\top, e_2 = (1,0)^\top$$



- Riesz basis

$$e_1 = (0,1)^\top, e_2 = \frac{1}{\sqrt{2}}(1,1)^\top$$



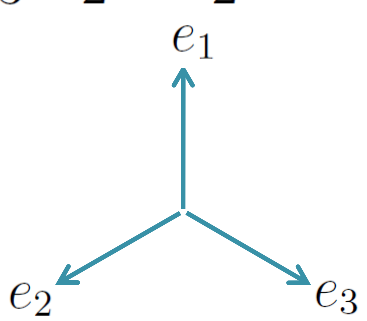
- Tight frame: Mercedes-Benz frame

$$e_1 = \sqrt{\frac{2}{3}}(0,1)^\top, e_2 = \sqrt{\frac{2}{3}}(-\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top, e_3 = \sqrt{\frac{2}{3}}(\frac{\sqrt{3}}{2}, -\frac{1}{2})^\top$$

- Expansions:

Unique $v = \alpha_1 e_1 + \alpha_2 e_2, \quad \forall v \in \mathbb{R}^2$

Not unique $v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \quad \forall v \in \mathbb{R}^2$



Wavelet Frames

- General frame system: $X = \{g_j : j \in \mathbb{Z}\} \subset L_2(\mathbb{R}^d)$
  - They are redundant systems satisfying

$$A\|f\|^2_{L_2(\mathbb{R}^d)} \le \sum_{j\in\mathbb{Z}} |\langle f, g_j\rangle|^2 \le B\|f\|^2_{L_2(\mathbb{R}^d)}, \quad \forall f \in L_2(\mathbb{R}^d)$$

  - and we have

$$f = \sum_{j\in\mathbb{Z}} \langle f, g_j\rangle \tilde{g}_j \quad \forall f \in L_2(\mathbb{R}^d)$$

- Wavelet frames: given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L_2(\mathbb{R}^d)$

$$X(\Psi) = \{\psi_{\ell,n,\boldsymbol{k}} : \ 1 \le \ell \le L; n \in \mathbb{Z}, \boldsymbol{k} \in \mathbb{Z}^d\}$$

where $\psi_{\ell,n,\boldsymbol{k}} := \begin{cases} 2^{\frac{nd}{2}}\psi_\ell(2^n \cdot - \boldsymbol{k}), & n \ge 0; \\ 2^{nd}\psi_\ell(2^n \cdot - 2^{n-J}\boldsymbol{k}), & n < 0. \end{cases}$ **Quasi-Affine system**

- A wavelet frame is called a tight wavelet frame if *A*=*B*=*1*

MRA-Based Tight Wavelet Frames

- Refinable and wavelet functions

$$\phi = 2^d \sum \boldsymbol{a}_0[\boldsymbol{k}]\phi(2\cdot - \boldsymbol{k}) \qquad \psi_\ell = 2^d \sum \boldsymbol{a}_\ell[\boldsymbol{k}]\phi(2\cdot - \boldsymbol{k}), \quad \ell = 1, 2, \ldots, q.$$

- Unitary extension principle (UEP) **[Ron and Shen, 1997]**:

$$\sum_{\ell=0}^{q} |\widehat{\boldsymbol{a}}_\ell(\xi)|^2 = 1 \quad \text{and} \quad \sum_{\ell=0}^{q} \widehat{\boldsymbol{a}}_\ell(\xi)\overline{\widehat{\boldsymbol{a}}_\ell(\xi+\nu)} = 0,$$

$$\nu \in \{0,\pi\}^d \setminus \{\mathbf{0}\} \text{ and } \xi \in [-\pi,\pi]^d$$

