Random Long Time Dynamics for Large Systems of Interacting Oscillators

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Overview:motivations

Aim of the talk

large time dynamics of systems of a large number N of interacting units

- Well known: large number of interacting units $\Longrightarrow$ PDE limit, when $N \to \infty$ and on a finite time horizon
- Is this PDE limit relevant for the large time dynamics of the finite N system? There is a clash behind the double appearance of *large* in the key sentence of this page
- In order to be concrete: focus on the limited set-up of interacting oscillators
- I will talk of (some sort of) mean field interactions:
  - exceptional results have been obtained with interactions of local type
  - but the mean field case allows to tackle a much larger spectrum of phenomenologies (phase transitions, pattern formations,...)
  - local interactions are often not the most natural from the modeling viewpoint, notably in life sciences

Focus on interacting rotator – Kuramoto model

$$\mathrm{d}\varphi_j(t) = \omega_j\,\mathrm{d}t + \frac{1}{N}\sum_{i=1}^{N} J_{i,j}\left(\varphi_j(t) - \varphi_i(t)\right)\mathrm{d}t + \sigma\,\mathrm{d}w_j(t) \qquad \text{(SD)}$$

for $j = 1, 2, \ldots, N$, where

1. $\sigma \geq 0$ and $\{w_j(\cdot)\}_{j=1,2,\ldots}$ are IID standard Brownian motions (*dynamical noise*, with law $\mathbf{P}$)
2. $J_{i,j}(\cdot) := -K_{i,j}\sin(\cdot)$, $K_{i,j} \geq 0$
3. ω_j are real numbers often chosen randomly as a sequence of independent identically distributed random variables (*disorder*, with law $\mathbb{P}$)

The $\varphi_j(t)$ are actually angles ($\to \varphi_j(t)\mathrm{mod}(2\pi) \in \mathbb{S}$) so $\{\varphi_j(\cdot)\}_{j=1,\ldots,N}$ may/should be viewed as a diffusion process on a manifold (degenerate if $\sigma = 0$) and: ω_j *natural frequencies*

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$$\mathrm{d}\varphi_j(t) = \omega_j \,\mathrm{d}t - \frac{1}{N}\sum_{i=1}^{N} K_{i,j} \sin\left(\varphi_j(t) - \varphi_i(t)\right) \mathrm{d}t + \sigma \,\mathrm{d}w_j(t)$$

1. Basic (stochastic) case: $\sigma = 1$, $K_{i,j} = K$, $\omega_j = \omega$ (hence $\omega = 0$ by a change of variables). This a statistical mechanics model: mean field plane rotators or mean field XY model
2. stochastic Kuramoto model: prototype model for synchronization phenomena
3. Inhomogeneous graph interaction

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3. Inhomogeneous graph interaction

For each of the items the scheme is:

- $N \to \infty$ limit for $t \in [0, T]$, T arbitrary but fixed
- Analysis of the case $T = T(N)$, with $\lim_{N\to\infty} T(N) = \infty$

Fundamental symmetry

The model is rotation invariant.

Basic stochastic case: mean field XY model

$K_{i,j} = K$, $\omega_j \equiv 0$ (and $\sigma = 1$):

1. The model has a unique invariant probability with a smooth density (with respect to the unifom measure on $\mathbb{S}^N$) and, for every initial condition, the law of the system converges to this invariant measure
2. The density of the invariant probability is proportional to

$$\exp\left(\frac{K}{N}\sum_{i,j=1}^{N}\cos(\varphi_i - \varphi_j)\right)$$

   Note that the rotation invariance is inherited by the invariant measure.

3. The stationary dynamics is stochastically reversible, that is the law of $\{\varphi_t\}_{t\in[0,T]}$ coincides with the law of $\{\varphi_{T-t}\}_{t\in[0,T]}$.
4. A reversible model is an *equilibrium statistical mechanics model* and a world of tools opens (at the expense of a richer phenomenology).

About representing φ

$$\mathrm{d}\varphi_j(t) = \frac{1}{N}\sum_{i=1}^{N} J\left(\varphi_j(t) - \varphi_i(t)\right)\mathrm{d}t + \sigma\,\mathrm{d}w_j(t) \tag{SD}$$

Useful tool for N large: the empirical measure

$$\nu_{N,t}(\mathrm{d}\theta) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t)}(\mathrm{d}\theta)$$

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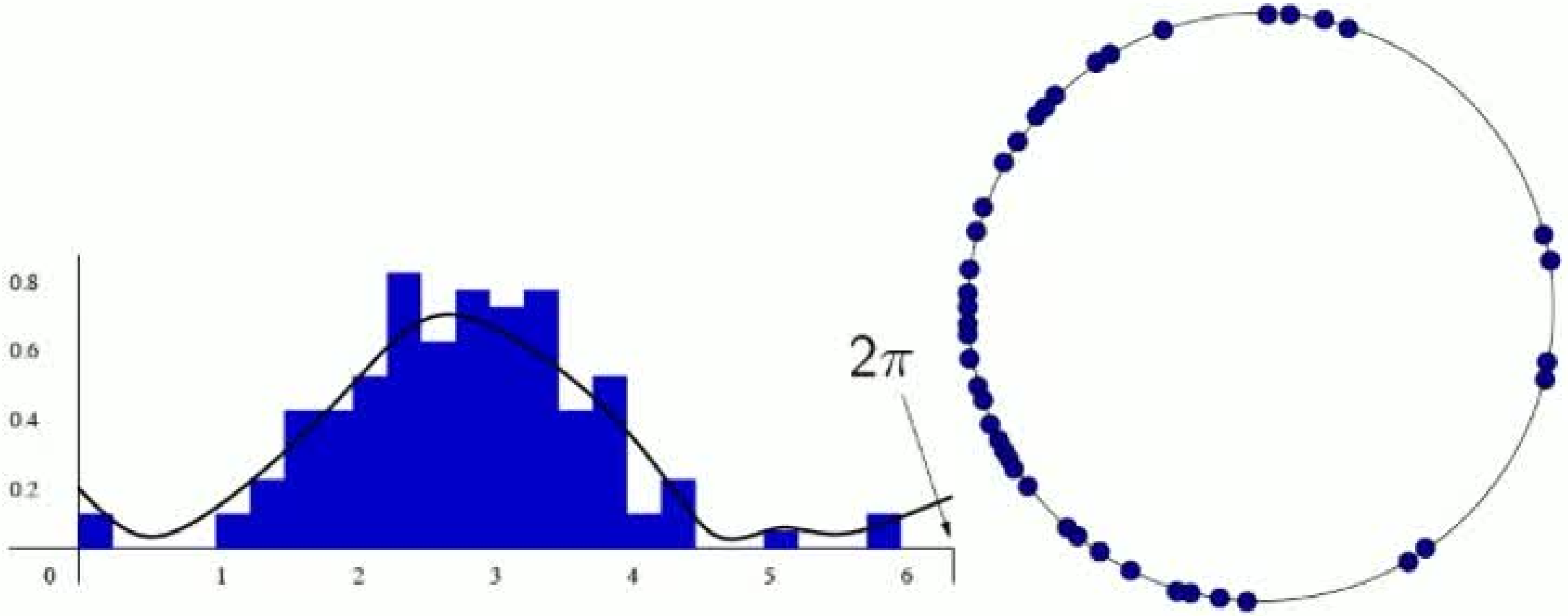

The empirical measure and the $N \to \infty$ limit

We start with

$$\mathrm{d}\varphi_j(t) = -\frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right) \mathrm{d}t + \sigma \,\mathrm{d}w_j(t) \qquad \text{(SD)}$$

If $\lim_{N\to\infty} \nu_{N,0}(\mathrm{d}\theta) = p_0(\theta)\,\mathrm{d}\theta$ then $\lim_{N\to\infty} \nu_{N,t}(\mathrm{d}\theta) = p_t(\theta)\,\mathrm{d}\theta$

$$\partial p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta\left[p_t(\theta)(J * p_t)(\theta)\right] \quad \text{with } J(\cdot) = -K\sin(\cdot) \qquad \text{(FP)}$$

Important observations

- No time rescaling
- The result holds for t finite, to be precise we can consider $\nu_{N,\cdot} \in C^0([0,T];\mathcal{M}_1)$, $\mathcal{M}_1$ space of probability measures with weak convergence, and the convergence is in this space (soft approach).
- (FP) inherits the rotation symmetry: $p_t(\cdot + \psi)$ solves (FP) too

A word about these $N \to \infty$ limits

General fact for mean field models

The evolution equation for the empirical measure is, for finite N, the (weak form of the) $N \to \infty$ PDE, plus (if $\sigma > 0$) a stochastic term whose quadratic variation vanishes as $N \to \infty$.

