Reduced Order Methods for PDEs: state of the art and perspectives with applications in Industry, Medicine and Environmental Sciences

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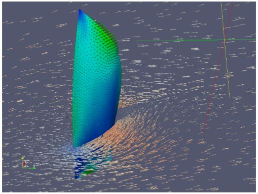
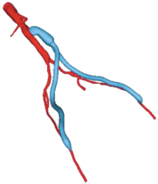
Leading Motivation: Computational Sciences challenges

- **Reduced order modelling** is a quickly emerging field in applied mathematics and computational science and engineering.
- Present and future efforts: towards **multiphysics** problems, as well as coupled systems.
- Growing demand of
  - ∗ **efficient computational tools** for
  - ∗ **many query** and **real time** computations,
  - ∗ **parametric formulations**,
  - ∗ simulations of increasingly **complex systems** with uncertain scenarios,

  by **industrial and clinical** research partners.
- The need of a computational collaboration rather than a competition between **High Performance Computing** (HPC) and **Reduced Order Methods** (ROM), as well as Full/High Order and Reduced Order Methods.





Overview
our current efforts, aims and perspectives at SISSA mathLab

A team developing **Advanced Reduced Order Methods** for parametric PDEs with a special focus on **Computational Fluid Dynamics**

Overview:
our current efforts, aims and perspectives at SISSA mathLab

A team developing **Advanced Reduced Order Methods** with special focus on **Computational Fluid Dynamics**:

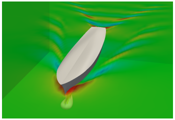
- to face and overcome **several limitations** of the **state of the art** for parametric ROM in CFD;
- to improve capabilities of reduced order methodologies for **more demanding applications** in **industrial, medical and applied sciences settings**;
- to carry out important methodological developments in **Numerical Analysis**, with special emphasis on mathematical modelling and a more extensive exploitation of **Computational Science and Engineering**;
- focus on **Computational Fluid Dynamics** as a central topic to enhance broader applications in **multiphysics** and **coupled settings**, as well as more realistic models (e.g. aeronautical, mechanical, naval, off-shore, wind, sport, biomedical engineering and also cardiovascular surgery planning).



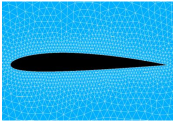
Overview of the physical problems

The interest is in **viscous parametrized incompressible flows**

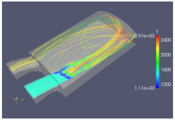
Industrial Flows



Naval Eng.

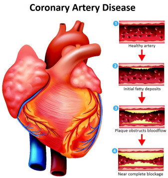


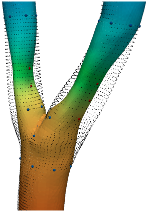
Aeronautics



Industrial App.

Biomedical Applications





Possible applications can be found in **naval** and **nautical** engineering, **aeronautical** engineering and **industrial** engineering.

In general any application dealing with incompressible fluid dynamic problems that has the response depending on **parameter changes** (Reynolds Number, Grashof Number, Geometrical parameters ..)

Overview: our current efforts, aims and perspectives

- Towards **Real-Time** Computing and Visualization, through an **Offline**–**Online** computational paradigm that combines

  High Performance Computing to **Advanced Reduced Order Modelling techniques**.

Offline:

HPC facilities, time demanding

"Science" driven

Online:

In situ, tablets or smartphones, real time

"Industrial needs" driven

- **Export numerical simulations and scientific computing** in fields and places where at the state of the art there is still little exploitation.
- Development of new open-source tools based on reduced order methods:
  - * **ITHACA**, **I**n real **T**ime **H**ighly **A**dvanced **C**omputational **A**pplications, as an add-on to integrate already well established CSE/CFD open-source software libraries (FV, SEM) with ROMs (OpenFoam, Nektar, FEniCS, Libmesh)
  - * **RBniCS** as educational initiative (FEM) for newcomer ROM users (training).
  - * **Argos** Advanced Reduced order modellinG Online **computational web server** for parametric Systems
  - * **ATLAS**

ITHACA ITHACA - FV ITHACA-SEM RBniCS multiphenics ARGOS ATLAS

http://mathlab.sissa.it/cse-software

History, present and future

Pioneers of RB 1980's: Noor, Peters, Brogan, Stern, Almroth, Fink, Rheinboldt... (**Extensive Scientific Computing** was still a dream...)

New mathematical and methodological developments 2000's: Maday, Patera, Willcox, Huerta, Ito, Ravindran, Peterson, Farhat, Quarteroni, Hesthaven, Benner, Sorensen, Volkwein, Kunish, Urban, Schilders, Mehrmann, Ohlberger, Chinesta, Cueto, Ladeveze, Iollo, Perotto, Diez, Iliescu, Bergmann, Borggaard, Gunzburger, Heinkenschloss, Dahmen, Haasdonk, Veroy, Grepl, Tezaur, Carlberg, Peherstorfer, Stamm, Kramer, Quaini, Mula, Chakir, Ern, Yano, Nguyen, Huynh
... And many others!

Applications 2010's: several groups around the world focusing on many different aspects and applications...

....and new generation(s) of younger talented scientists.

Today at SIAM CSE19: 20 minisimposia on model reduction...

Networking initiatives: COST EU-MORNET (Handbook)

Intrusive Reduced Order Methods in a nutshell

- $()^{\mathcal{N}}$: "truth" full order method (FEM, FV, FD, SEM) – to be accelerated
- $()_N$: reduced order method (ROM) – *the accelerator*

* Input parameters:

$$\boldsymbol{\mu} \text{ (geometry, physical properties, etc.)}$$

* Parametrized PDE:

$$\mathcal{A}(u(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \rightsquigarrow \quad \underbrace{\boldsymbol{A}^{\mathcal{N}}(\boldsymbol{\mu})\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}) = 0}_{\text{full order}} \quad \rightsquigarrow \quad \underbrace{\boldsymbol{A}_N(\boldsymbol{\mu})\boldsymbol{u}_N(\boldsymbol{\mu}) = 0}_{\text{reduced order}}$$

* Output:

$$s(\boldsymbol{\mu}) \quad \approx \quad \underbrace{s^{\mathcal{N}}(\boldsymbol{\mu})}_{\text{full order}} \quad \approx \quad \underbrace{s_N(\boldsymbol{\mu})}_{\text{reduced order}}$$

* Input-Output evaluation:

$$\boldsymbol{\mu} \quad \rightarrow \quad s^{\mathcal{N}}(\boldsymbol{\mu}) \quad \rightarrow \quad s_N(\boldsymbol{\mu})$$

- **Reduced Basis Method(RB)**: continuation method in non-linear structural mechanics...
- **Proper Orthogonal Decomposition(POD)**: transient and turbulent flows...
- **Other methodologies**: Proper Generalized Decomposition (PGD), Hierarchical Model Reduction (HiMod).

J. S. Hesthaven, G. Rozza, B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer, 2015

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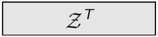
- **Offline**: very expensive preprocessing (full order): basis calculation (done *once*) after suitable parameters sampling (greedy, POD, ...)

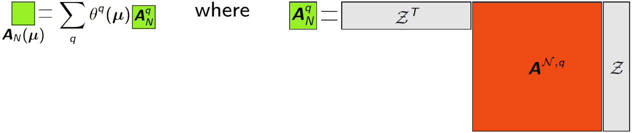


- **Online**: extremely fast (reduced order): real-time input-output evaluation

$$\boldsymbol{\mu} \quad \rightarrow \quad s_N(\boldsymbol{\mu})$$

thanks to an efficient assembly of problem operators

$$\boldsymbol{A}_N(\boldsymbol{\mu}) = \sum_q \theta^q(\boldsymbol{\mu}) \boldsymbol{A}_N^q, \text{ where } \boldsymbol{A}_N^q = \mathcal{Z}^T \boldsymbol{A}^{\mathcal{N},q} \mathcal{Z}$$



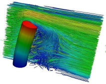
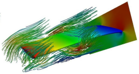
- Numerical issues: approximation stability, error bounds and stability factors, efficient (geometrical) parametrization, sampling, coupling, nonlinearities...

... reduction in parameter space

J. S. Hesthaven, G. Rozza, B. Stamm. *Certified Reduced Basis Methods for Parametrized Partial Differential Equations*. SpringerBriefs in Mathematics. Springer, 2015

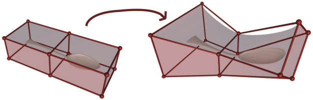
Overview on the topics: from intrusive to non-intrusive ROM

- ROMs exploit a parametrized formulation of the problem. In particular, an efficient **geometrical parametrization** is required when interested in the variation of the domain/interface, such as in **shape optimization or fluid-structure interaction problems** $\Omega_o(\boldsymbol{\mu}) = \boldsymbol{T}(\Omega; \boldsymbol{\mu})$.
- **Focus of this talk: show some state of the art and perspectives in parametric flow problems treated in the reduced order context** for
  - complex computational mechanics phenomena and bifurcations;
  - fluid-structure interaction (FSI) reduced problems;
  - **flow control**;
  - **uncertainty quantification (UQ)**;
  - inverse problems;
  - shape optimization;
  - and some perspectives and challenges.

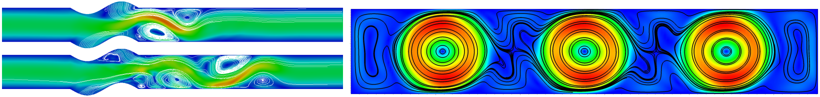

Shape parameterization for ROM

- Free-Form Deformations (FFD) [Lassila, Rozza, CMAME, 2010], [Salmoiraghi *et al.*, AMSES, 2016].
- Radial Basis Functions (RBF) [Manzoni *et al.*, IJNMBE, 2011].
- Transfinite Mapping (TM) [Løvgren, Maday, Rønquist, 2006], [Iapichino *et al.*, CMAME, 2012].
- Vascular shape parametrization [Ballarin *et al.*, JCP, 2016].
- Reduced inverse Distance Weighting [D'Amario et al, 2017].