- Discrete 2D transformation:

$$\boldsymbol{W}\boldsymbol{u} = \{\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u} : 0 \le l \le L-1, 0 \le i_1, i_2 \le r\}$$

$$\boxed{\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u} := \boldsymbol{a}_{l,\boldsymbol{i}}[-\cdot] \circledast \boldsymbol{u},}$$

$$\boldsymbol{a}_{\boldsymbol{i}}[\boldsymbol{k}] := \boldsymbol{a}_{i_1}[k_1]\boldsymbol{a}_{i_2}[k_2], \quad 0 \le i_1, i_2 \le r; \ (k_1, k_2) \in \mathbb{Z}^2.$$

$$\boldsymbol{a}_{l,\boldsymbol{i}} = \tilde{\boldsymbol{a}}_{l,\boldsymbol{i}} \circledast \tilde{\boldsymbol{a}}_{l-1,0} \circledast \ldots \circledast \tilde{\boldsymbol{a}}_{0,0} \quad \text{with} \quad \tilde{\boldsymbol{a}}_{l,\boldsymbol{i}}[\boldsymbol{k}] = \begin{cases} \boldsymbol{a}_{\boldsymbol{i}}[2^{-l}\boldsymbol{k}], & \boldsymbol{k} \in 2^l\mathbb{Z}^2; \\ 0, & \boldsymbol{k} \notin 2^l\mathbb{Z}^2. \end{cases}$$

- Lecture notes: **[Dong and Shen, *MRA-Based Wavelet Frames and Applications*, IAS Lecture Notes Series,2012]**

Connections: Analysis Based Model and Variational Model

- **[Cai, Dong, Osher and Shen, *JAMS*, 2012]**:

$$\|\boldsymbol{\lambda} \cdot Wu\|_1 + \frac{1}{2}\|Au - f\|_2^2 \xrightarrow{\text{Converges}} \lambda\|\boldsymbol{D}(u)\|_1 + \frac{1}{2}\|Au - f\|_{L_2(\Omega)}^2$$

For any differential operator when proper parameter is chosen.

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For any differential operator when proper parameter is chosen.

Theorem. Let the objective functionals of the analysis based model and the variational model be $E_n(u)$ and $E(u)$ respectively, then:

(1) $E_n(u) \to E(u)$ for each $u \in W_1^s(\Omega)$;

(2) $E_n(u_n) \to E(u)$ for every sequence $u_n \to u$. Consequently, E_n Γ-converges to E;

(3) If $u_n^\star$ is an ϵ-optimal solution to E_n, i.e. $E_n(u_n^\star) \le \inf_u E_n(\boldsymbol{u}) + \epsilon$, then

$$\limsup_n E_n(u_n^\star) \le \inf_u E(u) + \epsilon.$$

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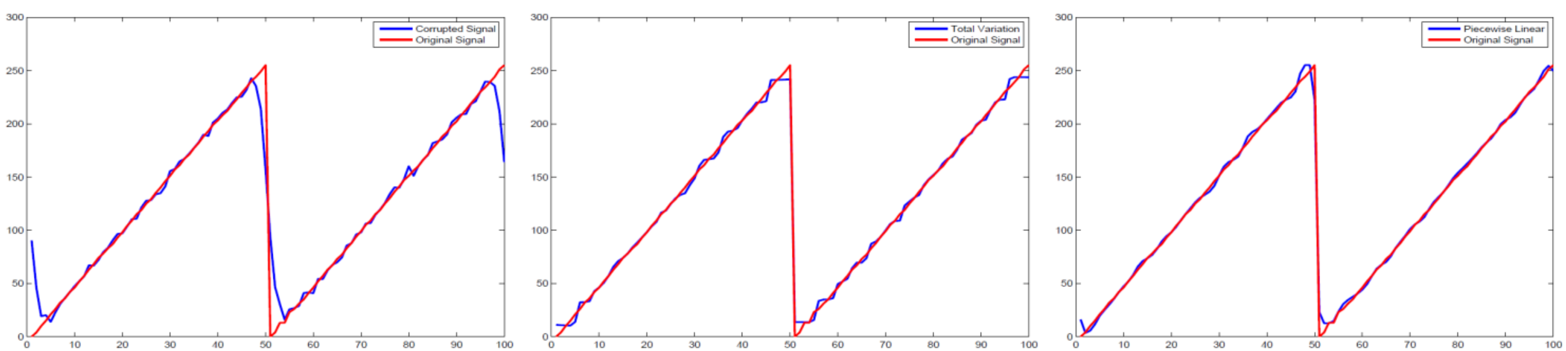
For any differential operator when proper parameter is chosen.