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- (FP) inherits the rotation symmetry: $p_t(\cdot + \psi)$ solves (FP) too
- The limit PDE result holds also for $\sigma = 0$

A word about these $N \to \infty$ limits

General fact for mean field models

The evolution equation for the empirical measure is, for finite N, the (weak form of the) $N \to \infty$ PDE, plus (if $\sigma > 0$) a stochastic term whose quadratic variation vanishes as $N \to \infty$. No mystery: just differentiate (in t) $\int h(\theta)\nu_{N,t}(\mathrm{d}\theta) = \frac{1}{N}\sum_{j=1}^{N} h(\varphi_j(t))$ and the weak form of the PDE appears.

Consequences:

1. If $\sigma = 0$ the PDE limit is therefore just the fact that (FP) is well-posed for initial data that are probability measures (Neunzert, Dobrushin 70s)
2. If $\sigma > 0$ the argument is slightly more involved (Oelschläger 80s)

Different approach ($\sigma \geq 0$): propagation of chaos (Kac, McKean, and many others).

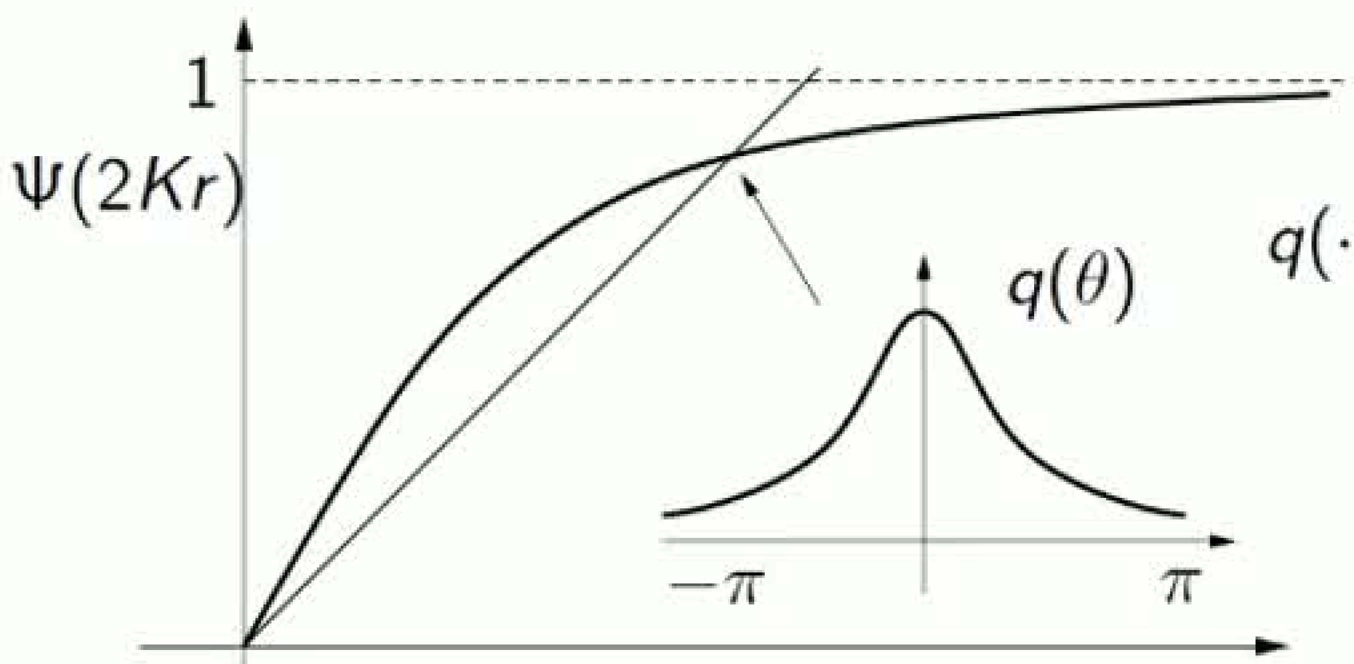
Long time behavior of the PDE

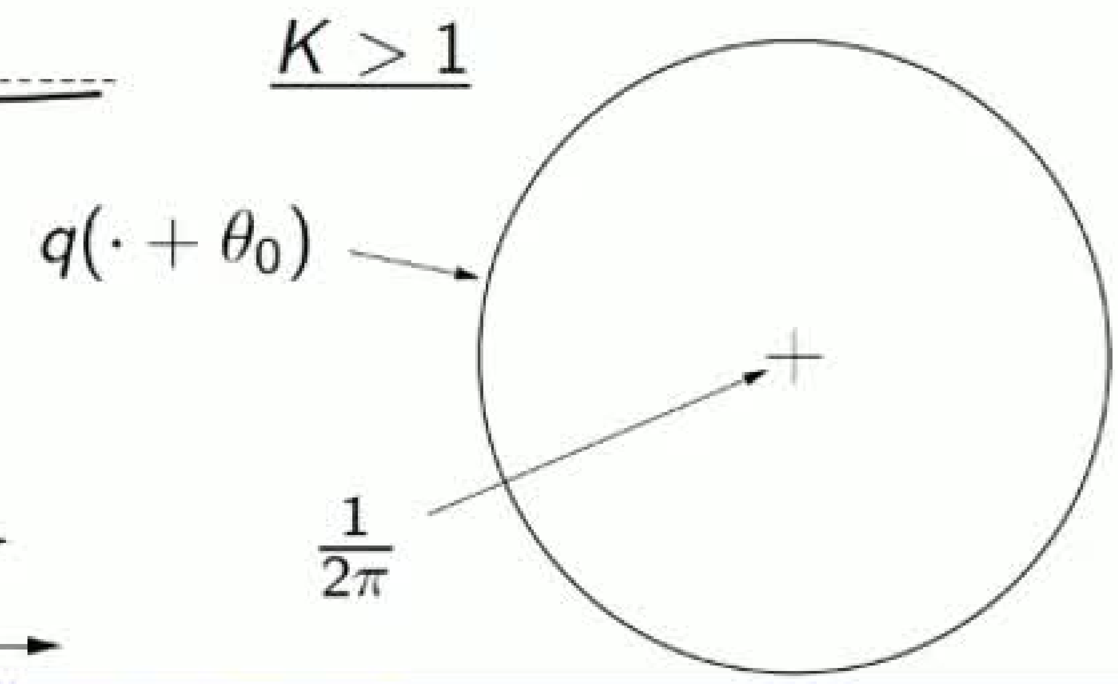
Two important facts for the (FP) PDE ($\sigma > 0$)

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta \left[p_t(\theta)(J * p_t)(\theta)\right] \qquad \text{(FP)}$$

1. Statmech: (FP) is the gradient flow of a free energy functional
2. All stationary solutions are up to rotation invariance

$$q(\theta) \propto \exp(2Kr\cos(\theta)), \qquad \text{with } r = \Psi(2Kr)$$ and $\Psi(\cdot)$ explicit with $\Psi'(0) = 1/2$





On the Fokker-Planck PDE [Bertini, G., Pakdaman 2010], [G., Pakdaman, Pellegrin 2012]

For $K > 1$ the stable manifold

$$M_0 := \{q(\cdot + \psi) =: q_\psi(\cdot) : \psi \in \mathbb{S}\}$$

attracts everything, except the stationary unstable profile $\frac{1}{2\pi}$ and whatever is attracted to it, that is

$$U := \left\{p : \int_{\mathbb{S}} p(\theta) \exp(i\theta)\, \mathrm{d}\theta = 0\right\}$$

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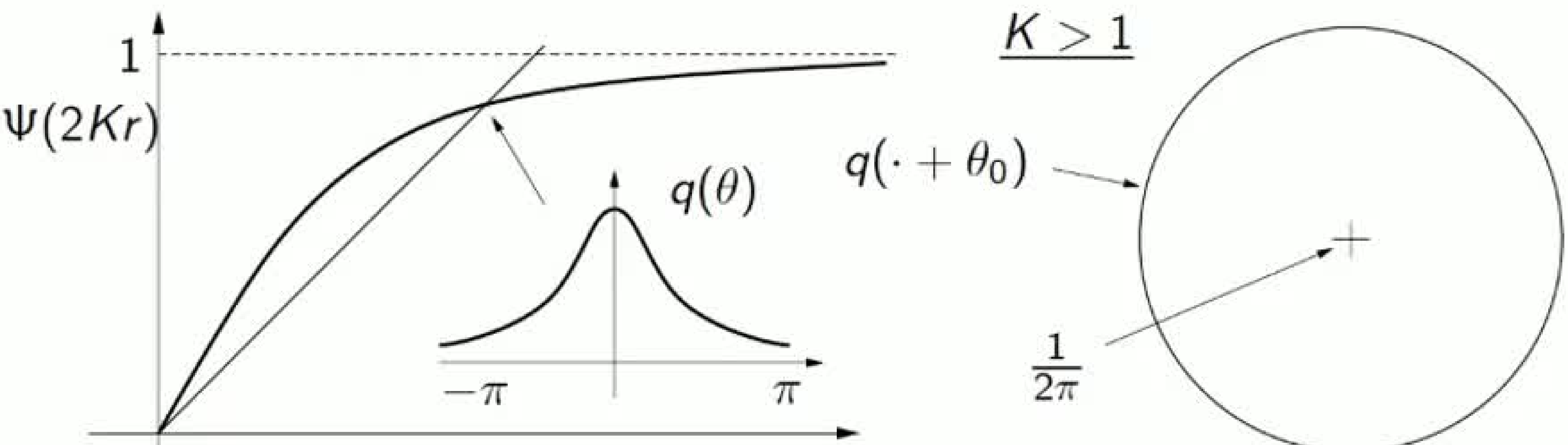