#CFD #intrusive #ROM

ROM and stability
for fluid mechanics problems
with Francesco Ballarin, Giovanni Stabile, Shafqat Ali, Enrique Delgado



Reduced-order numerical simulations by POD-Galerkin ROM

- parametrized formulation of Navier-Stokes equations (modelling Newtonian fluid);
- offline stage:
  - intensive phase, on HPC architectures, to be done once;
  - Finite Element approximation of the problem for few values of the parameters (snapshots):

for $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}), \underline{\mathbf{p}}^{\mathcal{N}}(\boldsymbol{\mu})) \in \mathbb{R}^{\mathcal{N}_{\mathbf{u}}} \times \mathbb{R}^{\mathcal{N}_p}$,

large $\mathcal{N}$

$$\begin{bmatrix} A^{\mathcal{N}}(\boldsymbol{\mu}) + C^{\mathcal{N}}(\underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) & B^{\mathcal{N}}(\boldsymbol{\mu})^T \\ B^{\mathcal{N}}(\boldsymbol{\mu}) & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{p}}^{\mathcal{N}}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \underline{f}^{\mathcal{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{0}} \end{bmatrix}$$

  - [POD] Proper Orthogonal Decomposition (based on singular value decomposition) to extract optimal basis functions from the set of numerical simulations (snapshots) of the system to build $\mathcal{Z}$. [RB] Greedy as an alternative.

    Aubry et al. J. Fluid Mech. 1988; Ravindran, Int. J. Numer. Meth. Fluids, 2000
  - online stage

Reduced-order numerical simulations by POD-Galerkin ROM

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  - Finite Element approximation of the problem for few values of the parameters (snapshots)
  - [POD] Proper Orthogonal Decomposition (based on singular value decomposition) to extract optimal basis functions from the set of numerical simulations (snapshots) of the system to build $\mathcal{Z}$.

  Build the correlation matrices $\mathbf{C}^{\boldsymbol{u}}, \mathbf{C}^{\boldsymbol{p}} \in \mathbb{R}^{N_{train} \times N_{train}}$, where N_{train} is the dimension of the training set and

  $$\mathbf{C}^{\boldsymbol{u}}_{ij} = (\underline{\boldsymbol{u}}^{\mathcal{N}}(\boldsymbol{\mu}_i), \underline{\boldsymbol{u}}^{\mathcal{N}}(\boldsymbol{\mu}_j)) \text{ and } \mathbf{C}^{\boldsymbol{p}}_{ij} = (\underline{\boldsymbol{p}}^{\mathcal{N}}(\boldsymbol{\mu}_i), \underline{\boldsymbol{p}}^{\mathcal{N}}(\boldsymbol{\mu}_j)) \qquad i, j = 1, \dots, N_{train}.$$

  Then we find $(\lambda_i^{\boldsymbol{u}}, v_i^{\boldsymbol{u}})$ and $(\lambda_i^{\boldsymbol{p}}, v_i^{\boldsymbol{p}})$ such that $\mathbf{C}^{\boldsymbol{u}} v_i^{\boldsymbol{u}} = \lambda_i^{\boldsymbol{u}} v_i^{\boldsymbol{u}}$ and $\mathbf{C}^{\boldsymbol{p}} v_i^{\boldsymbol{p}} = \lambda_i^{\boldsymbol{p}} v_i^{\boldsymbol{p}}$. We retain only the first $N_{\boldsymbol{u}}$ and $N_{\boldsymbol{p}}$ eigenvalues for pressure and velocity, respectively.
  The reduced space is span$\{\Phi_1^{\boldsymbol{u}}, \dots, \Phi_{N_{\boldsymbol{u}}}^{\boldsymbol{u}}, \Phi_1^{\boldsymbol{p}}, \dots, \Phi_{N_{\boldsymbol{p}}}^{\boldsymbol{p}}\}$, where the basis function $\Phi_i^{\boldsymbol{u}}$ and $\Phi_i^{\boldsymbol{p}}$ are the eigenvectors of $\lambda_i^{\boldsymbol{u}}$ and $\lambda_i^{\boldsymbol{p}}$, respectively.
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  - online stage

Reduced-order numerical simulations by POD-Galerkin ROM

- parametrized formulation of Navier-Stokes equations (modelling Newtonian fluid);
- offline stage
- online stage:
  - inexpensive and very fast, on a laptop, to be done multiple times (for each new value of the parameters);
  - Galerkin projection over a reduced basis space:

for $\boldsymbol{\mu} \in \mathcal{D}$, find $(\underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}), \underline{\mathbf{p}}_{\mathbf{N}}(\boldsymbol{\mu})) \in \mathbb{R}^{N_{\mathbf{u}}} \times \mathbb{R}^{N_p}$, $\boxed{N = N_{\mathbf{u}} + N_p \ll \mathcal{N}}$

$$\begin{bmatrix} A_{\mathbf{N}}(\boldsymbol{\mu}) + C_{\mathbf{N}}(\underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) & B_{\mathbf{N}}(\boldsymbol{\mu})^T \\ B_{\mathbf{N}}(\boldsymbol{\mu}) & O \end{bmatrix} \begin{bmatrix} \underline{\mathbf{u}}_{\mathbf{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{p}}_{\mathbf{N}}(\boldsymbol{\mu}) \end{bmatrix} = \begin{bmatrix} \underline{f}_{\mathbf{N}}(\boldsymbol{\mu}) \\ \underline{\mathbf{0}} \end{bmatrix}$$

Inf-sup stabilization and pressure recovery

- inf-sup condition is **not** necessarily preserved by Galerkin projection in the online phase.
- reduced velocity space **enrichment** by supremizer solutions,

$$V_N = POD(\{\boldsymbol{u}^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_{\mathbf{u}}) \;\oplus\; POD(\{S^{\boldsymbol{\mu}^i} p^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_{\mathbf{s}}),$$
$$Q_N = POD(\{p^{\mathcal{N}}(\boldsymbol{\mu}^i)\}_{i=1}^{N_{train}}; N_p),$$

where $S^{\boldsymbol{\mu}} : Q^{\mathcal{N}} \to V^{\mathcal{N}}$ is the **supremizer operator** given by

$$(S^{\boldsymbol{\mu}} p^{\mathcal{N}}, \boldsymbol{w}^{\mathcal{N}})_V = b(p^{\mathcal{N}}, \boldsymbol{w}^{\mathcal{N}}; \boldsymbol{\mu}), \quad \forall \boldsymbol{w} \in V^{\mathcal{N}}.$$

where $b(\cdot,\cdot;\boldsymbol{\mu}) = \int_\Omega p \mathrm{div} \boldsymbol{w} \, d\Omega$ (pressure-divergence term)

In order to fulfill an ***inf-sup condition at the reduced-order level*** too:

$$\beta_N(\boldsymbol{\mu}) = \inf_{\underline{\mathbf{q}}_N \neq \underline{\mathbf{0}}} \sup_{\underline{\mathbf{v}}_N \neq \underline{\mathbf{0}}} \frac{\underline{\mathbf{q}}_N^T B_N(\boldsymbol{\mu}) \underline{\mathbf{v}}_N}{\|\underline{\mathbf{v}}_N\|_{V_N} \|\underline{\mathbf{q}}_N\|_{Q_N}} \geq \tilde{\beta}_N > 0 \qquad \forall \boldsymbol{\mu} \in \mathcal{D}.$$

where $B_N(\boldsymbol{\mu})$ is the reduced-order matrix associated to the divergence term. ***(Rozza, Veroy. CMAME, 2007, Rozza et al, Numerische Mathematik, 2013. Ballarin et al. IJNME, 2015)***. Other options: residual-based stabilization procedures for POD-Galerkin ***(Caiazzo, Iliescu et al. JCP, 2014)***, Petrov-Galerkin ***(Dahmen; Carlberg; Abdulle, Budac)***, div-free approach ***(Lovgren et al.)***.

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Some ROM challenges in CFD: to higher Reynolds parametrized flows

- ROMs of parametrized viscous flows for low and moderate Reynolds number are well developed: we need to **increase Reynolds number** for several industrial applications.

- Offline–Online **stabilization techniques** for parametrized flows (geometry, physics) is derived from streamline upwind Petrov-Galerkin (SUPG), ...

$$\sup_{\boldsymbol{v}_N \neq 0} \frac{b(\boldsymbol{v}_N, q_N; \boldsymbol{\mu})}{\|\boldsymbol{v}_N\|_{V_N}} + s(q_N, q_N)^{\frac{1}{2}} \geq \tilde{\beta}_N \|q_N\|_{Q_N} > 0, \quad \forall q_N \in Q_N, \forall \boldsymbol{\mu} \in \mathcal{D} \quad \text{(Generalized inf-sup).}$$

- A ROM **variational multiscale** approach in parametrized context towards turbulence modelling and a Smagorinski turbulent model have been recently proposed

  G. Stabile, F. Ballarin, G. Zuccarino and G. Rozza. A reduced order variational multiscale approach for turbulent flows, submitted, 2018, https://arxiv.org/abs/1809.11101

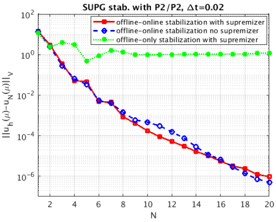
  F. Ballarin, T. Chacón Rebollo, E. Delgado Ávila, M. Gómez Mármol, and G. Rozza, *Certified Reduced Basis VMS-Smagorinsky model for natural convection flow in a cavity with variable height.* ArXiV preprint. http://arxiv.org/abs/1902.05729

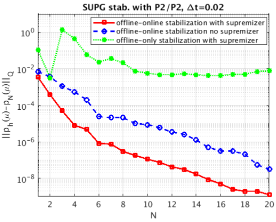
  T. Chacón Rebollo, E. Delgado Ávila, M. Gómez Mármol, F. Ballarin, and G. Rozza *On a certified Smagorinsky reduced basis turbulence model.* SIAM Journal on Numerical Analysis, 55 (2017) pp. 3047-3067

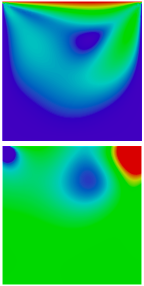
- Important expectations and needs dealing with **industrial and cardiovascular flows**.
- ROM developments in FV and also higher order methods.