- The connections give us
  - Geometric interpretations of the wavelet frame transform (WFT)
  - WFT provides flexible and good discretization for differential operators



Standard Discretization

Piecewise Linear WFT

Connections: Analysis Based Model and Variational Model

- **[Cai, Dong, Osher and Shen, *JAMS*, 2012]**:

$$\|\boldsymbol{\lambda} W u\|_1 + \frac{1}{2}\|Au - f\|_2^2 \xrightarrow{\text{Converges}} \lambda\|\boldsymbol{D}(u)\|_1 + \frac{1}{2}\|Au - f\|_{L_2(\Omega)}^2$$

For any differential operator when proper parameter is chosen.

- The connections give us
  - Geometric interpretations of the wavelet frame transform (WFT)
  - WFT provides flexible and good discretization for differential operators
  - Different discretizations affect reconstruction results
  - Good regularization should contain differential operators with varied orders (e.g., total generalized variation [Bredies, Kunisch, and Pock, 2010])

- Leads to new applications of wavelet frames:
  - Image segmentation: **[Dong, Chien and Shen, 2010]**
  - Surface reconstruction from point clouds: **[Dong and Shen, 2011]**

Wavelet Shrinkage and Nonlinear PDEs

- **[Dong, Jiang and Shen, preprint, 2015]**

$$\boldsymbol{u}^k = \widetilde{\boldsymbol{W}}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}), \quad k = 1, 2, \cdots$$

$$u_t = \sum_{\ell=1}^{L} \frac{\partial^{\boldsymbol{\alpha}_\ell}}{\partial x^{\boldsymbol{\alpha}_\ell}} \Phi_\ell(\boldsymbol{D}u, u), \quad \text{with } \boldsymbol{D}u = \left(\frac{\partial^{\boldsymbol{\beta}_1}}{\partial x^{\boldsymbol{\beta}_1}}, \ldots, \frac{\partial^{\boldsymbol{\beta}_L}}{\partial x^{\boldsymbol{\beta}_L}}\right)$$

- Theoretical justification available for quasilinear parabolic equations.
- Lead to new PDE models such as:

$$u_{tt} + Cu_t = \sum_{\ell=1}^{L} (-1)^{1+|\boldsymbol{\beta}_\ell|} \frac{\partial^{\boldsymbol{\beta}_\ell}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_\ell}} \left[g_\ell\left(u, \frac{\partial^{\boldsymbol{\beta}_1} u}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_1}}, \cdots, \frac{\partial^{\boldsymbol{\beta}_L} u}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_L}}\right) \frac{\partial^{\boldsymbol{\beta}_\ell}}{\partial \boldsymbol{x}^{\boldsymbol{\beta}_\ell}} u \right] - \kappa A^\top (Au - f)$$

- Lead to new wavelet frame shrinkage algorithms:

$$\boldsymbol{u}^k = (I - \mu \boldsymbol{A}^\top \boldsymbol{A}) \boldsymbol{W}^\top \boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) + \mu \boldsymbol{A}^\top \boldsymbol{f}$$

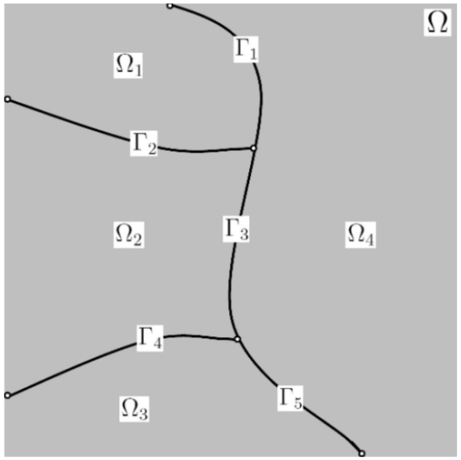
where

$$\boldsymbol{S}_{\boldsymbol{\alpha}^{k-1}}(\boldsymbol{W}\boldsymbol{u}^{k-1}) = \{ S_{\alpha_{l,\ell,\boldsymbol{n}}(\boldsymbol{W}_l \boldsymbol{u}^{k-1})}(\boldsymbol{W}_l \boldsymbol{u}^{k-1}) : \ 0 \le l \le \mathrm{Lev} - 1, 1 \le \ell \le L \}$$

$$S_{\boldsymbol{\alpha}_{\ell,\boldsymbol{n}}(\boldsymbol{d})}(d_{1,\boldsymbol{n}}, d_{2,\boldsymbol{n}}) = d_{\ell,\boldsymbol{n}} \left(1 - \frac{4\tau}{h^2} g\left(\frac{4(d_{1,\boldsymbol{n}})^2 + 4(d_{2,\boldsymbol{n}})^2}{h^2}\right)\right)$$