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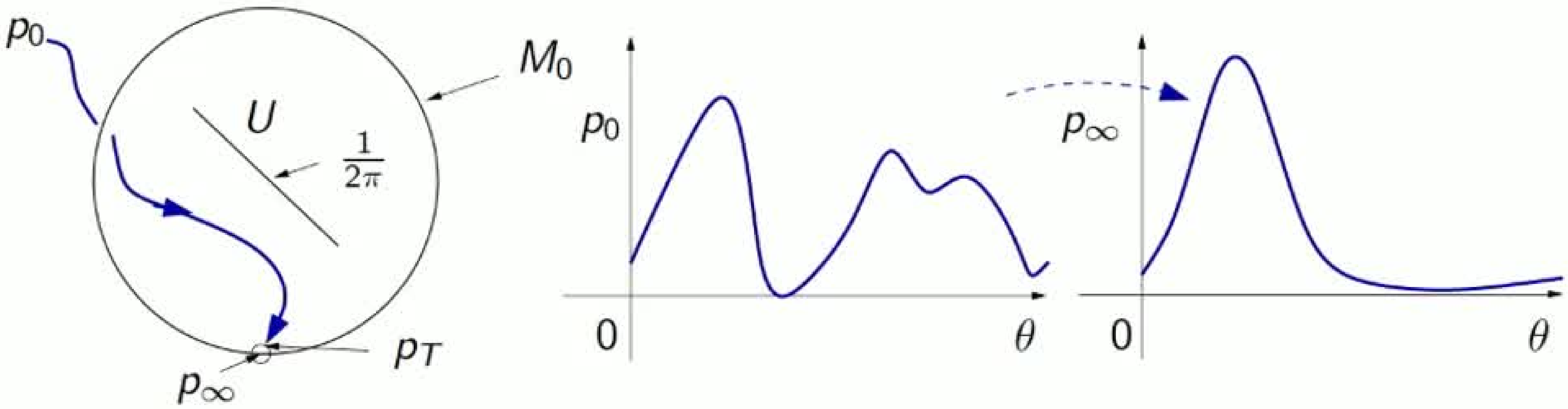

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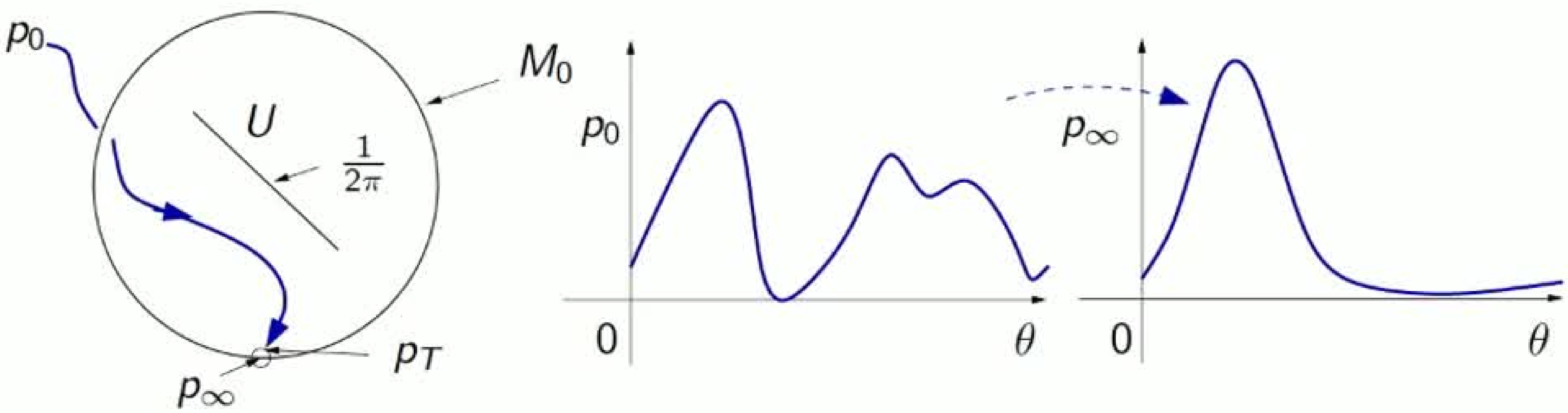
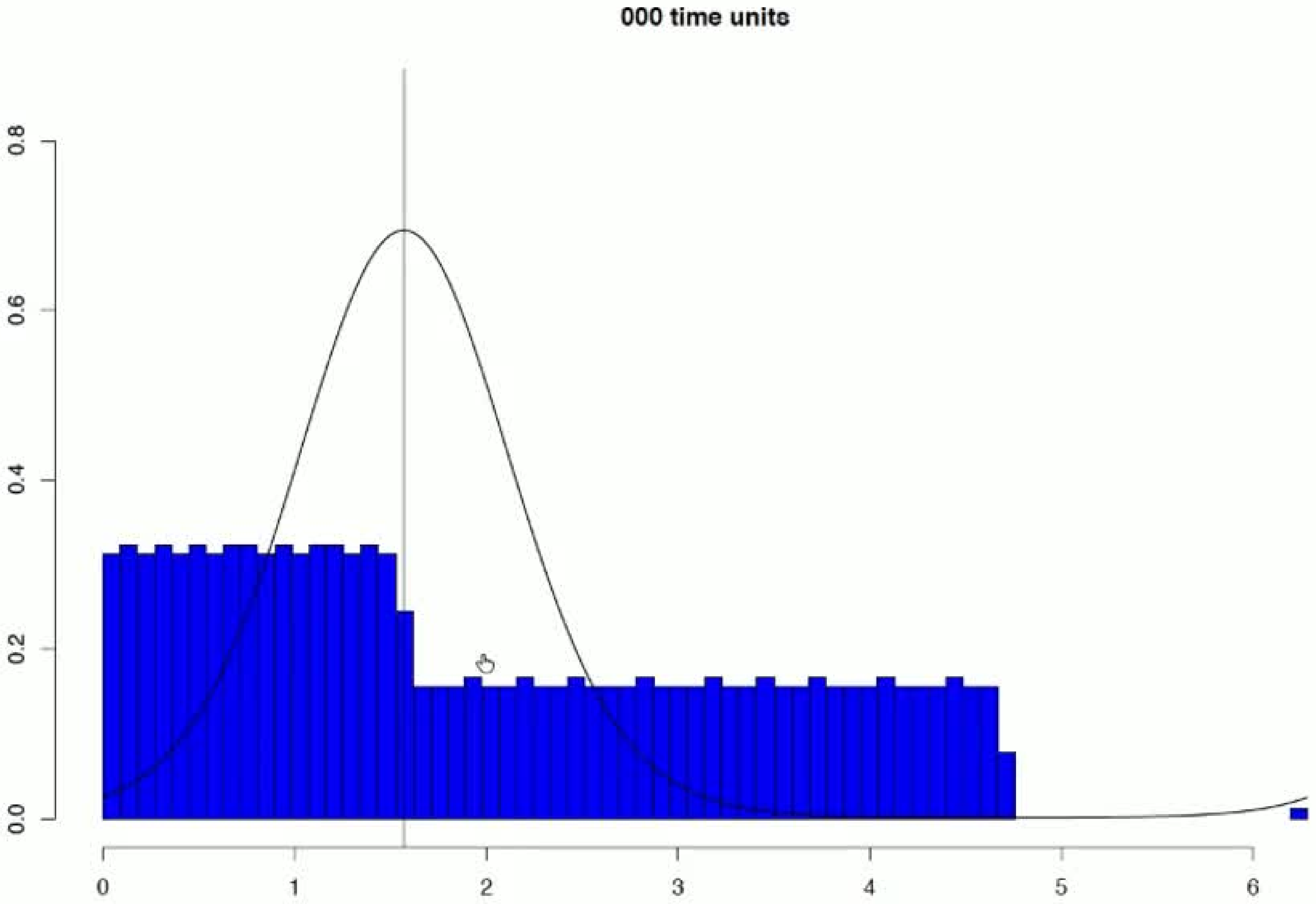