Unsteady incompressible Navier-Stokes equations

- Numerical simulations on a lid driven cavity using FE discretization $\mathbb{P}_2/\mathbb{P}_2$.
- Classical **stabilization** technique is implemented in the high order and then projected on reduced basis.
- **RB stabilization** is based on Streamline Upwind Petrov Galerkin (SUPG) and compared with the supremizer enrichment approach
- A significant reduction in the computational cost of **offline-online stabilization** without supremizer could be achieved





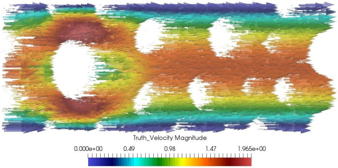


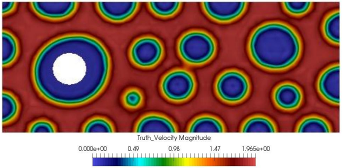
Error comparison for Velocity (left) and Pressure (right). Parameter range in offline stage is $Re \in [100, 200]$, FE dimension $\mathcal{N}$=3327, RB dimension, $N = 60$ (with supremizer), $N = 40$ (without supremizer).

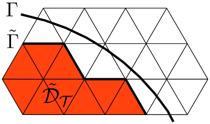
S. Ali, S. Hijazi, G. Stabile, F. Ballarin, G. Rozza, The effort of increasing Reynolds number in POD-Galerkin Reduced Order Methods: from laminar to turbulent flows, for the special volume of the FEF conference, in press, 2019.

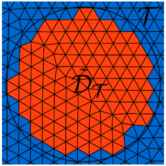
#Fluid Dynamics #Embedded-Immersed FEMs #ROMS on a Background Geometry

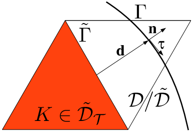
with Efthymios Karatzas, Francesco Ballarin, Giovanni Stabile
Guglielmo Scovazzi, Leo Nouveau, Nabil Atallah





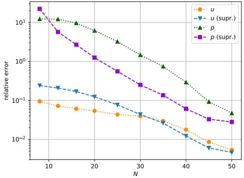


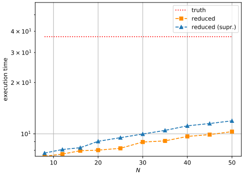


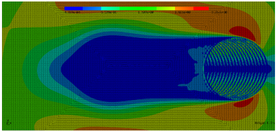


ROMS for systems with Parametric Geometry and Embedded FEMs

- **Equations: Multiphase fluid dynamics, viscous steady and unsteady incompressible flows,** Stokes, Navier-Stokes, Cahn-Hilliard
- **Methodology:** SBM, CutFEM, EBM/IBM, differences/**advantages** with respect to a **reference domain approach**





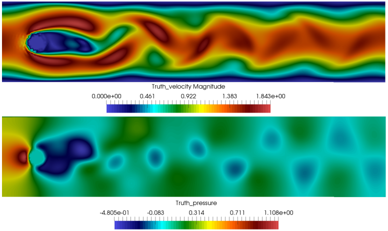


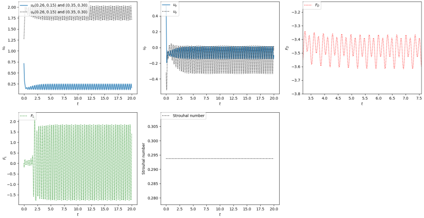
Supremizers enrichment:	No		Yes	
Number of modes	relative error u	relative error p	relative error u	relative error p
8	0.0947158	12.309881	0.2406999	22.319781
12	0.0723268	12.133591	0.2078557	5.7159319
16	0.0610052	9.6652163	0.1692787	2.6962056
20	0.0538906	6.1692750	0.1243368	1.2535779
25	0.0434925	3.2331644	0.0770726	0.5568314
30	0.0396132	1.4693532	0.0437348	0.2504069
35	0.0298269	0.7455038	0.0262345	0.1356788
40	0.0177170	0.2918072	0.0121903	0.0611154
45	0.0085905	0.0923509	0.0060355	0.0330206
50	0.0053882	0.0473412	0.0046300	0.0279857

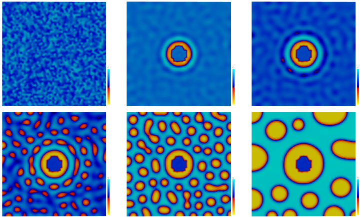
E.N. Karatzas, G. Stabile, L. Nouveau, G. Scovazzi, G. Rozza. *A reduced basis approach for PDEs on parametrized geometries based on the shifted boundary finite element method and application to a Stokes flow*, CMAME, (347), pp. 568-587, 2019

Some ROM challenges for Embedded FEMs

Model for multiphase fluids dynamics: hydrodynamic effects/polymer fluids



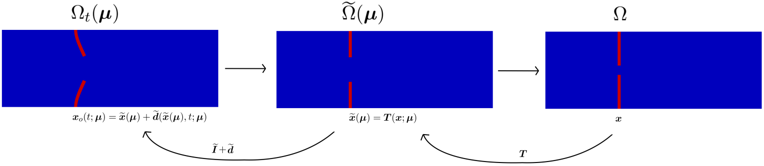




- $c_t - \frac{1}{Pe}\nabla\cdot(b(c)\nabla w) + \mathbf{u}\cdot\nabla c = 0,$
  $w = \Phi'(c) - \gamma^2\Delta c,\ \Phi(c) = \frac{1}{4}(1-c^2)^2$
  $\mathbf{u}_t - \frac{1}{Re}\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p + Kc\nabla w = 0$,
  $\nabla\cdot\mathbf{u} = 0,\ \mathbf{u} = \frac{1+c}{2}\mathbf{u}_1 + \frac{1-c}{2}\mathbf{u}_2$
  Notation: concentration c, chemical potential w, capillary number K, interface parameter γ, mobility function $b(\cdot)$, Péclet adv/diff transport rate number Pe.
- **Software** packages used, **Offline**: Nalu, ngsxfem, **Online**: ITHACA, RBniCS.

#FSI

Monolithic ROMs for FSI problems with Francesco Ballarin, Monica Nonino and Yvon Maday



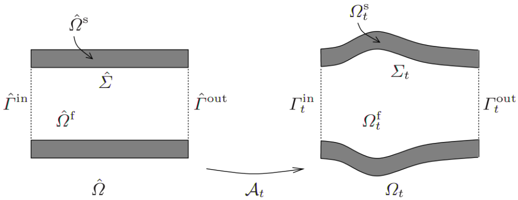
Formulation of FSI problems

- Fluid variables: $(\boldsymbol{u}_f, p, \boldsymbol{d}_f)$,
- Structure variables: $(\boldsymbol{u}_s, \boldsymbol{d}_s)$,
- Fluid-structure interaction problem three-fields formulation:

$$\begin{cases} F(\boldsymbol{u}_f, p, \boldsymbol{d}_f; \boldsymbol{d}_s) = 0, & \text{Fluid} \\ S(\boldsymbol{u}_s, \boldsymbol{d}_s) = 0, & \text{Structure} \\ I(\boldsymbol{d}_f, \boldsymbol{d}_s) = 0, & \text{Interface} \end{cases}$$

subject to interface (coupling) conditions

$$\begin{cases} \boldsymbol{d}_s - \boldsymbol{d}_f = 0 & \text{on } \Gamma, & \text{geometric continuity} \\ \boldsymbol{u}_s - \boldsymbol{u}_f = 0 & \text{on } \Gamma, & \text{velocity continuity} \\ \sigma_f \cdot n_f + \sigma_s \cdot n_s = 0 & \text{on } \Gamma, & \text{balance of normal forces.} \end{cases}$$



Reduced order monolithic formulation of FSI problems

Truth Finite Element discretization (P2-P1 Taylor-Hood)

For $\boldsymbol{\mu} \in \mathcal{D}$, solve large $\mathcal{N}$

$$F^{\mathcal{N}}(\boldsymbol{u}_f^{\mathcal{N}}(\boldsymbol{\mu}), p^{\mathcal{N}}(\boldsymbol{\mu}), \boldsymbol{d}_f^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{d}_s^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \text{Fluid}$$

$$S^{\mathcal{N}}(\boldsymbol{u}_s^{\mathcal{N}}(\boldsymbol{\mu}), \boldsymbol{d}_s^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \text{Structure}$$

$$I^{\mathcal{N}}(\boldsymbol{d}_f^{\mathcal{N}}(\boldsymbol{\mu}), \boldsymbol{d}_s^{\mathcal{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \text{Interface, coupled conditions}$$

OFFLINE – Space construction and matrices assembling

- Space construction by **Proper Orthogonal Decomposition** for **global** variables.
- Additional computations related to inf-sup stabilization procedure by means of supremizer enrichment → accurate pressure recovery for balance of normal forces. [Ballarin et al., 2015], [Rozza *et al.*, 2012], [Rozza, Veroy, 2007].

ONLINE – Galerkin projection over the enriched space

For $\boldsymbol{\mu} \in \mathcal{D}$, solve $\mathbf{N} \ll \mathcal{N}$

$$F^{\mathbf{N}}(\boldsymbol{u}_f^{\mathbf{N}}(\boldsymbol{\mu}), p^{\mathbf{N}}(\boldsymbol{\mu}), \boldsymbol{d}_f^{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{d}_s^{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \textbf{Reduced}\text{ fluid}$$

$$S^{\mathbf{N}}(\boldsymbol{u}_s^{\mathbf{N}}(\boldsymbol{\mu}), \boldsymbol{d}_s^{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \textbf{Reduced}\text{ structure}$$

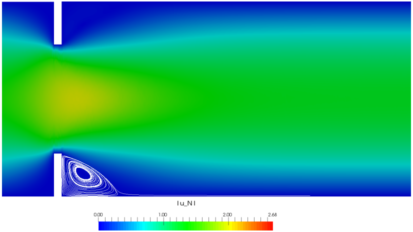
$$I^{\mathbf{N}}(\boldsymbol{d}_f^{\mathbf{N}}(\boldsymbol{\mu}), \boldsymbol{d}_s^{\mathbf{N}}(\boldsymbol{\mu}); \boldsymbol{\mu}) = 0 \quad \textbf{Reduced}\text{ interface, coupled conditions}$$

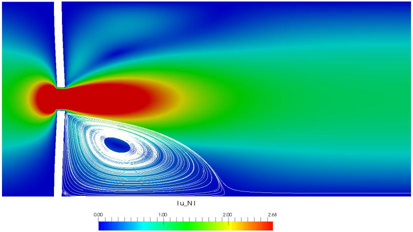
Our approach:

- POD–Galerkin method for **global** variables $\boldsymbol{u}, p, \boldsymbol{d}$ (monolithic approach), time dependent,
- capability to parametrize the initial configuration (geometry).