Modeling Images

- Existing generic image models
  - Functions in BV space (variational and PDE models)
  - Functions in bounded TGV space
  - Functions in Besov spaces (wavelets and wavelet frames)
  - Functions in SBV space (Mumford-Shah)
- Modeling images as **piecewise smooth functions**



Wavelet Frame Based Model for Piecewise Smooth Function

- Wavelet frame based image restoration model

$$\inf_{\boldsymbol{u},\,\boldsymbol{\Gamma}} \|[\boldsymbol{\lambda}\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}^c}\|_2^2 + \|[\gamma\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}}\|_1 + \frac{1}{2}\|\boldsymbol{A}\boldsymbol{u}-\boldsymbol{f}\|_2^2$$

where

$$\|[\boldsymbol{\lambda}\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}^c}\|_2^2 := \sum_{\boldsymbol{k}\in\mathbb{O}^2\setminus\boldsymbol{\Gamma}}\sum_{l=0}^{L-1}\sum_{i\in\mathbb{B}}\lambda_{l,i}[\boldsymbol{k}]\left|(\boldsymbol{W}_{l,i}\boldsymbol{u})[\boldsymbol{k}]\right|^2$$

$$\|[\gamma\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}}\|_1 := \sum_{\boldsymbol{k}\in\boldsymbol{\Gamma}}\left[\sum_{l=0}^{L-1}\left(\sum_{i\in\mathbb{B}}\gamma_{l,i}[\boldsymbol{k}]\left|(\boldsymbol{W}_{l,i}\boldsymbol{u})[\boldsymbol{k}]\right|^2\right)^{\frac{1}{2}}\right]$$

  - When $\boldsymbol{\Gamma}=\emptyset$, the model reduces to Tikhonov regularization model (over-smoothing)
  - When $\boldsymbol{\Gamma}^c=\emptyset$, the model reduces to the analysis based model (introducing unwanted singularities)
  - The term $\|[\boldsymbol{\lambda}\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}^c}\|_2^2$ is to introduce enough smoothness away from singularities
  - The term $\|[\gamma\cdot\boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}}\|_1$ regularizes both jumps and hidden jumps
  - The model is solved by alternative optimization strategy

Fast Algorithm

- Alternative optimization
  - Fixing jump set, recover image

$$\boldsymbol{u}^k = \arg\min_{\boldsymbol{u}\in\mathcal{I}_2} \left\| [\boldsymbol{\lambda}\cdot \boldsymbol{W}\boldsymbol{u}]_{(\boldsymbol{\Gamma}^{k-1})^c} \right\|_2^2 + \|[\boldsymbol{\gamma}\cdot \boldsymbol{W}\boldsymbol{u}]_{\boldsymbol{\Gamma}^{k-1}}\|_1 + \frac{1}{2}\|\boldsymbol{A}\boldsymbol{u}-\boldsymbol{f}\|_2^2$$

$$\begin{cases} \boldsymbol{u}^{k,j} = \arg\min_{\boldsymbol{u}} \frac{1}{2}\|\boldsymbol{A}\boldsymbol{u}-\boldsymbol{f}\|_2^2 + \frac{\mu}{2}\|\boldsymbol{W}\boldsymbol{u}-\boldsymbol{d}^{j-1}+\boldsymbol{b}^{j-1}\|_2^2, \\ \boldsymbol{d}^j = \arg\min_{\boldsymbol{d}} \left\| [\boldsymbol{\lambda}\cdot \boldsymbol{d}]_{(\boldsymbol{\Gamma}^{k-1})^c} \right\|_2^2 + \left\| [\gamma\cdot \boldsymbol{d}]_{(\boldsymbol{\Gamma}^{k-1})} \right\|_1 + \frac{\mu}{2}\|\boldsymbol{d}-\boldsymbol{W}\boldsymbol{u}^{k,j}-\boldsymbol{b}^{j-1}\|_2^2, \\ \boldsymbol{b}^j = \boldsymbol{b}^{j-1} + (\boldsymbol{W}\boldsymbol{u}^{k,j}-\boldsymbol{d}^j). \end{cases}$$