$N = 1000$, $K = 2$, $\sigma = 1$



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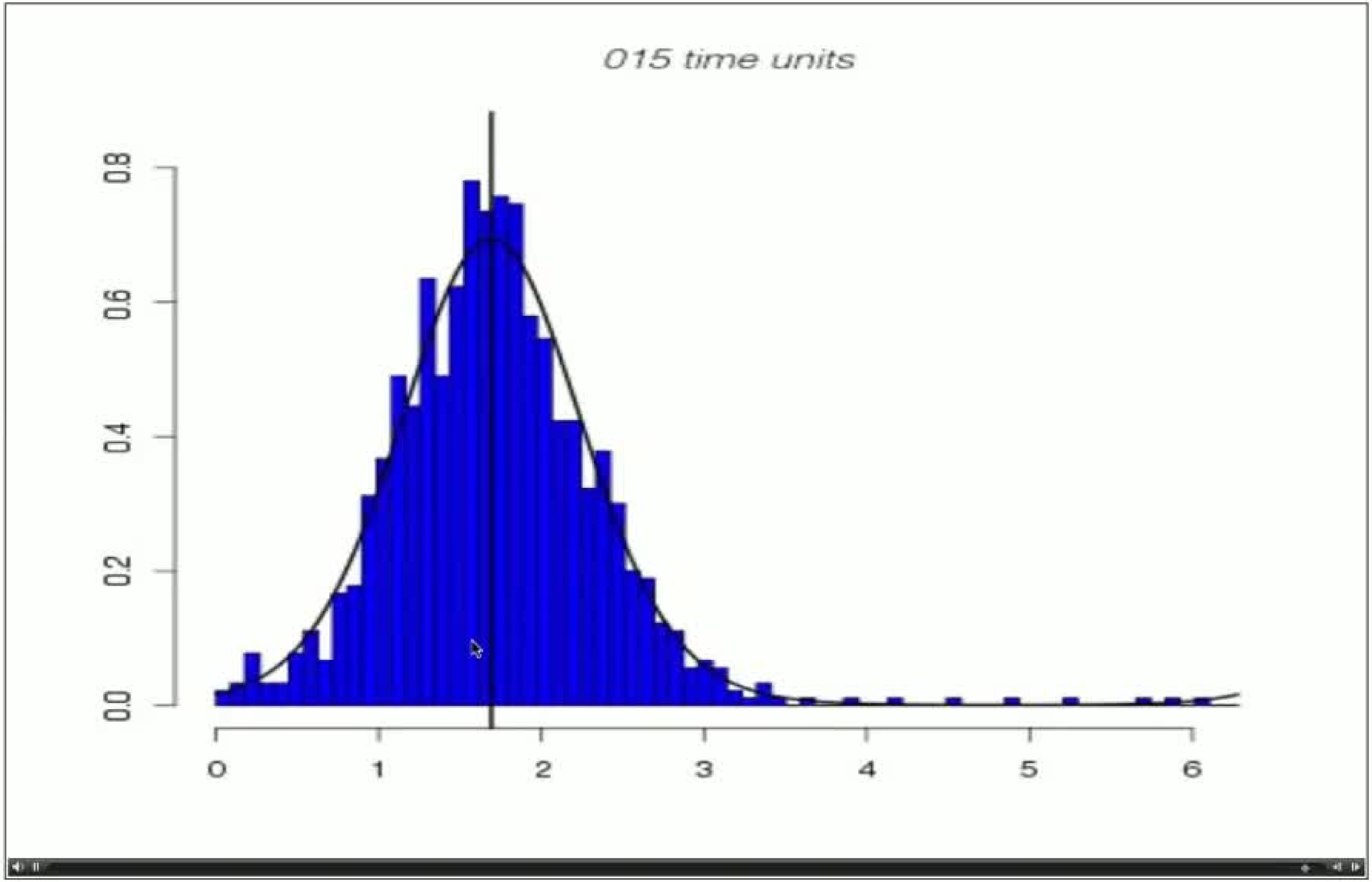

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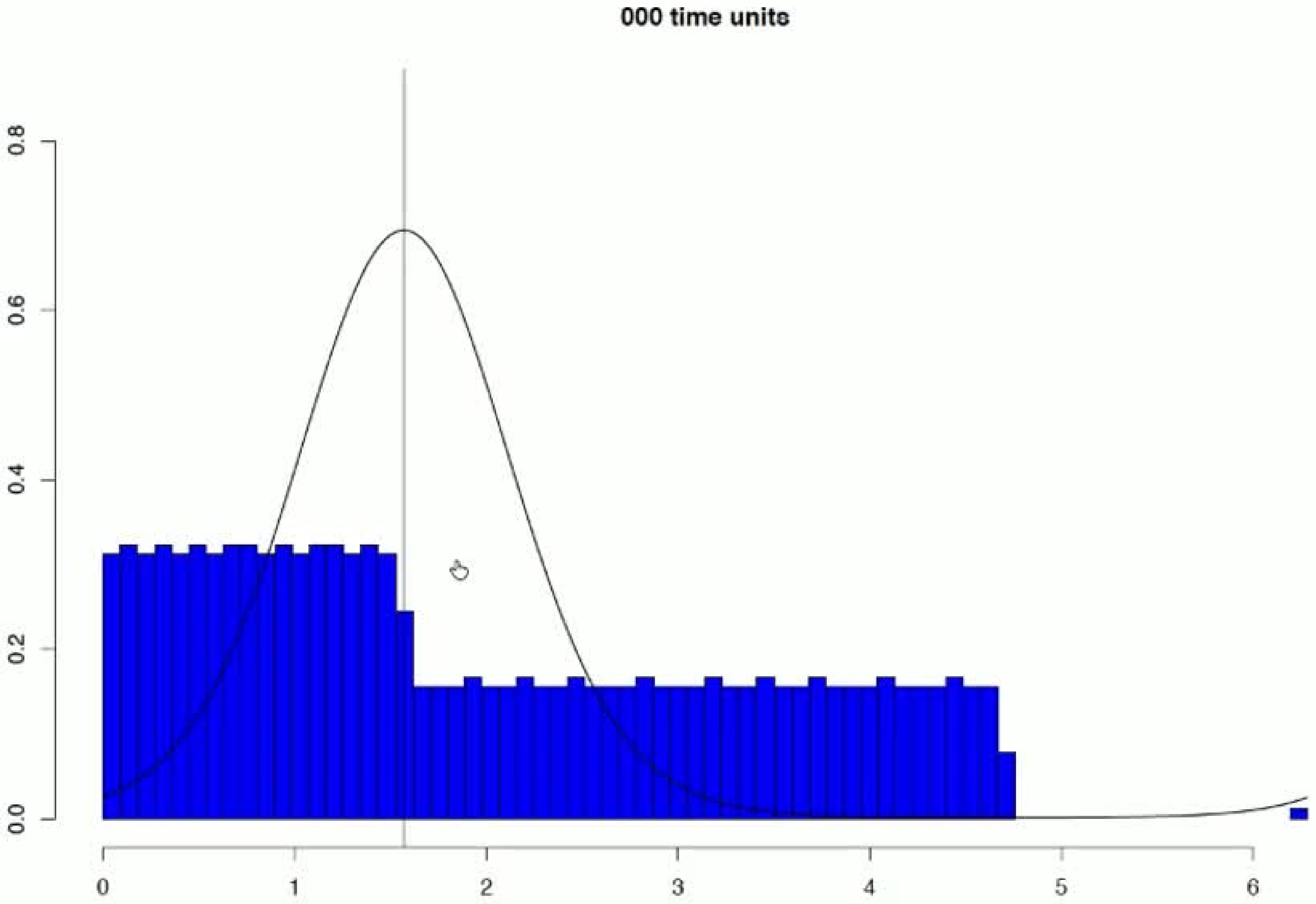