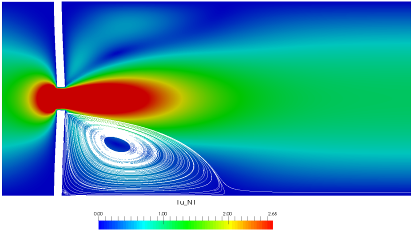
Ongoing applications to cardiovascular modelling

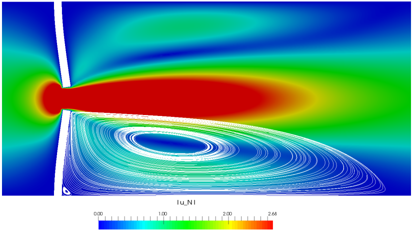
Increase leaflet length: *(same inlet velocity)*





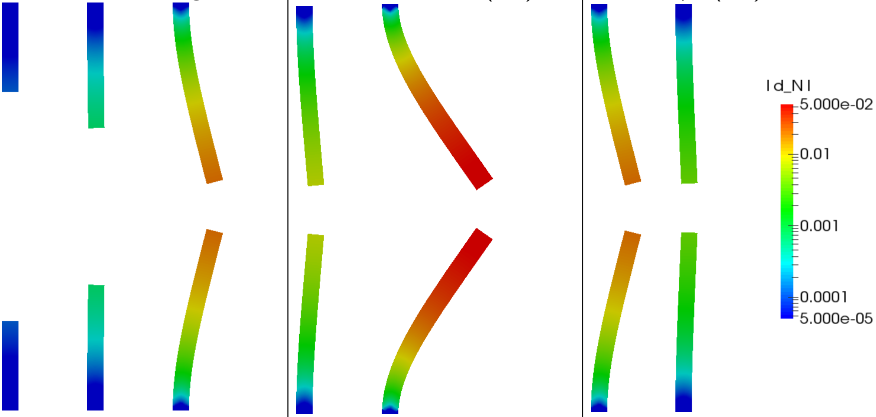
Increase inlet velocity: *(same leaflet length)*





Ongoing applications to cardiovascular modelling

Increase leaflet length: | *Increase inlet vel.* $(5\times)$*:* | *Increase* μ_s $(8\times)$*:*



(same inlet velocity, same material properties) | *(same leaflet length, same material properties)* | *(same leaflet length, same inlet velocity)*

- *Ballarin, Rozza. POD–Galerkin monolithic reduced order models for parametrized fluid-structure interaction problems. IJNMF, 82(12):1010–1034, 2016.*
- *F. Ballarin, G. Rozza, Y. Maday. Reduced-order semi-implicit schemes for fluid-structure interaction problems. MS&A, vol. 17, 2017. Springer [segregated approach]*

Reduction of Kolmogorov n-width

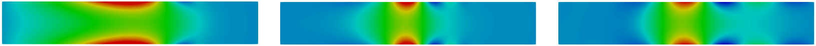
- Coupled problem: tube with a fluid, and solid walls at the top and at the bottom.
- Time $t \in [0, T]$ is the only parameter; the problem is transport dominated.

Example: pressure ($t = 0.0024, 0.006, 0.011$)*:*

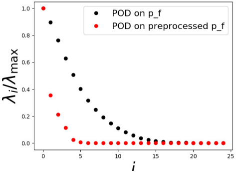
Pressure wave travelling through the domain, causing a slower decay of the Kolmogorov n-width of the solution manifold.

- Offline step (preprocessing): store the snapshots and then stretch them so that we move the peak of the pressure wave at the same point.

Example: preprocessed pressure ($t = 0.0024, 0.006, 0.011$)*:*



N. Cagniart, Y. Maday, B. Stamm. Model Order Reduction for problems with large convection effects. https://hal.upmc.fr/hal-01395571. 2016.

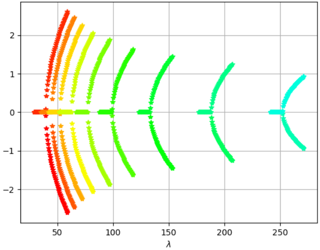


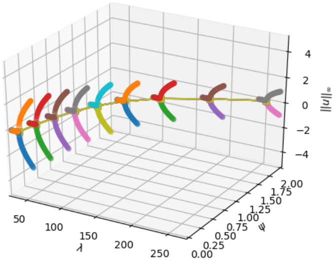
Results: comparison of the rate of decay of the singular values of the POD on the pressure

- Plateau around 10^{-4} with the preprocessing: the reason for this behaviour is still under investigation;
- testing different suitable stretching maps. They have to be smooth and invertible (possibly in an easy way);
- More FSI test cases.

#MS188
#Advanced #CFD & #Structural #Mechanics

ROM for Stability and Bifurcations Studies
with Martin Hess, Federico Pichi
Annalisa Quaini, Max Gunzburger and Anthony Patera

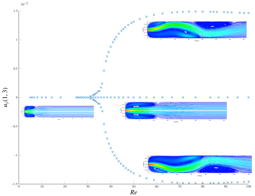


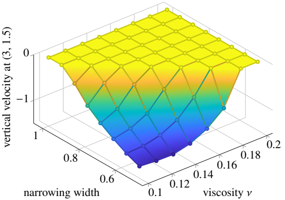


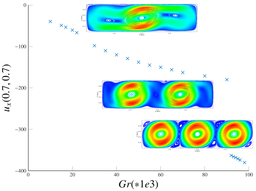
Bifurcation analysis with ROMs in fluid dynamics

Bifurcation and stability analysis of parametrized Navier-Stokes models by global and localized ROMs

- **Stability studies** in nonlinear problems are very expensive.
- Understand and detect complex phenomena, such as **bifurcations**, leading to loss of uniqueness with changing geometry and physical parameters, using spectral element simulations.
- Efficient reduced numerical techniques to detect **steady and Hopf bifurcations and branching**.
- continuation, eigenvalues analysis, long-term goal: use in multi-physics studies.





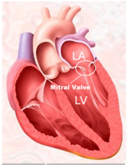


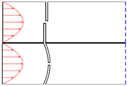
- this work is presented MS188

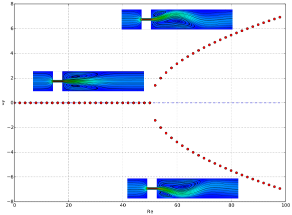
– M. Hess, A. Quaini, G. Rozza. "Reduced Basis Model Order Reduction for Navier-Stokes equations in domains with walls of varying curvature", 2019, ArXiv arxiv.org/abs/1901.03708
– M. Hess, A. Quaini, and G. Rozza, “A Spectral Element Reduced Basis Method for Navier-Stokes Equations with Geometric Variations”, 2018, ArXiv arxiv.org/abs/1812.11051
– M. Hess, A. Alla, A. Quaini, G. Rozza, and M. Gunzburger, “A Localized Reduced-Order Modeling Approach for PDEs with Bifurcating Solutions”, 2018, ArXiv arxiv.org/abs/1807.08851

Some ROM challenges in CFD

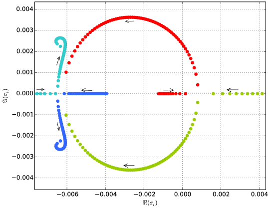
- **Complex CFD problems** in 3D setting characterized by bifurcations, e.g. Coanda effect during **mitral valves** regurgitation, influence of more complex geometries and multiphysics on bifurcations and stability.

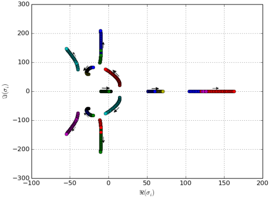






Investigations on bifurcations and loss of uniqueness of the solution require ROM for parametrized eigenvalue analysis. [Pitton, Rozza, 2017, Journal of Scientific Computing; Pitton, Quaini, Rozza, 2017, Journal of Computational physics]

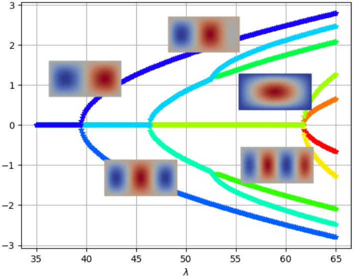


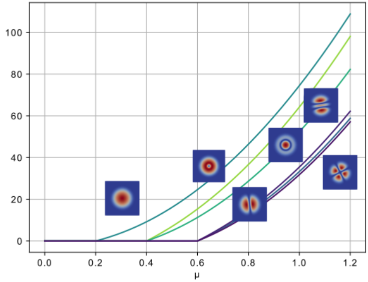


Some ROM challenges in Structural Mechanics

- **Computational mechanics problems** to study the deformation of a plate under compression (load λ and shape ψ) and the bifurcations of the Von Kaŕmań model through the linearized eigenproblem.

