Topological Analysis of Mapper and Multiscale Mapper

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Joint work with
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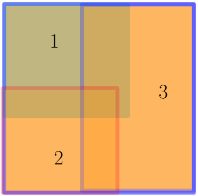

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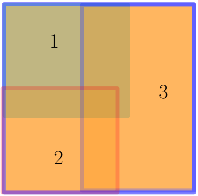
- Nerve of $\mathcal{U}$: $N(\mathcal{U})$ with vertex set A, iff $U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_k} \neq \emptyset$.



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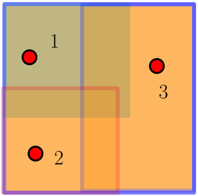

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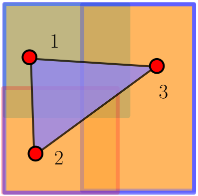

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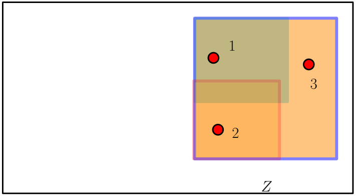

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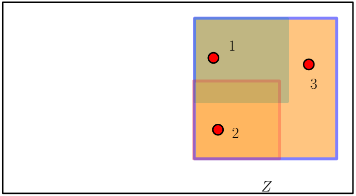

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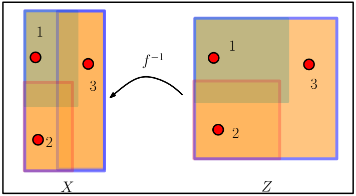

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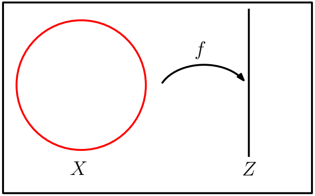

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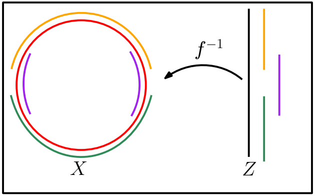

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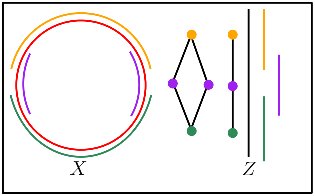

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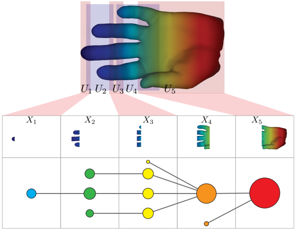
Mapper



Definition (Mapper)

[Singh-Carlsson-Mémoli] Let $f : X \to Z$ be continuous and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a finite open covering of Z. The Mapper is

$$\mathrm{M}(\mathcal{U}, f) := N(f^*(\mathcal{U}))$$

Maps between covers

- Consider covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ and a map of sets $\xi : A \to B$ satisfying $U_\alpha \subseteq V_{\xi(\alpha)}$ for all $\alpha \in A$

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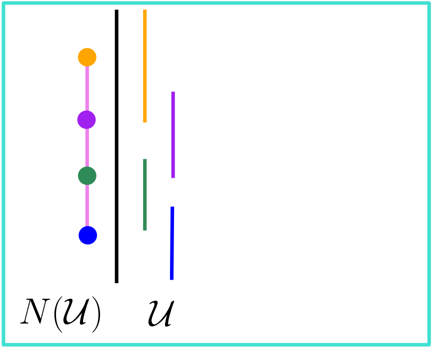

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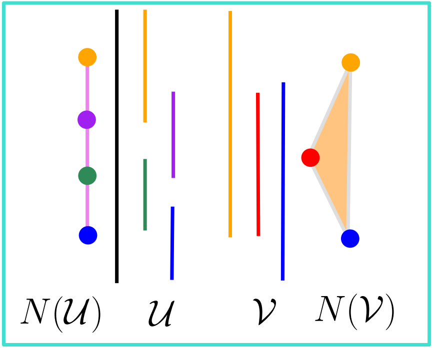

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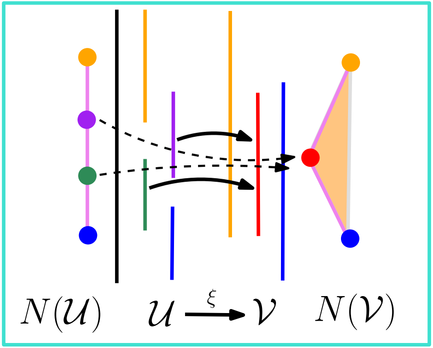