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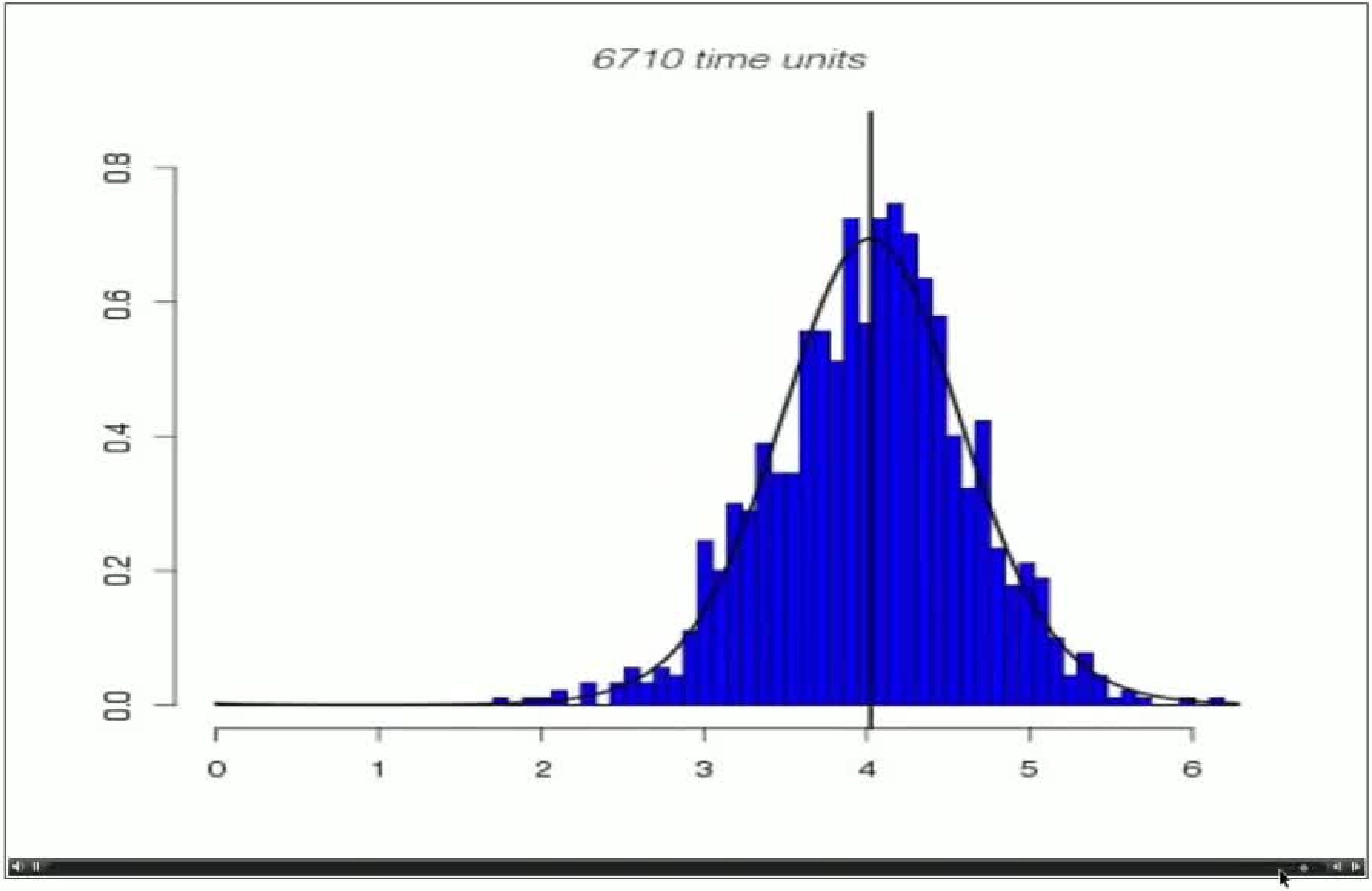

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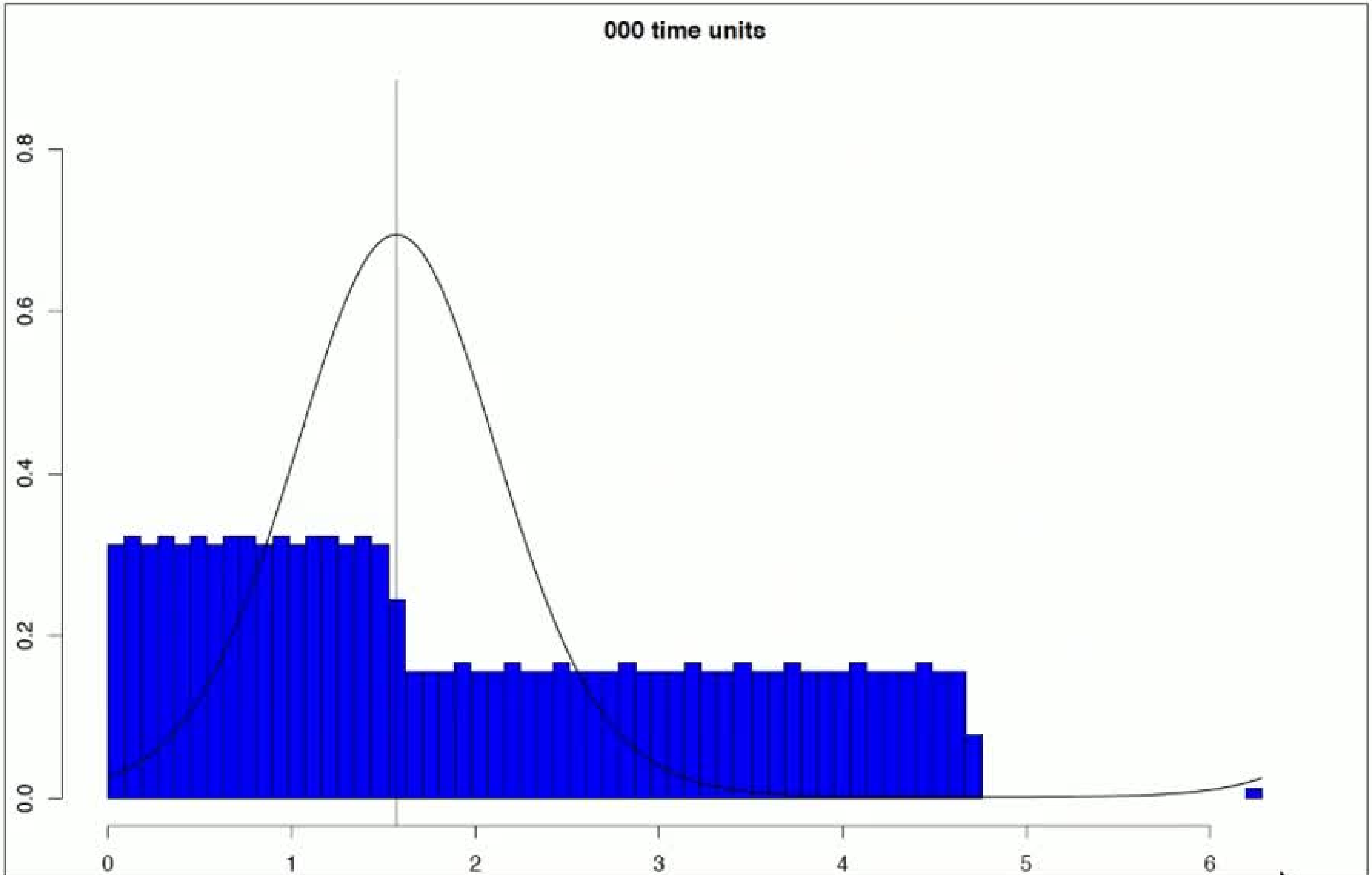

Theorem (Bertini, G, Poquet (PTRF 2014), Dahms (2002))

Set (w.l.o.g.) $\sigma = 1$ *and choose a positive constant* τ_f *and a probability density* $p_0(\cdot) \notin U$, *so* $\lim_{t\to\infty} p_t(\cdot) = q_{\psi_0}(\cdot)$. *If for every* $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,0} - p_0\|_{-1} \le \varepsilon \right) = 1$$

then for every $\varepsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[\varepsilon_N,\tau_f]} \left\|\mu_{N,\tau N} - q_{\psi_0 + D_K B^N_\tau}\right\|_{-1} \le \varepsilon \right) = 1$$

where $\varepsilon_N := C_\varepsilon / N$ *and* $B^N_\cdot$ *is adapted to the processes* $\tau \mapsto w_j(\tau N)$, $j = 1, \ldots, N$, *and converges to a standard Brownian motion for* $N \to \infty$. *Moreover*

$$D_K := \frac{1}{\sqrt{1 - (I_0(2Kr))^{-2}}}$$

with $I_0(\cdot)$ *the modified Bessel function of first kind and 0 order*

The stochastic Kuramoto model

The Kuramoto model is the mean field XY model with natural frequencies:

$$\mathrm{d}\varphi_j(t) = \omega_j \,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right)\mathrm{d}t + \mathrm{d}w_j(t)$$

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Major difference: the natural frequencies break the reversibility.

Nevertheless, the model is mean field and the evolution can be written in terms of the empirical measure

$$\nu_{N,t}(\,\mathrm{d}\theta,\,\mathrm{d}\omega) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t),\omega_j}(\,\mathrm{d}\theta,\,\mathrm{d}\omega)$$

and the FP PDE limit follows from the same proceudure as in the reversible case, but statistical mechanics structure is no longer available!