  [Pichi, F. , Rozza, G., 2018, Reduced basis approaches for parametrized bifurcation problems held by non-linear Von *Kármán* equations, arXiv:1804.02014]

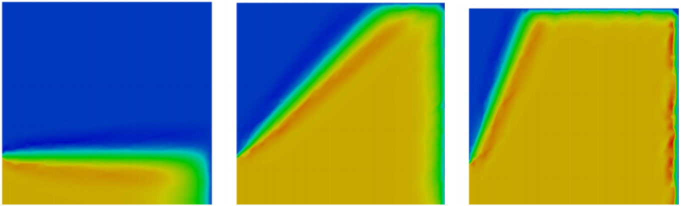




- Secondary bifurcations, better computations of parametric stability factors
- **Quantum mechanics problems** to study the Gross–Pitaevskii equation that describes the ground state of a quantum system of identical bosons.
  [Pichi, F. , Quaini, A., Rozza, G., 2019, Reduced deflation technique in bifurcating phenomena: application to the Gross–Pitaevskii equation, In progress]

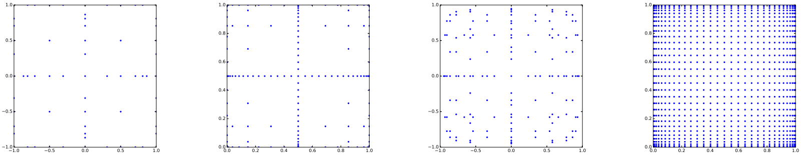
Further investigations on: Empirical Interpolation techniques, Neo-Hookean beam 2D/3D problem and a posteriori error estimate are in progress.
FVG - MIT project ROM2S

#UQ #CFD # FEM
Weighted reduced methods for parametrized problems with random inputs
with Francesco Ballarin, Davide Torlo and Luca Venturi



Uncertainty quantification problems with weighted reduced approach

- extension of (deterministic) **reduced order methods** to **stochastic PDEs**:
  - **weighting** to account for the **probability space**;
  - **sampling** from representative **probability distribution**;
  - exploitation of **sparse grids** to reduce the computational cost for **high dimensional parameter spaces** to break the curse of dimensionality;
- application to **advection dominated problems** by means of **reduced order stabilization techniques**.



L. Venturi, D. Torlo, F. Ballarin, and G. Rozza. “Weighted reduced order methods for parametrized partial differential equations with random inputs”. Uncertainty Modeling for Engineering Applications, Springer, 2019.

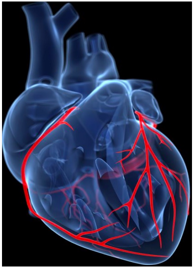
L. Venturi, F. Ballarin, and G. Rozza. “A weighted POD method for elliptic PDEs with random inputs”. Journal of Scientific Computing, in press, 2019.

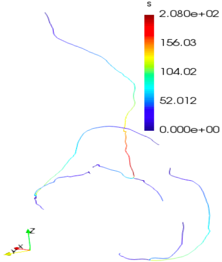
D. Torlo, F. Ballarin, and G. Rozza. “Stabilized weighted reduced basis methods for parametrized advection dominated problems with random inputs”. SIAM/ASA Journal on Uncertainty Quantification, 2018.

Former works in collaboration with A. Quarteroni and C. Peng

#MS130 #MS188 #Applications
ROMs # CFD # OFCPs # CAD # CABGs

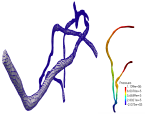
Parametrized reduced order optimal control for blood flows in patients' specific geometries
with Zakia Zainib, Francesco Ballarin
Piero Triverio, Laura Jiménez, Stephen Fremes



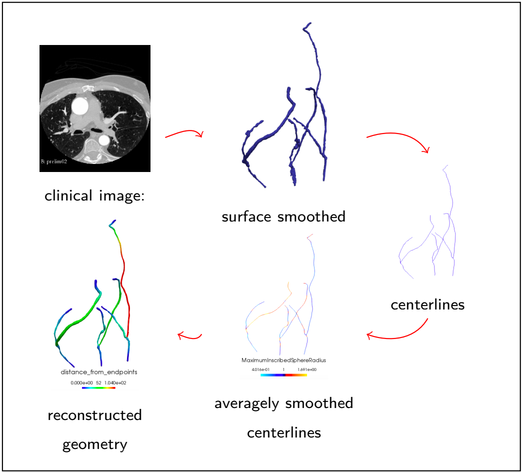


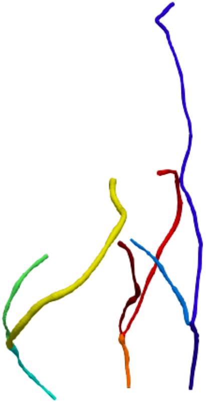
Mysteries of the heart: U of T Engineering professor developing solutions for coronary artery disease with mathematical models

Professor Piero Triverio (ECE) at centre, and collaborators Drs. Stephen Fremes (right), Laura Jiménez-Juan (left) of Sunnybrook Health Sciences Centre and Professor Gianluigi Rozza of SISSA are developing a study that could provide surgeons with better information about coronary artery disease (Credit: Jessica MacInnis)



Geometrical data assimilation: from Toronto partners





Smooth 3D geometry preserving patient-specific anatomical structure

Ballarin, Faggiano, Manzoni, Quarteroni, Rozza, Ippolito, Antona, and Scrofani, Biomech. Model. Mechan., 2017.

Ballarin, Faggiano, Ippolito, Manzoni, Quarteroni, Rozza, and Scrofani, J. Comput. Phys., 2016.

Physiological data assimilation

Problem description:

Navier-Stokes equations constrained **boundary control** with **physical parametrization**, and **geometrical and physiological data assimilation**.

$$\min_{(\boldsymbol{v},\boldsymbol{u})} \mathcal{J}\left(\boldsymbol{v}(\mu), p(\mu), \boldsymbol{u}(\mu)\right) = \frac{1}{2}\int_{\Omega} |\boldsymbol{v}(\mu) - \boldsymbol{v_d}|^2 + \frac{\alpha}{2}\int_{\Gamma_{out}} |\boldsymbol{u}(\mu)|^2$$

$$\text{subject to} \begin{cases} -\eta\Delta\boldsymbol{v}(\mu) + (\boldsymbol{v}(\mu)\cdot\nabla)\boldsymbol{v}(\mu) + \nabla p(\mu) = \boldsymbol{0}, & \text{in } \Omega \\ \nabla\cdot\boldsymbol{v}(\mu) = 0, & \text{in } \Omega \\ \boldsymbol{v}(\mu) = \boldsymbol{v}_{in}, & \text{on } \Gamma_{in} \\ \boldsymbol{v}(\mu) = \boldsymbol{0}, & \text{on } \Gamma_{wall} \\ \eta\nabla\boldsymbol{v}(\mu)\cdot\boldsymbol{n} - p(\mu)\boldsymbol{n} = \boldsymbol{u}(\mu) & \text{on } \Gamma_{out} \end{cases}$$

- Patient-specific computational domain Ω.
- Patient-specific physiological data $\boldsymbol{v_d}$ acquired through 4D-MRI.

The goal: rely on simplier Neumann boundary conditions, but tune $\boldsymbol{u}(\mu)$ to best match $\boldsymbol{v_d}$ acquired by measurement of the velocity profile.

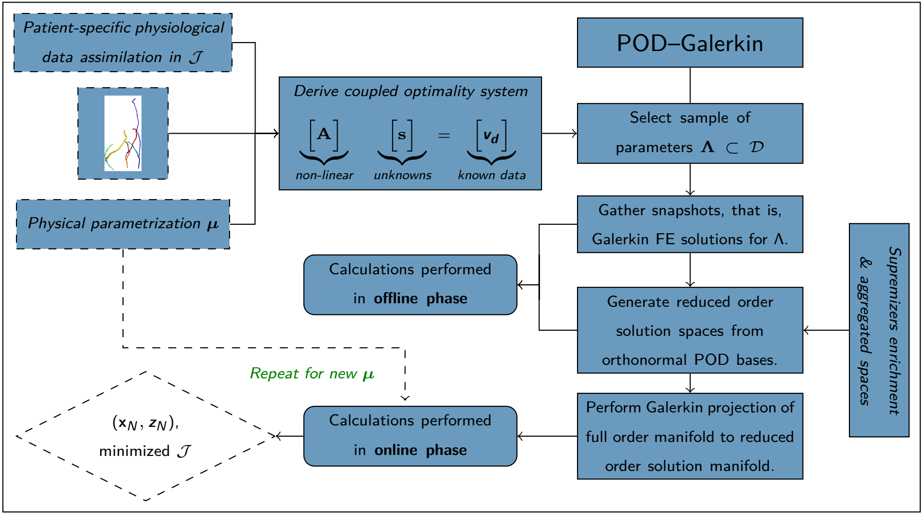


Figure: Reduced order optimal control pipeline: an overview

Negri, Manzoni, and Rozza, Comp. Math. App., 2015.

Negri, Rozza, Manzoni, and Quarteroni, SIAM J. Sci. Comp., 2013.

Reliability of reduced order model

Test case: Graft connection between RIMA and LAD, and saphenuous vein and OM1, Reynolds number as physical parameters.

$$\boldsymbol{\mu} = \left(\boldsymbol{\mu}_{\Gamma_{in_1}}, \boldsymbol{\mu}_{\Gamma_{in_2}}\right) \in ([45, 50], [70, 80])$$

$\Gamma_{in_1} :=$ inlets of vein graft and OM1.

$\Gamma_{in_2} :=$ inlets of RIMA and LAD.

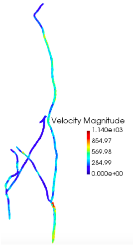


Figure: FE approx. of velocity

Figure: ROM approx. of velocity

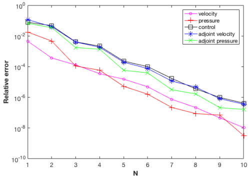


Figure: Relative error b/w FE and POD approx. of variables

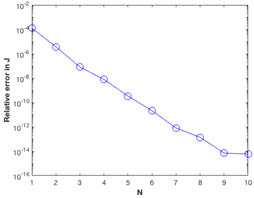


Figure: Relative error b/w FE and POD reduction of $\mathcal{J}$

Computational efficiency of reduced order model

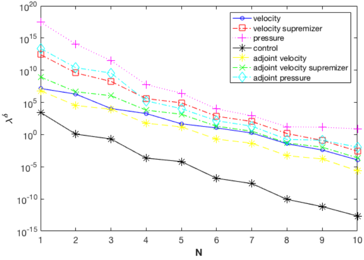


Figure: Eigenvalues' reduction

mathlab.sissa.it/rbnics
mathlab.sissa.it/multiphenics

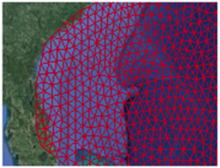
	FE approx.	ROM approx.
Mesh size	605451	-
Degrees of freedom	715462	73
CPU time (secs)	1848.13	202.27 (online)

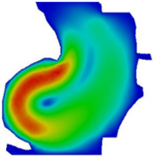
Table: Computational performance

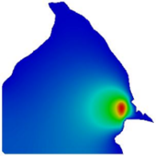
Zainib, Ballarin, Rozza, Triverio, Jiménez-Juan, and Fremes, Reduced order methods for parametric optimal flow control in coronary bypass grafts: patient-specific data assimilation and geometrical reconstruction. *In preparation*, 2019.