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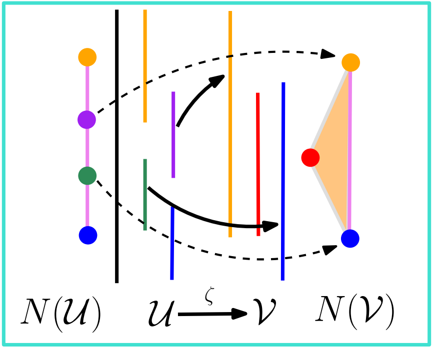

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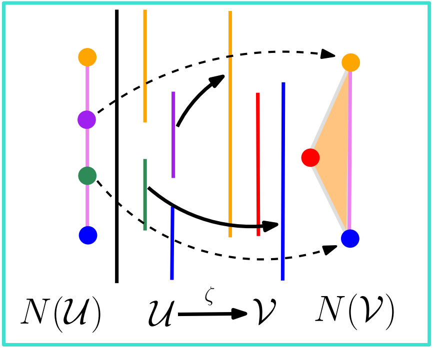
- ξ induces a simplicial map $N(\xi) : N(\mathcal{U}) \to N(\mathcal{V})$
- if $\mathcal{U} \xrightarrow{\xi_1} \mathcal{V} \xrightarrow{\xi_2} \mathcal{W}$, then $N(\xi_2 \circ \xi_1) = N(\xi_2) \circ N(\xi_1)$



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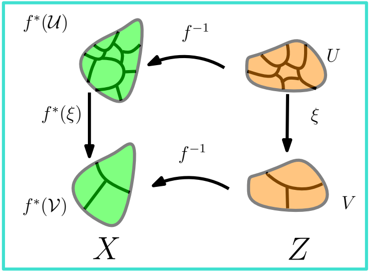

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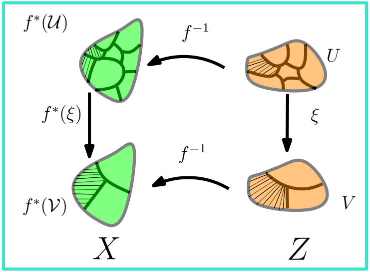

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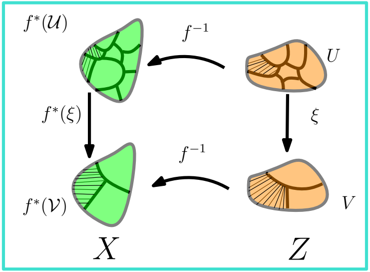

- if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\zeta} \mathcal{W}$, then $f^*(\zeta \circ \xi) = f^*(\zeta) \circ f^*(\xi)$

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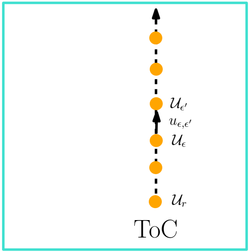

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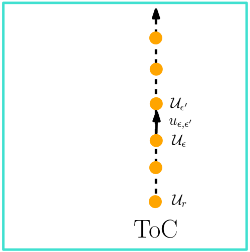

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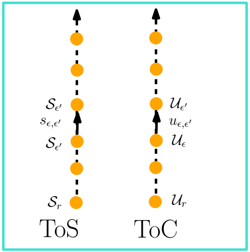

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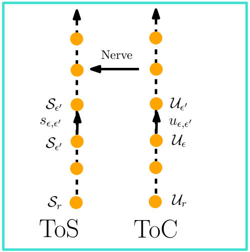

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- Then, $f^*(\mathfrak{U})$ is ToC of X and $N(f^*(\mathfrak{U}))$is ToS

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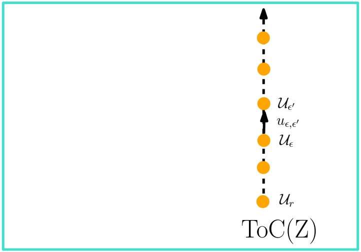