The stochastic Kuramoto model

For every $\theta \in \mathbb{S}$ and every ω in the support of the law μ of ω_1

$$\partial_t p_t(\theta, \omega) = \frac{1}{2}\Delta p_t(\theta, \omega) - \partial_\theta\Big(p_t(\theta, \omega)(\langle J * p_t\rangle_\mu(\theta) + \omega)\Big)$$

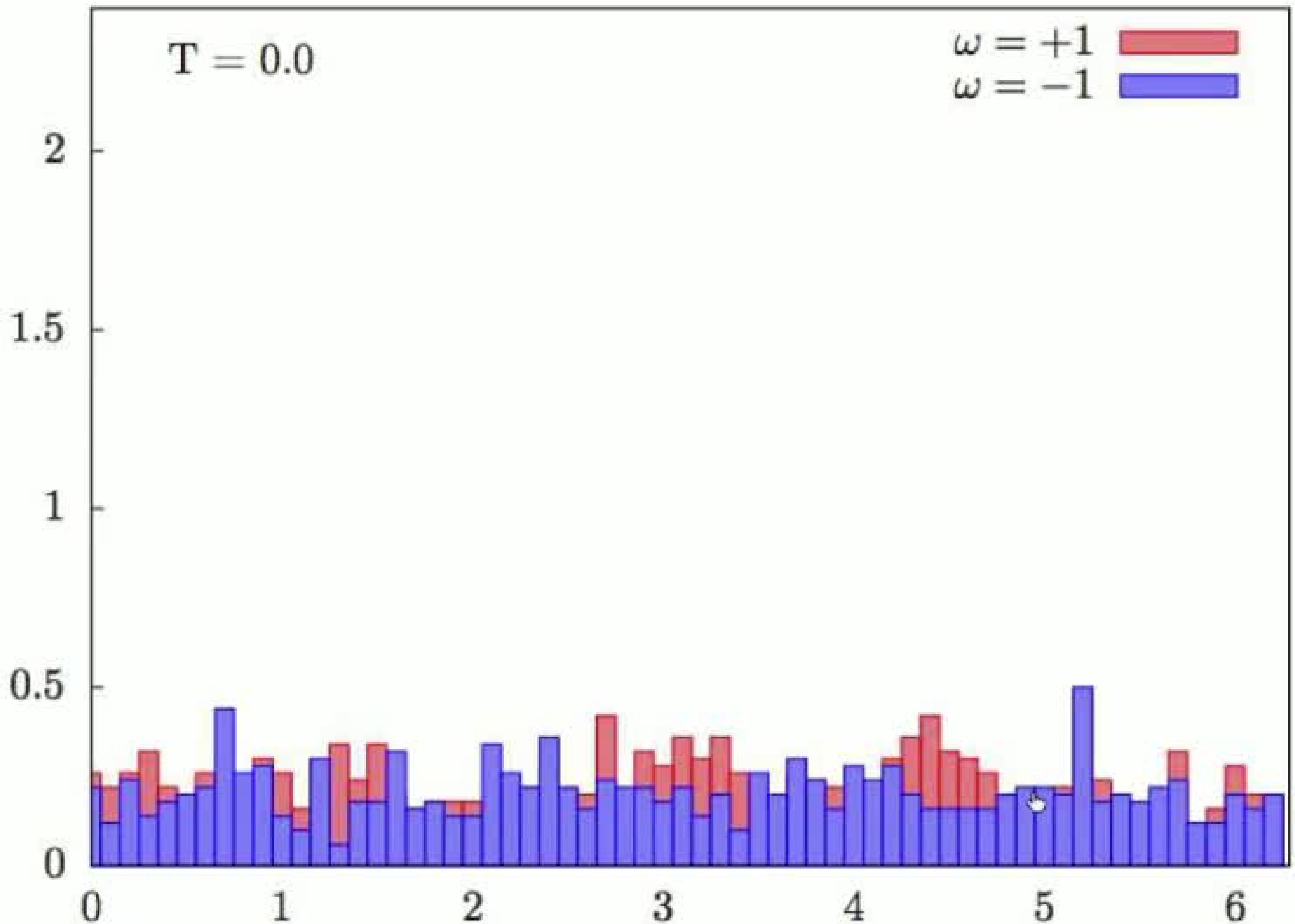
$$\langle J * u\rangle_\mu(\theta) = \int_{\mathbb{R}}\int_{\mathbb{S}} J(\theta - \theta')u(\theta', \omega)\, \mathrm{d}\theta\, \mu(\,\mathrm{d}\omega)$$

[Kuramoto 1975],. . . , [Dai Pra, den Hollander 1996, Luçon 2011]

PDE is substantially more complex than the reversible case, but all stationary measures can be written if μ is symmetric in a way that is formally equivalent to the reversible case, but the fixed point problem is considerably harder.

Nevertheless one can go very far when the support of μ is discrete and in $[-\delta, \delta]$, for δ small [Luçon and Poquet AIHP 2017]: PDE validity breaks down on time scale $\sqrt{N}$

$N = 700$ ($\sqrt{700} = 26.45\ldots$, $K = 5$, $\sigma = 1$



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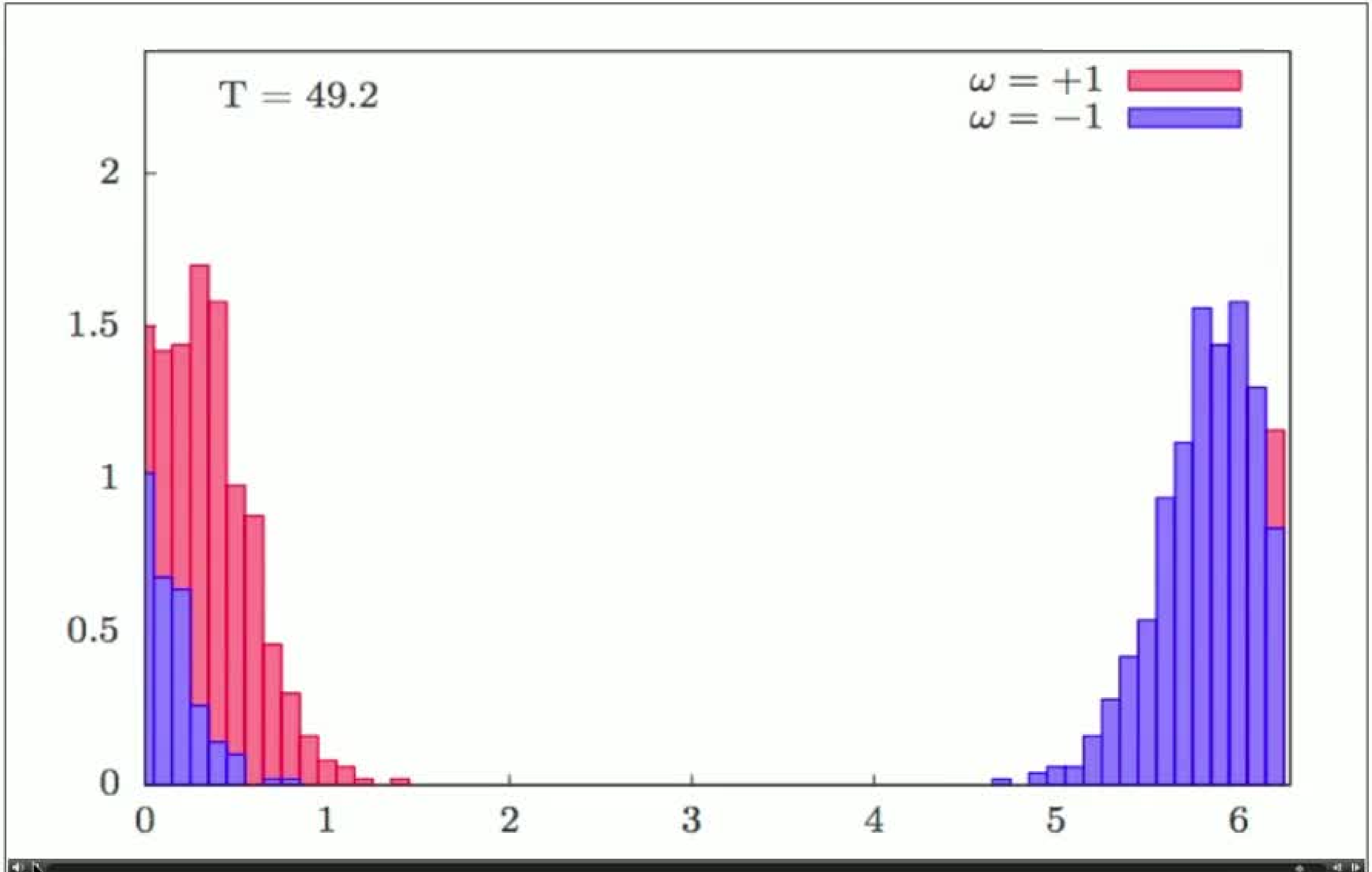