  - Fixing image, estimate jump set

$$\boldsymbol{\Gamma}^k = \arg\min_{\boldsymbol{\Gamma}\subset\mathbb{O}^2} \left\| \left[\boldsymbol{\lambda}\cdot \boldsymbol{W}\boldsymbol{u}^k\right]_{\boldsymbol{\Gamma}^c} \right\|_2^2 + \left\| \left[\gamma\cdot \boldsymbol{W}\boldsymbol{u}^k\right]_{\boldsymbol{\Gamma}} \right\|_1$$

$$\boldsymbol{\Gamma}^k = \left\{ \boldsymbol{p}\in\mathbb{O}^2 : \sum_{l=0}^{L-1}\left(\sum_{\boldsymbol{i}\in\mathbb{B}} \gamma_{l,\boldsymbol{i}}[\boldsymbol{p}]\left|(\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u}^k)[\boldsymbol{p}]\right|^2\right)^{\frac{1}{2}} \leq \sum_{l=0}^{L-1}\sum_{\boldsymbol{i}\in\mathbb{B}} \lambda_{l,\boldsymbol{i}}[\boldsymbol{p}]\left|(\boldsymbol{W}_{l,\boldsymbol{i}}\boldsymbol{u}^k)[\boldsymbol{p}]\right|^2 \right\}$$

Piecewise Sobolev Space

- *Piecewise Sobolev space:* $\cup_j \Omega_j = \Omega, \quad \cup_{j'} \Omega_{j,j'} = \Omega_j$

$$\mathcal{H}^{1,s}(\{\Omega_{j,\tilde{j}}\}) := \{f \in L_2(\Omega) : \|f\|_{\mathcal{H}^{1,s}(\{\Omega_{j,\tilde{j}}\})} < \infty\}$$

$$\|f\|_{\mathcal{H}^{1,s}(\{\Omega_{j,\tilde{j}}\})} := \sum_{j=1}^{m} \left[\|f\|_{H^1(\Omega_j)} + \sum_{\tilde{j}=1}^{m_j} \|f\|_{H^{s_{j,\tilde{j}}}(\Omega_{j,\tilde{j}})} \right]$$

- *Trace operator* is a linear bounded operator defined on

$$C^\infty(\overline{B}) \subset H^s(B) \text{ as: } \mathfrak{T}(u) = u|_{\partial B} \text{ for } u \in C^\infty(\overline{B}).$$

- *Key observations:* $\langle u, \boldsymbol{D}^\top \varphi_{n,\boldsymbol{k}} \rangle = \langle u, \psi_{n,\boldsymbol{k}} \rangle$ and integration by parts

Proposition. Let $u \in H^s(B)$ and $\varphi \in C^s(\overline{B})$ with $B \subset \Omega$ a Lipschitz domain with piecewise C^1 boundary ∂B. Then, for any $1 \le |\boldsymbol{i}| \le s$, we have the following formula of integration by parts

$$\langle u, D_{\boldsymbol{i}}\varphi \rangle = \sum_{\boldsymbol{j}_l \in \mathbb{D}_{\boldsymbol{i}}, 1 \le l \le |\boldsymbol{i}|} (-1)^{l-1} \int_{\partial B} \mathfrak{T}(D_{\boldsymbol{j}_l} u) D_{\boldsymbol{i}-\boldsymbol{j}_{l+1}} \varphi \boldsymbol{n}_{\boldsymbol{j}_{l+1}-\boldsymbol{j}_l} ds + (-1)^{|\boldsymbol{i}|} \langle D_{\boldsymbol{i}} u, \varphi \rangle,$$

where $\mathfrak{T}(\cdot)$ is the trace operator defined on $H^s(B)$, and the set $\mathbb{D}_{\boldsymbol{i}}$ indicates the type of differential operators that appears on u at the boundary after the operation of integration by parts:

$$\mathbb{D}_{\boldsymbol{i}} := \{\boldsymbol{j}_l < \boldsymbol{i} : |\boldsymbol{j}_l| = l-1; \boldsymbol{j}_l < \boldsymbol{j}_{l+1}; l = 1, 2, \ldots, |\boldsymbol{i}|\}.$$