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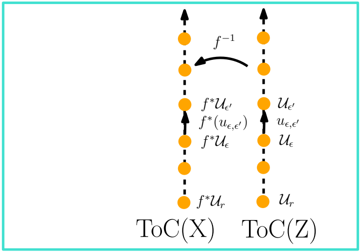

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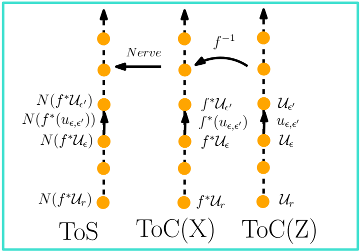

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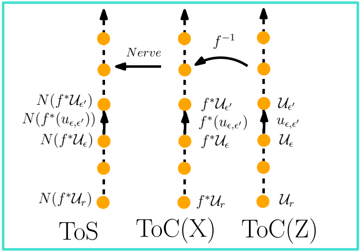


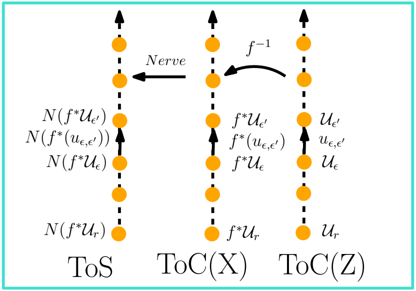
Multiscale Mapper:

$$\mathrm{MM}(\mathfrak{U}, f) := N(f^*(\mathfrak{U}))$$

Persistence diagram of MM

- $\mathrm{D_k MM}(\mathfrak{U}, f)$= persistence diagram of:

$$\mathrm{H}_k\left(N(f^*(\mathcal{U}_{\varepsilon_1}))\right) \to \mathrm{H}_k\left(N(f^*(\mathcal{U}_{\varepsilon_2}))\right) \to \cdots \to \mathrm{H}_k\left(N(f^*(\mathcal{U}_{\varepsilon_n}))\right)$$



Stability against perturbation of ToCs

Lemma

Given

- $\mathfrak{U}, \mathfrak{V}$ *be two ToCs with same resolution*
- $\mathfrak{U}$ *and* $\mathfrak{V}$ *are* η*-interleaved.*

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Then,

- $\mathrm{MM}(\mathfrak{U}, f)$ *and* $\mathrm{MM}(\mathfrak{V}, f)$ *are* η*-interleaved*
- $d_b(\mathrm{D}_k\mathrm{MM}(\mathfrak{U}, f), \mathrm{D}_k\mathrm{MM}(\mathfrak{V}, f)) \leq \eta$ *by strong interleaving [CGGO09]*

Stability against function perturbation

- $f, g : X \to Z$ s.t. $\max_{x \in X} d_Z(f(x), g(x)) = \delta$
- $\mathfrak{W}$ is (c, s)-good ToC of Z and $\varepsilon_0 = \max(1, s)$

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$$\log\big(2c\max(\delta, s) + c\big) - \textit{interleaved}$$

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$$\log(2c\max(s, \delta) + c) + \max(0, \log\frac{1}{s})$$

What does MM compute?

What does persistence of MM mean?

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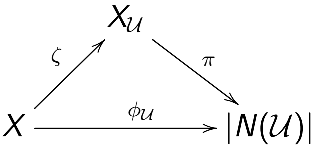
- Answered partially [DMW, SoCG 17]

From space to nerve

- X a path connected, paracompact space
- $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, a path connected cover, $X_{\mathcal{U}}$: blowup space
- $\phi_{\mathcal{U}} : X \to |N(\mathcal{U})|$ is a map where $\phi_{\mathcal{U}} = \pi \circ \zeta$



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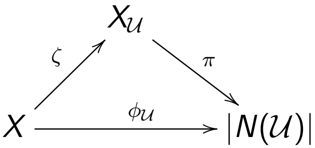

Theorem

$\phi_{\mathcal{U}*} : H_1(X) \to H_1(|N(\mathcal{U})|)$ *is a surjection.*

From nerve to nerve

Proposition

$\mathcal{U}$ and $\mathcal{V}$ be two covers of X with a cover map $\mathcal{U} \xrightarrow{\theta} \mathcal{V}$. Then, $\phi_{\mathcal{V}} = \hat{\tau} \circ \phi_{\mathcal{U}}$ where $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ is induced by θ.