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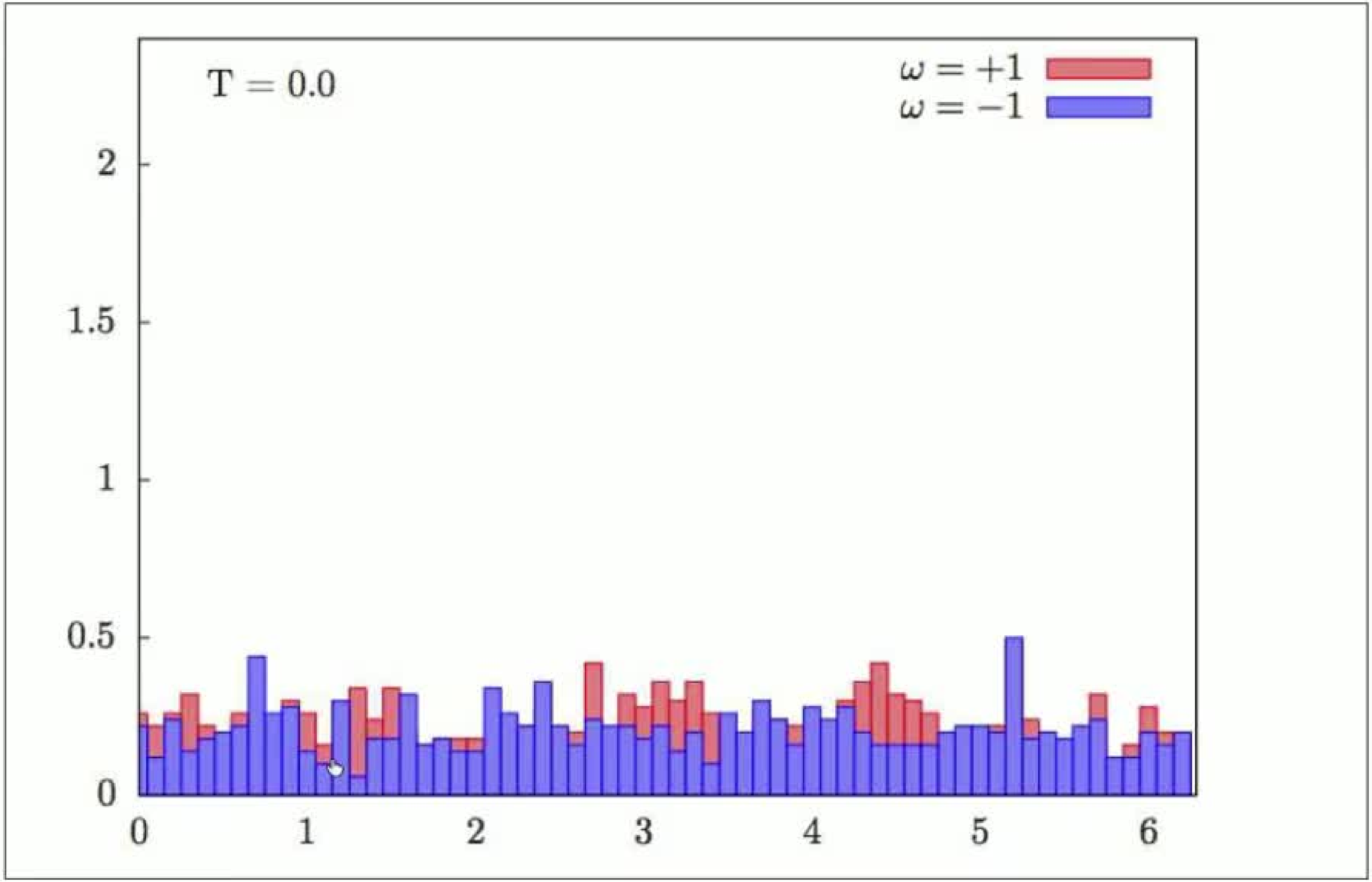

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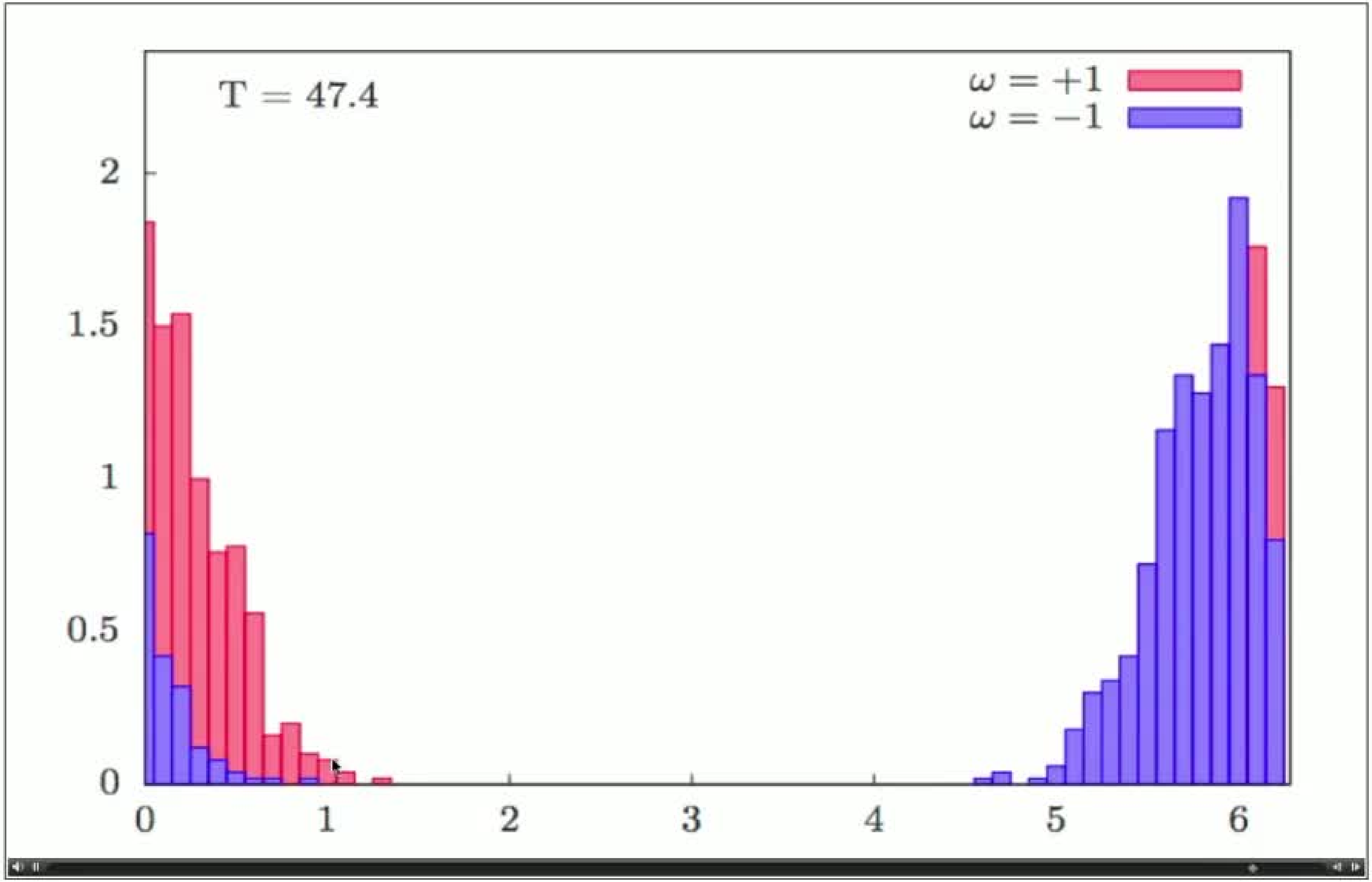