#Applications #Environmental #CFD #DataAssimilation #InverseProblems

Reduced Order Methods for Parametrized Optimal Flow Control in Environmental Marine Sciences with Maria Strazzullo and Francesco Ballarin

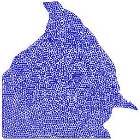






Pollutant Control on Gulf of Trieste, Italy

Motivations: forecasting, data assimilation, ecological and touristic and geographical interest.
Collaborations: National Institute of Oceanography and Applied Geophysics, OGS, Trieste, Italy.
Problem formulation
$y \in H^1_{\Gamma_D}(\Omega), u \in \mathbb{R}, y_d \in \mathbb{R}$ (safeguard threshold)



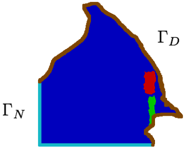
Weak formulation

Minimise with respect to
$(y(\boldsymbol{\mu}), u(\boldsymbol{\mu})) \in Y \times U$

$$\frac{1}{2}\int_{\Omega_{OBS}} (y(\boldsymbol{\mu}) - y_d)^2 \, d\Omega_y + \frac{\alpha}{2}\int_{\Omega_u} u(\boldsymbol{\mu})^2 \, d\Omega_u$$

constrained to an advection-diffusion state equation:

$$a(y(\boldsymbol{\mu}), q) = c(u(\boldsymbol{\mu}), q), \qquad \forall q \in Q.$$



Boundaries:
Γ_D = coasts, Γ_N = Adriatic Sea.
Subdomains:
Ω_{OBS} = Natural area of Miramare;
Ω_u = Source of pollutant (in front of the city of Trieste).

Weak Formulation of the Parametric Inverse problem

- $a : Y \times Q \to \mathbb{R}$:

$$a(y, q, \boldsymbol{\mu}) = \int_{\Omega} (\nu(\boldsymbol{\mu}) \nabla y \cdot \nabla q + \boldsymbol{\beta}(\boldsymbol{\mu}) \cdot \nabla y q) \, d\Omega,$$

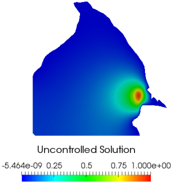
- $c : U \times Q \to \mathbb{R}$:

$$c(u, q) = L u \int_{\Omega_u} q \, d\Omega_u, \quad [L = 10^3 \to \text{ non-dimensional system}]$$

Parameters ($\mathcal{D} = [0.5, 1] \times [-1, 1] \times [-1, 1]$)

$\nu(\boldsymbol{\mu}) \equiv \mu_1$ is the diffusivity parameter,

$\boldsymbol{\beta}(\boldsymbol{\mu}) = [\beta_1(\mu_2), \beta_2(\mu_3)]$ is the transport field,

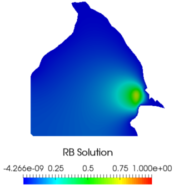


Uncontrolled Solution

Control and cost functional value for several parameters

	$\boldsymbol{\mu}$	u	J_r
No wind	(1,0,0)	$7.6901 \cdot 10^{-1}$	$5.1320 \cdot 10^{-5}$
Bora	(1,-1,1)	$7.3698 \cdot 10^{-1}$	$4.9167 \cdot 10^{-5}$
Scirocco	(1,1,-1)	$8.0800 \cdot 10^{-1}$	$5.3417 \cdot 10^{-5}$

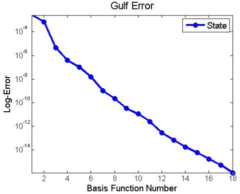
Time of a run: $t_{\mathcal{N}} = 2.79s, t_N = 2.41 \cdot 10^{-2} s$.
Dimensions: $\mathcal{N} = 5639$ and $N = 20$.

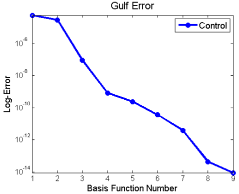


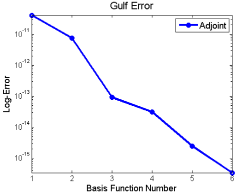
RB Solution

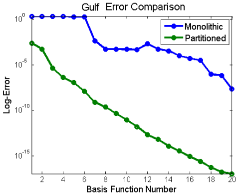
[Strazzullo et al., Model Reduction for Parametrized Optimal Control Problems in Environmental Marine Sciences and Engineering. SIAM SISC, 40:4, B1055-B1079, 2018]

Numerical Results: FE – POD Errors









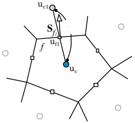
Parameter: $\boldsymbol{\mu} = (1, -1, 1)$.
Sampling distribution for POD: Uniform.
Training set dimension: 100.

Bora Errors. *Bottom left*: monolithic (one POD for $U(\boldsymbol{\mu}) = (y(\boldsymbol{\mu}), u(\boldsymbol{\mu}), q(\boldsymbol{\mu}))$) and partitioned (different POD reductions for state, control and adjoint variables) error comparison.

[Strazzullo et al., Model Reduction for Parametrized Optimal Control Problems in Environmental Marine Sciences and Engineering. SIAM SISC, 40:4, B1055-B1079, 2018]

#CFD #FV

ROM for Finite Volume Discretization of viscous flows with stable pressure with Giovanni Stabile, Saddam Hijazi, Andrea Lario and Matteo Zancanaro



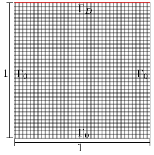
Why Finite Volumes?

- It became the standard for **real world applications** in several engineering fields (Aeronautics, Industrial flows, Automotive, Naval Engineering)
- For increasing Reynolds numbers there are less problems concerning stability and several **turbulence models** are already available.

Numerical examples

The lid driven cavity problem

The first proposed benchmark consists into the well known lid driven cavity problem:



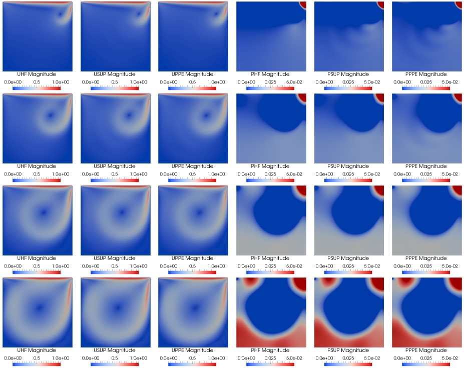
	Γ_D	Γ_0
$\boldsymbol{u}$	$\boldsymbol{u} = (1, 0)$	$\boldsymbol{u} = (0, 0)$
p	$\nabla p \cdot \boldsymbol{n} = 0$	$\nabla p \cdot \boldsymbol{n} = 0$

The mesh is structured and counts 40000 quadrilateral cells, 200 on each dimension of the square. The kinematic viscosity is equal to $\nu = 1 \times 10^{-4} \mathrm{m}^2/\mathrm{s}$ that leads to a Reynolds number of 10000.

In this case no parametrization is introduced.

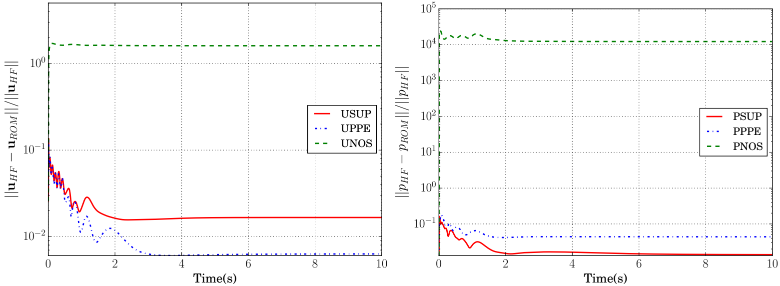
Comparison of the velocity and pressure fields for high fidelity, SUP-ROM and PPE-ROM.

The fields are depicted for different time instant equal to $t = 0.2\mathrm{s}, 0.5\mathrm{s}, 1\mathrm{s}$ and $5\mathrm{s}$.



Numerical examples

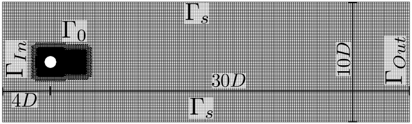
The L^2 norm of the relative error over time for three different models.

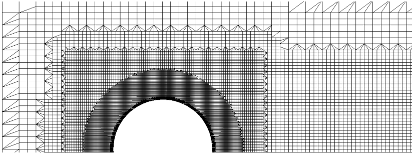


The table contains the cumulative eigenvalues for the lid driven cavity test. The last column contains the value of the inf-sup constant, in the supremizer stabilization case, for different different number of supremizer modes and with a fixed number of velocity and pressure modes.

N Modes	$\boldsymbol{u}$	p	$\boldsymbol{s}$	β
1	0.978946	0.975406	0.980260	9.264e-05
2	0.994184	0.991528	0.995232	9.264e-05
3	0.997737	0.995385	0.997912	7.175e-04
4	0.998990	0.998116	0.999400	7.175e-04
5	0.999483	0.999270	0.999844	7.175e-04
10	0.999971	0.999971	0.999997	1.551e-02

The flow around a circular cylinder



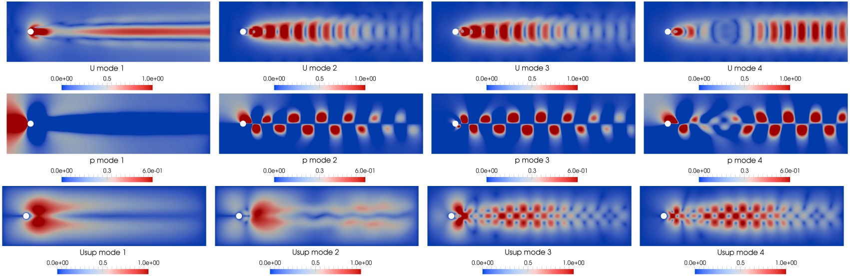


	Γ_{In}	Γ_0	Γ_s	Γ_{Out}
$\boldsymbol{u}$	$\boldsymbol{u} = (1,0)$	$\boldsymbol{u} = (0,0)$	$\boldsymbol{u} \cdot \boldsymbol{n} = 0$	$\nabla \boldsymbol{u} \cdot \boldsymbol{n} = 0$
p	$\nabla p \cdot \boldsymbol{n} = 0$	$\nabla p \cdot \boldsymbol{n} = 0$	$\nabla p \cdot \boldsymbol{n} = 0$	$p = 0$

The properties of the presented algorithms have been tested also with the benchmark of the **laminar flow around a circular cylinder**. In this case the viscosity have been parametrized and results refer to a parameter non experimented in the full order simulations. The parameter space is given by **5 different** values of the viscosity: $\nu \in [0.005, 0.01]$. These values of viscosity result into the values of the Reynolds number $\mathrm{Re} \in [100, 200]$.