Asymptotic Analysis: Linking to Continuum

- We proved that the discrete objective function Gamma-converges to the energy functional of the following (new) variational problem

$$\inf_{u\in\mathcal{H}^{1,s}(\{\Omega_{j,\tilde{j}}\}),\ \{\Gamma_j\},\ \{\Gamma_{j,\tilde{j}}\}} \|\boldsymbol{\nu}\cdot\boldsymbol{D}u\|_2^2 + \sum_{j=1}^{\tilde{m}}\Bigg[\mu_1\int_{\Gamma_j}\left|\mathfrak{T}_j^{+}(u)-\mathfrak{T}_j^{-}(u)\right|ds$$

$$+\mu_2\sum_{\tilde{j}=1}^{\tilde{m}_j}\int_{\Gamma_{j,\tilde{j}}}\left(\sum_{|\boldsymbol{i}|=1}\left|\mathfrak{T}_{j,\tilde{j}}^{+}(D_{\boldsymbol{i}}u)-\mathfrak{T}_{j,\tilde{j}}^{-}(D_{\boldsymbol{i}}u)\right|^2\right)^{\frac{1}{2}}ds\Bigg]+\frac{1}{2}\|Au-f\|_{L_2(\Omega)}^2$$

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Joint vanishing moment = 1

$$\inf_{u\in\mathcal{H}^{1,s}(\{\Omega_{j,\tilde{j}}\}),\ \{\Gamma_j\},\ \{\Gamma_{j,\tilde{j}}\}} \|\boldsymbol{\nu}\cdot\boldsymbol{D}u\|_2^2+\sum_{j=1}^{\tilde{m}}\Bigg[\mu_1\int_{\Gamma_j}\left|\mathfrak{T}_j^+(u)-\mathfrak{T}_j^-(u)\right|ds + \mu_2\sum_{\tilde{j}=1}^{\tilde{m}_j}\int_{\Gamma_{j,\tilde{j}}}\left(\sum_{|\boldsymbol{i}|=1}\left|\mathfrak{T}_{j,\tilde{j}}^+(D_{\boldsymbol{i}}u)-\mathfrak{T}_{j,\tilde{j}}^-(D_{\boldsymbol{i}}u)\right|^2\right)^{\frac{1}{2}}ds\Bigg]+\frac{1}{2}\|Au-f\|_{L_2(\Omega)}^2$$

  - The term $\int_{\Gamma_j}\left|\mathfrak{T}_j^+(u)-\mathfrak{T}_j^-(u)\right|ds$ takes care of the jumps

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Joint vanishing moment = 1

Joint vanishing moment = 2

  - The term $\int_{\Gamma_j}\left|\mathfrak{T}_j^+(u)-\mathfrak{T}_j^-(u)\right|ds$ takes care of the jumps
  - The term $\int_{\Gamma_{j,\tilde{j}}}\left(\sum_{|i|=1}\left|\mathfrak{T}_{j,\tilde{j}}^+(D_iu)-\mathfrak{T}_{j,\tilde{j}}^-(D_iu)\right|^2\right)^{\frac{1}{2}}ds$ takes care of first order hidden jumps

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- The discrete model has far richer structure in general, whose corresponding variational model in continuum is more complicated