Corollary

The maps $\phi_{\mathcal{U}*} : H_k(X) \to H_k(|N(\mathcal{U})|)$, $\phi_{\mathcal{V}*} : H_k(X) \to H_k(|N(\mathcal{V})|)$, and $\hat{\tau}_* : H_k(|N(\mathcal{U})|) \to H_k(|N(\mathcal{V})|)$ commute, that is, $\phi_{\mathcal{V}*} = \hat{\tau}_* \circ \phi_{\mathcal{U}*}$.

Theorem

Let $\tau : N(\mathcal{U}) \to N(\mathcal{V})$ be induced by a cover map $\mathcal{U} \to \mathcal{V}$. Then, $\tau_* : H_1(N(\mathcal{U})) \to H_1(N(\mathcal{V}))$ is a surjection.

Implication for multiscale mapper

Theorem

Consider the following multiscale mapper:

$$N(f^*\mathcal{U}_0) \to N(f^*\mathcal{U}_1) \to \cdots \to N(f^*\mathcal{U}_n)$$

- *surjection from $H_1(X)$ to $H_1(N(f^*\mathcal{U}_i))$ for each $i \in [0, n]$.*
- *For H_1-persistence module:*

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_0)\big) \to \mathrm{H}_1\big(N(f^*\mathcal{U}_1)\big) \to \cdots \to \mathrm{H}_1\big(N(f^*\mathcal{U}_n)\big)$$

all connecting maps are surjections.

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Lebesgue number of a cover:

$$\lambda(\mathcal{U}) = \sup\{\delta \mid \forall X' \subseteq X \text{ with } s(X') \leq \delta, \exists U_\alpha \in \mathcal{U} \text{ where } U_\alpha \supseteq X'\}$$

Persistent H_1-classes

Theorem

Let $z_1, z_2, \ldots, z_g$ be a minimal generator basis of $H_1(X)$ ordered with increasing sizes.

i. *Let $\ell \in [1, g]$ be the smallest integer so that $s(z_\ell) > \lambda(\mathcal{U})$. If $\ell \neq 1$, the class $\bar{\phi}_{\mathcal{U}*}[z_j] = 0$ for $j = 1, \ldots, \ell - 1$. Moreover, the classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell,\ldots,g}$ generate $H_1(N(\mathcal{U}))$.*

ii. *The classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell',\ldots,g}$ are linearly independent where $s(z_{\ell'}) > 4s_{max}(\mathcal{U})$.*

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Implication: Just like in Reeb graphs, only vertical homology classes survive in Reeb spaces (extension of a result of [D.-Wang 14])

Persistent H_1-classes in multiscale mapper

- $f : X \to Z$ where (Z, d_Z) a metric space
- $d_f(x, x') := \inf_{\gamma \in \Gamma_X(x,x')} \operatorname{diam}_Z(f \circ \gamma)$.

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Theorem

Consider a H_1-persistence module of a multiscale mapper induced by a tower of path connected covers:

$$\mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_0})\big) \xrightarrow{s_{1*}} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_1})\big) \xrightarrow{s_{2*}} \cdots \xrightarrow{s_{n*}} \mathrm{H}_1\big(N(f^*\mathcal{U}_{\varepsilon_n})\big)$$

Let $\hat{s}_{i*} = s_{i*} \circ s_{(i-1)*} \circ \cdots \circ \bar{\phi}_{\mathcal{U}_{\varepsilon_0}*}$. Then, $\hat{s}_{i*}$ renders the small classes of $H_1(X)$ trivial in $H_1(N(f^*\mathcal{U}_{\varepsilon_i}))$ as detailed in previous theorem.

Higher dimensional homology

- $\exists$ a metric d_δ on mapper $N(\mathcal{U})$ so that $d_{GH}((N(\mathcal{U}), d_\delta), (X, d_f)) \leq 5\delta$
  - convergence of Reeb space to mappers [MW16]
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- Persistence diagram of (X, d_f) and $MM(\mathfrak{U}, f)$ are close
- Persistence diagrams of mapper and multiscale mapper are similar under an appropriate map-induced metric

Conclusion/Questions

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Conjecture: If t-wise intersections in $\mathcal{U}$ for all $t > 0$ have $\tilde{H}_{\leq k-t} = 0$, then is $\phi_{\mathcal{U}*}$ surjective for H_k?

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Thank You