Numerical examples

First four modes for velocity pressure and supremizers

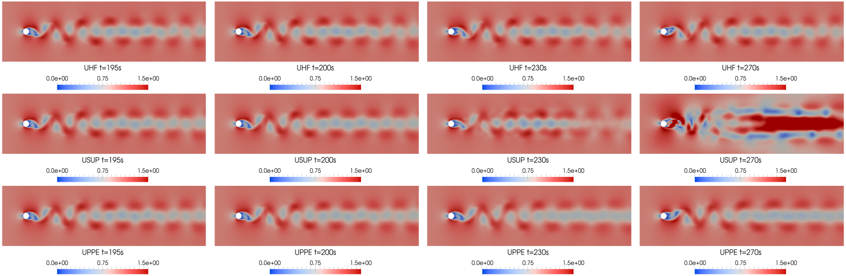


Cumulative eigenvalues

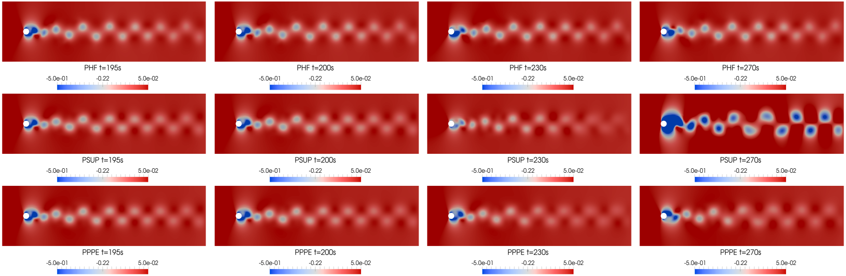
N Modes	$\boldsymbol{u}$	p	$\boldsymbol{s}$	β
1	0.390813	0.793239	0.921046	2.608e-04
2	0.598176	0.85809	0.941746	4.492e-04
3	0.802176	0.911636	0.961438	7.869e-03
4	0.879096	0.934997	0.978072	1.662e-02
5	0.949519	0.955578	0.98669	1.662e-02
10	0.986025	0.992347	0.998307	1.098e-01
15	0.995922	0.997994	0.999732	1.199e-01

Numerical examples

Comparison of the velocity field

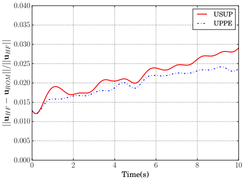


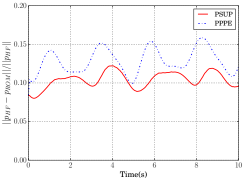
Comparison of the pressure field



Numerical examples

Comparison on the same time window ans computational costs





	Cavity Exp.	Cylinder Exp.
HF	25min	18.5min $\times$ 6proc.
SUP-ROM	7.64s	3.14s
PPE-ROM	4.86s	0.971s

- The **velocity** field is reproduced in a more accurate way using the **Poisson equation** approach. This is due to the "pollution" given by the non-necessary supremizer modes.
- On the other side the **pressure** field is better reproduced using a supremizer approach.
- The **cavity** example has run serially with OpenFOAM 5.0 (i7 laptop).
- The **cylinder** example has run in parallel with OpenFOAM 5.0.
- The reduced order models have run in serial in **ITHACA-FV**. It is available on github `https://github.com/mathLab/ITHACA-FV`.
- In the worst case the speed up is equal to approx. 200.

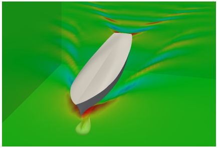
ITHACA - FV

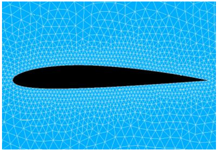
G. Stabile and G. Rozza, Stabilized Reduced order POD-Galerkin techniques for finite volume approximation of the parametrized Navier–Stokes equations, Computer & Fluids, 2018.

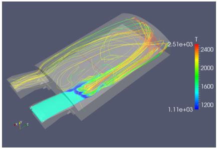
S. Ali, S. Hijazi, G. Stabile, F. Ballarin, G. Rozza, The effort of increasing Reynolds number in POD-Galerkin Reduced Order Methods: from laminar to turbulent flows, FEF, 2017, in press, 2019.

#CFD #turbulence #FV
#datadriven #ROM

ROM and Finite Volume Discretization
for fluid mechanics of turbulent flows
with Saddam Hijazi, Giovanni Stabile and Andrea Mola







Numerical results : Flow around a cylinder, unsteady case

- Results for the mixed **Data-Driven** and projection-based Reduced Order Model (DD-ROM) proved accuracy and efficiency compared to the ones obtained from a fully projection-based strategy.

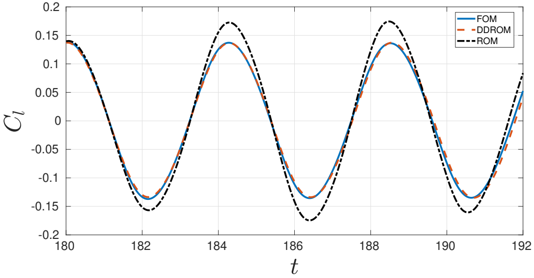


Figure: FOM,ROM and DD-ROM lift coefficients for the forces acting on the cylinder, in this case $Re = 10^4$. Turbulence model: K-omega.

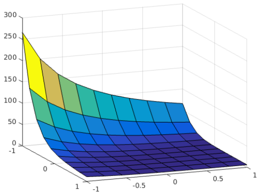
- DD-ROM relative error is in the range of $1-5$ %, while ROM has a relative error of 20%, $T_{CPU_{FOM}} = 525.32\ s$, $T_{CPU_{DD-ROM}} = 1.095\ s$, Speed up of 479.

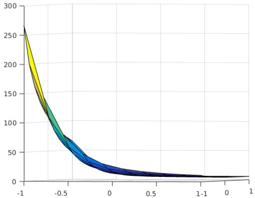
S. Hijazi, G. Stabile, A. Mola and G. Rozza (2018) Data-Driven POD–Galerkin reduced order model for turbulent flows POD–Galerkin reduced order model for turbulent flows, *In Preparation*

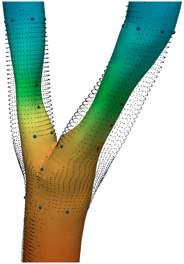
#Shape parametrisation #Active Subspaces #POD-Galerkin

Combined parameter and model reduction

with Marco Tezzele and Francesco Ballarin







Tezzele et al. “Combined parameter and model reduction of cardiovascular problems by means of active subspaces and POD-Galerkin methods”. 2018

Tezzele et al. “Dimension reduction in heterogeneous parametric spaces with application to naval engineering shape design problems”. 2018

Active subspaces property

In many cases the dimension of the parametrised problem is only artificially high

- Active subspaces property identifies a **set of important directions** in the space of all inputs

$$f : \mathbb{R}^m \to \mathbb{R} \qquad \mathbf{x} \in \mathbb{R}^m$$

$$\mathbf{C} = \mathbb{E}\left[\nabla_{\mathbf{x}} f \, \nabla_{\mathbf{x}} f^T\right] = \int (\nabla_{\mathbf{x}} f)(\nabla_{\mathbf{x}} f)^T \rho \, d\mathbf{x}$$

$$\mathbf{C} = \mathbf{W} \boldsymbol{\Lambda} \mathbf{W}^T$$

f is a scalar function that takes as arguments the parameters x

C is the uncentered covariance matrix of the gradients of f, symmetric, positive semidefinite

E is the expected value and rho a probability density function

- We define the active subspace to be the range of the first n eigenvectors of W

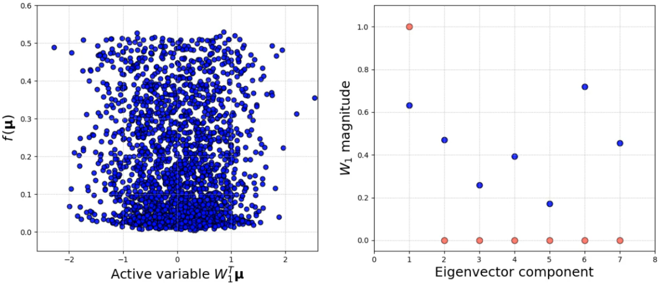
$$\mathbf{W} = [\mathbf{W_1} \quad \mathbf{W_2}] \in \mathbb{M}^{m \times m} \qquad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda_1} & \\ & \boldsymbol{\Lambda_2} \end{bmatrix}$$

- With the basis identified, we can map forward to the active subspace. So y is the active variable and z the inactive one. The surrogate model g is used to approximate f

$$\mathbf{y} = \mathbf{W_1^T} \mathbf{x} \in \mathbb{R}^n \qquad \mathbf{z} = \mathbf{W_2^T} \mathbf{x} \in \mathbb{R}^{m-n} \qquad f(\mathbf{x}) \approx g(\mathbf{W_1^T} \mathbf{x}) = g(\mathbf{y})$$

Constantine. “Active subspaces: Emerging ideas for dimension reduction in parameter studies.” SIAM, 2015.