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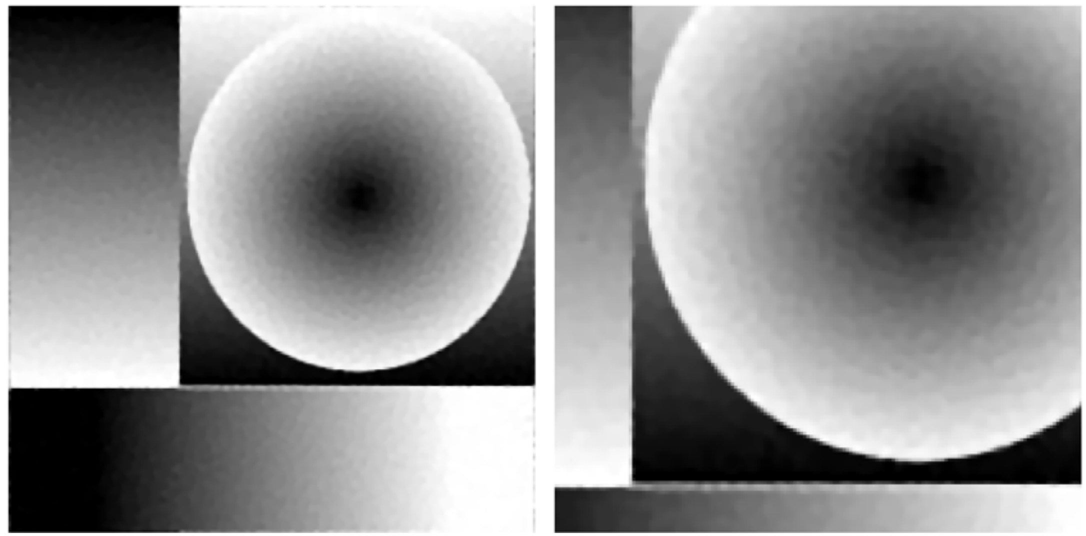
- The term $\int_{\Gamma_{j,\tilde{j}}}\left(\sum_{|i|=1}\left|\mathfrak{T}_{j,\tilde{j}}^+(D_iu)-\mathfrak{T}_{j,\tilde{j}}^-(D_iu)\right|^2\right)^{\frac{1}{2}}ds$ takes care of first order hidden jumps
- The discrete model has far richer structure in general, whose corresponding variational model in continuum is more complicated
- A special case of the above variational model is related to the well-known Mumford-Shah functional

$$\nu\int_{\Omega\setminus\Gamma}|\nabla u|^2+\mu|\Gamma|+\frac{1}{2}\|u-f\|_{L_2(\Omega)}^2$$

Numerical Results: Deblurring

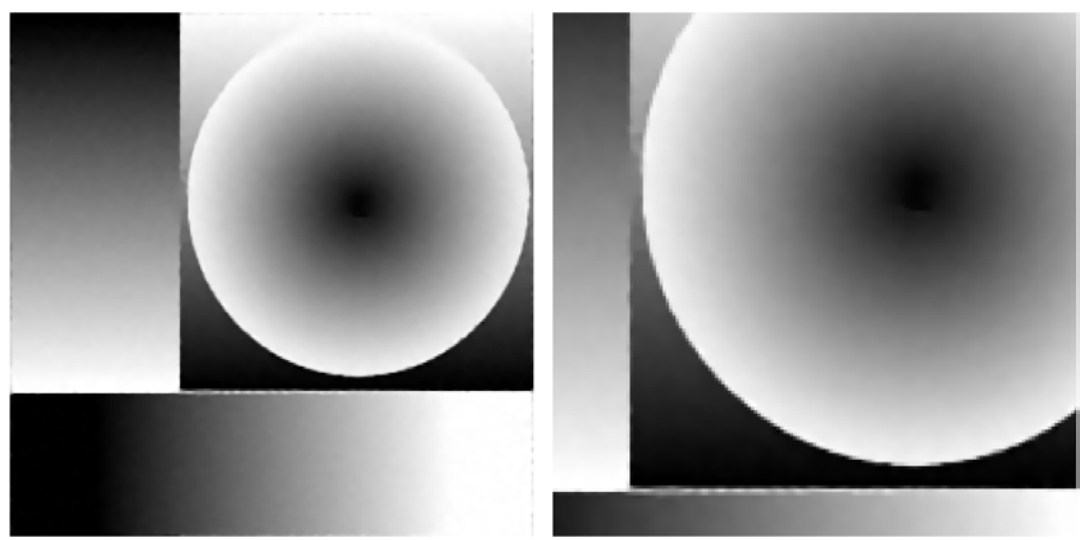
Analysis based model

PSNR=31.72



Piecewise smooth model

PSNR=34.27



Numerical Results: Deblurring

Car

Goldgate

Interior

Pitt

Samantha

Deblurring Results

Image Name	Analysis Based Model	Our Approach
Car	27.3194	27.5443
Goldgate	27.5312	27.8618
Interior	29.6087	30.0355
Pitt	29.4654	29.6716
Samantha	30.9207	31.0085

Conclusions

- What we have done:
  - Piecewise smooth image restoration model
  - Asymptotic analysis and relation to Mumford-Shah functional
  - Numerical experiments support our modeling concept
- What yet need to be done:
  - Regularization on the jump set
  - Full asymptotic analysis without assuming jump set is known
  - Application to image segmentation

Thanks for Your Attention
and
Questions?

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