Active Subspaces - A quadratic example

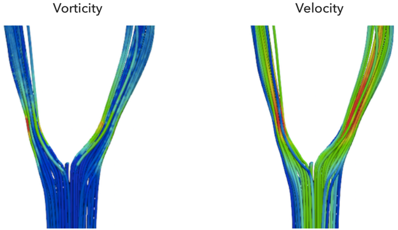


M. Tezzele, F. Ballarin and G. Rozza "Combined parameter and model reduction of cardiovascular problems by means of active subspaces and POD-Galerkin methods". 2018

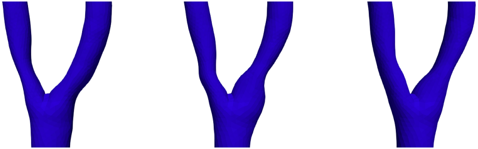
M. Tezzele, F. Salmoiraghi, A. Mola, G. Rozza. "Dimension reduction in heterogeneous parametric spaces with application to naval engineering shape design problems". 2018

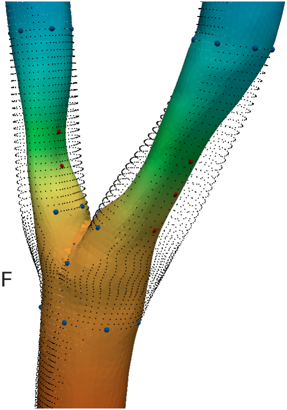
Flow across parametrised carotid bifurcations

- Vessels geometry strongly influences hemodynamics behaviour.
- The output function is the **relative pressure drop** of the two branches, computing the integral of the pressure on selected sections.



- We deform the carotid after the bifurcation moving 10 RBF control points (in red) solving an interpolation system.

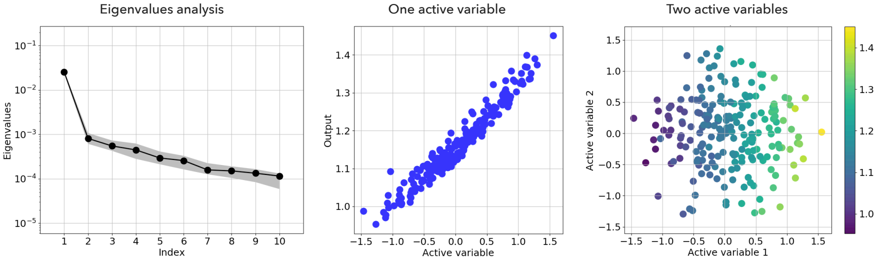




Deformed carotid with the deforming control points (red) and the undeformed state (black)

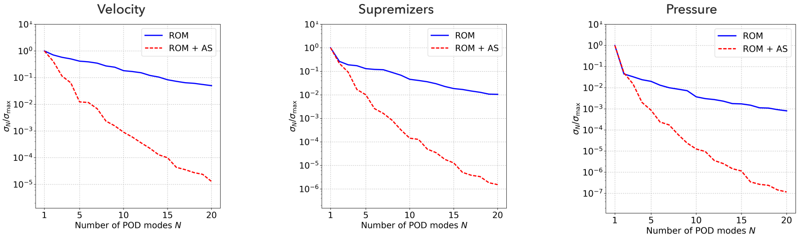
Spectral analysis

- The two dimensional active subspace spanned by the first two eigenvectors of the covariance matrix seems to better capture the behaviour of the output function. We use this information perform a further reduction by a **POD-Galerkin ROM**.
- We exploit a **2-dimensional active subspace to compute the POD snapshots** in a reduced space with respect to the full 10-dimensional parameter space.
- Typical reduced space dimensions and computational speedup for cardiovascular flows: **500:1**.

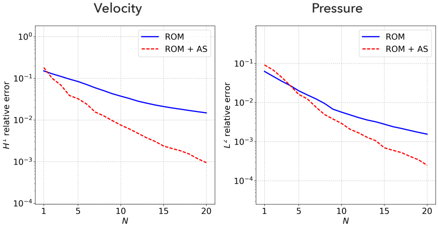


POD analysis

Here the POD singular values for velocity, supremizers and pressure, as a function of the number N of selected POD modes:

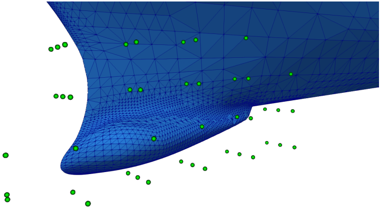


- The standard approach presents a slower decay, meaning that it has to deal with a considerably larger solution manifold.
- The combined methodology is able to reach relative errors which are up to **one order of magnitude smaller** when compared to the standard one, for both velocity and pressure when N = 20.



#GeometricalMorphing
#Industrial #applications #FFD
A full data-driven computational pipeline

with Marco Tezzele, Nicola Demo, Andrea Mola

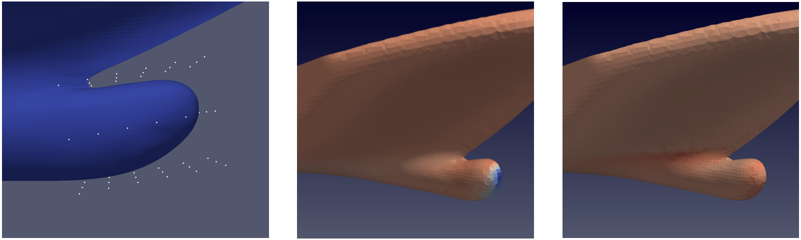


Hull optimization pipeline - From FFD to PODI

original hull → FFD → FV → DMD → PODI → OPT → optimized hull

- Given the time-average solutions, we combine the vectors containing the average force coefficients with the proper orthogonal decomposition (POD) interpolation technique implemented in EZyRB.
- Here we have the first POD modes and the optimized bulb.

PyDMD EZyRB

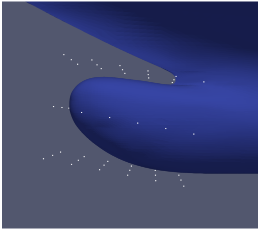


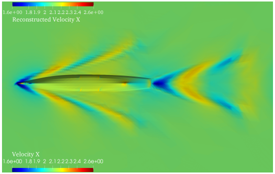
Reduced Order Model for industrial shape problems

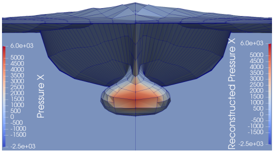
original hull → FFD → FV → DMD → PODI → OPT → optimized hull

In collaboration with Fincantieri, leader in cruise ship manifacturing, we developed an innovative pipeline involving **data-driven** reduced order modeling techniques for shape optimization in naval problems.

- **Shape parametrization** (FFD)
- **Proper orthogonal decomposition with interpolation**
- **Dynamic mode decomposition**

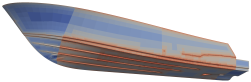


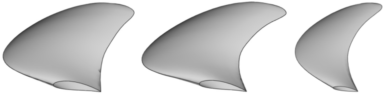




Reduced Order Model for industrial shape problems

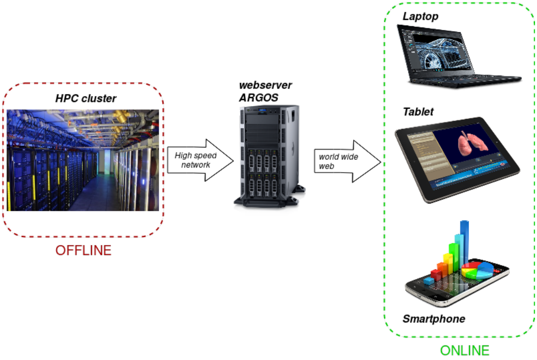
- **POR FESR: SOPHYA** the main goal of the project is to **improve planing yacht** hulls the performance in **non-calm sea** conditions. A set of specific methodologies have been developed to be able to **parameterize the hull** geometry and carry out a **shape optimization** campaign based both on high fidelity **RANS** and non-intrusive **ROM** simulations.
- **POR FESR: PRELICA** the main goal of the project is to **improve ship propeller** performance both in terms of **thrust** and **acoustic emissions**. A specific python package (**BladeX**) has been developed to generate **parametrized propeller** geometries. The optimal propeller shape has been identified making use of both high order **LES** and non-intrusive **ROM** hydroacoustic simulations.





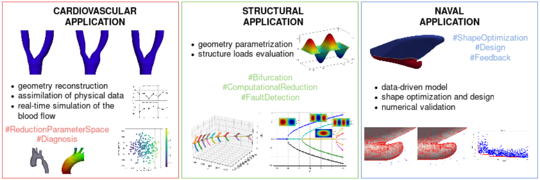
Vision and Perspective: to real-time computing

Model order reduction for web server: from biomedical to naval applications



CSE-Apps

- HPC, data science
- Web computing
- Digital twin
- 3D printing
- SMACT Industry4.0



Webserver for real-time computing

- **ARGOS** is a Web Platform that gives the opportunity to use the reduced order modelling techniques directly via web browser;
- it does not require any installation/configuration;
- it relies on the already developed libraries (ITHACA, ...);

Conclusion

- It is time to better integrate **Data, Modelling, Analysis, Numerics, Control, Optimization and Uncertainty Quantification** in a **new parametrized, reduced and coupled paradigm**.
- We need to draw the attention to the fact that **“Science and Industry advance with Mathematics”**.
- **Applied Mathematics as propeller for Innovation** and Technology Transfer by a new generation of computational scientists.

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Thanks for your attention!

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- **European Research Council Executive Agency**, ERC CoG 2015 AROMA-CFD, GA 681447, 2016-2021.
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- INDAM-GNCS 2015, "Computational Reduction Strategies for CFD and Fluid-Structure Interaction Problems"
- INDAM-GNCS 2016-2017 "Numerical methods for model order reduction of PDEs"
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- POR-FESR, 2014-2020, Regione Friuli Venezia Giulia, SOPHYA, PRELICA, UBE 2
- TRIM, INSEAN-CNR, 2016
- HPC resources: CINECA, INFN, SISSA-ICTP
- MIUR FARE-X-AROMA-CFD project
- MIT projects: "Probabilistic Multi-disciplinary Ship Design using Reduced Order Methods and Machine Learning Tools", "ROM2S Reduced Order Methods at MIT and SISSA".

Thanks for your attention!