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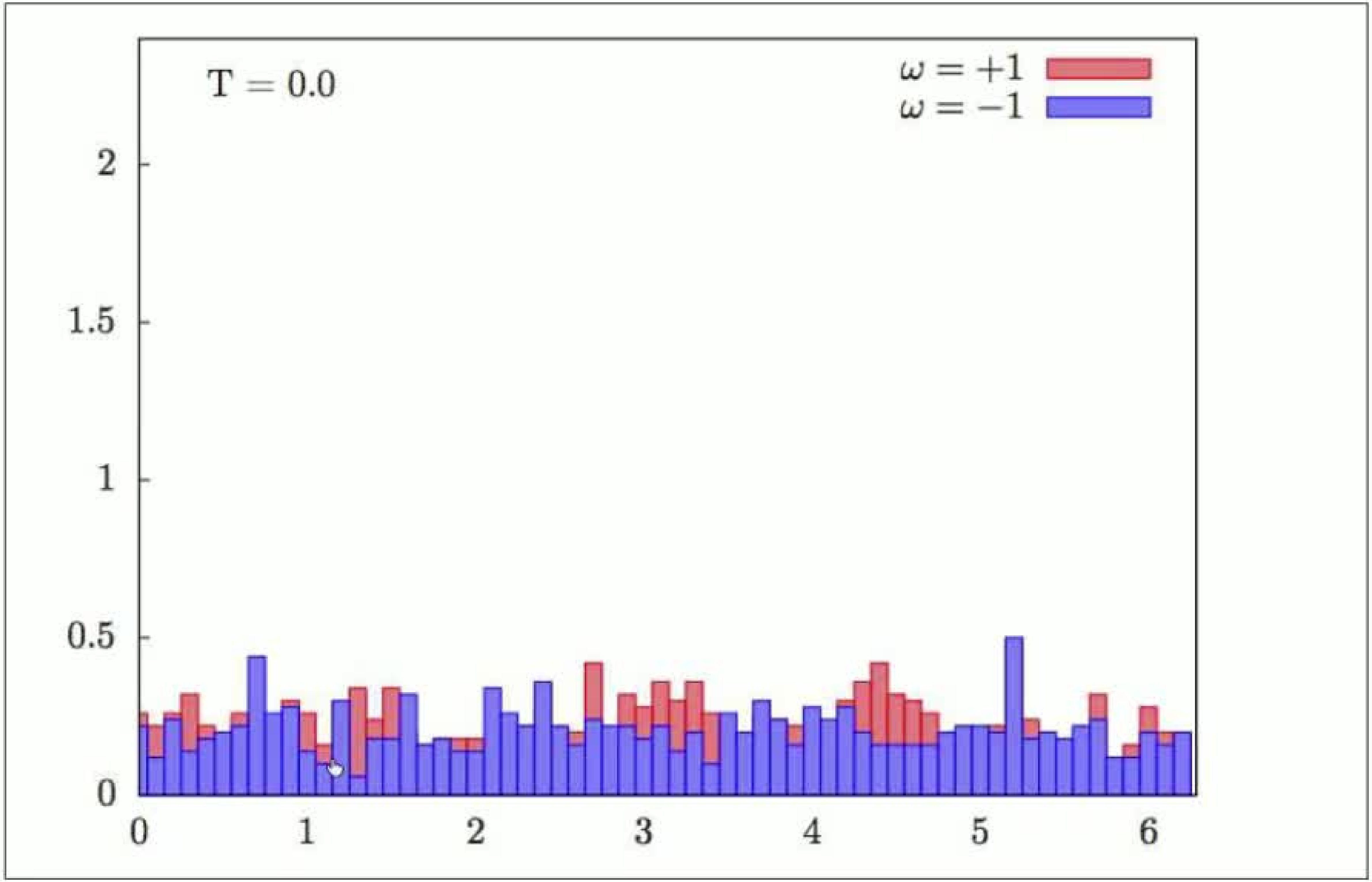

The inhomogeneous graph case

Just one example from [Delattre, G., Luçon JSP2016]

$$\mathrm{d}\varphi_j(t) = \cancel{\omega_j \mathrm{d}t} - \frac{1}{N}\sum_{i=1}^{N} K_{i,j} \sin\left(\varphi_j(t) - \varphi_i(t)\right) \mathrm{d}t + \sigma \,\mathrm{d}w_j(t)$$

with $K_{i,j} = K\xi_{i,j}$ and $\{\xi_{i,j}\}_{1\leq i,j\leq N}$ an adjacency matrix:

$$\xi_{i,j} \in \{0,1\} \text{ and } \xi_{i,j} = 1 \;\text{ if and only if } i \longrightarrow j$$

Call $d_i = \sum_j \xi_{i,j}$ of the degree of i and assume that

$$\lim_N \sup_{i=1,\dots,N} \left| \frac{d_i}{N} - \frac{1}{2} \right| = 0$$

Example: $\{\xi_{i,j}\}_{i,j}$ IID random variable of mean 1/2

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Nevertheless the propagation of chaos argument works and one can show that if $\{\varphi_j(0)\}_{j=1,\ldots,N}$ are independent and identically distributed with density $p_0(\theta)$ (strong hypothesis on initial condition!)

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Example: $\{\xi_{i,j}\}_{i,j}$ IID random variable of mean $1/2$

1. Seems awfully close to the mean field case with K replaced by $K/2$
2. Cannot write the evolution via empirical measure

The inhomogeneous graph case

Nevertheless the propagation of chaos argument works and one can show that if $\{\varphi_j(0)\}_{j=1,\ldots,N}$ are independent and identically distributed with density $p_0(\theta)$ (strong hypothesis on initial condition!) then the empirical measure converges as $N \to \infty$ (in the same sense as before, that is for $t \in [0, T]$) to the solution of

$$\partial p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta \left[p_t(\theta)(J * p_t)(\theta)\right] \qquad \text{(FP)}$$

with $J(\cdot) = -(K/2)\sin(\cdot)$. Other versions of this result in [Bhamidi, Budhiraja, Wu], [Chiba, Medvedev], just to mention mathematical results

I do not know how to push this result beyond times $O(\log N)$. But this is not necessarily bad because the result is VERY troublesome!

One very troublesome example: N even, $\xi_{i,j} = 1$ if and only if

$$i, j \in \{1, \ldots, N/2\} \quad \text{or} \quad i, j \in \{N/2+1, \ldots, N\}$$

In the sense of chaos propagation a system made of two disconnected K mean field systems is indistinguishable from a $K/2$ mean field system!

Sum-up and conclusions

- I have presented some systems of many interacting units (oscillators) and addressed the issue of how faithfully the PDE that emerges in the $N \to \infty$ limit captures the behavior of the finite N system.
- We have seen that on suitable time scales stochastic effects become macroscopic. For stochastic systems this is not surprising (Large Deviations!), but the deviations we presented happen on a much shorter time scale than the $\exp(cN)$ scale of Large Deviations.

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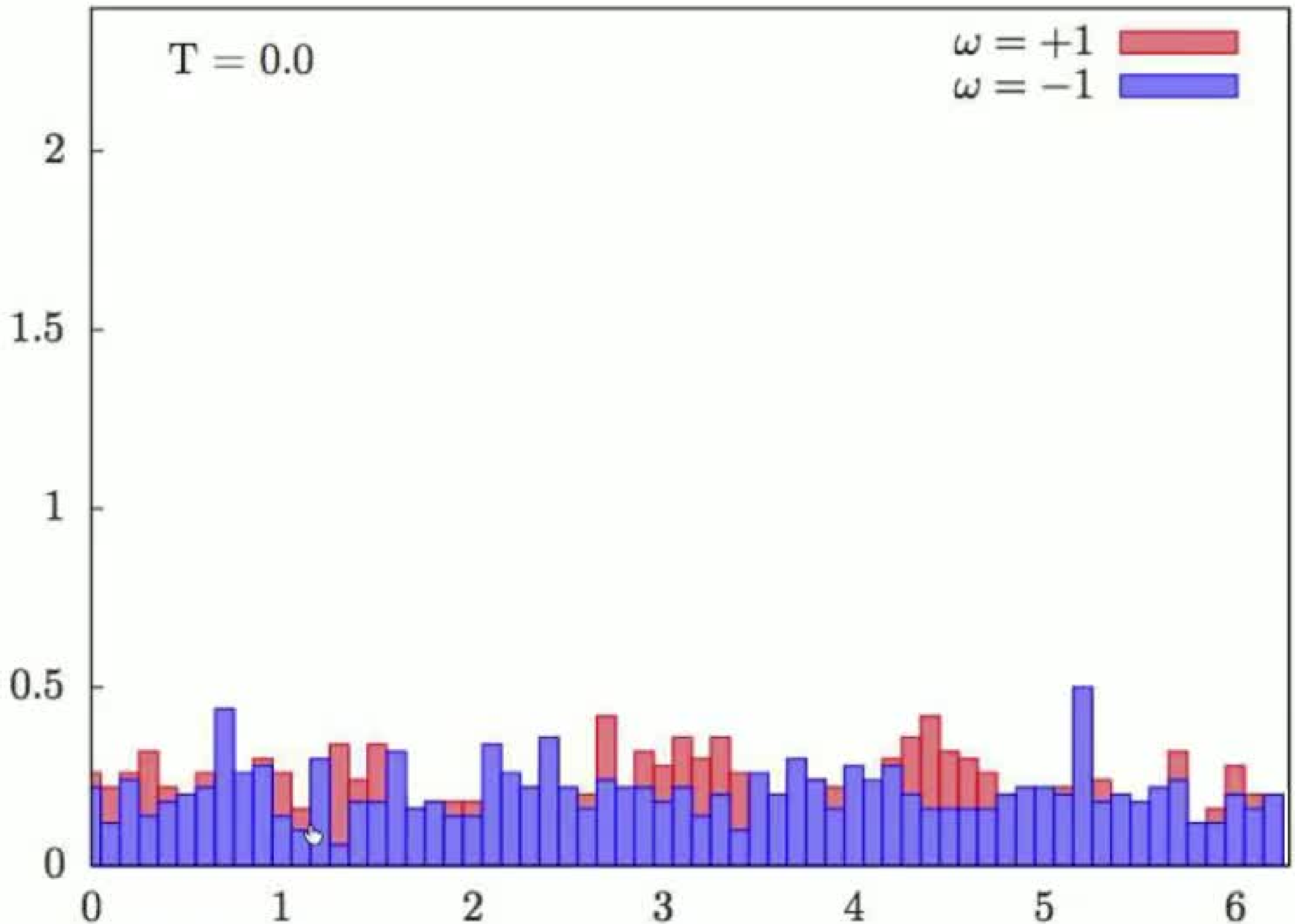

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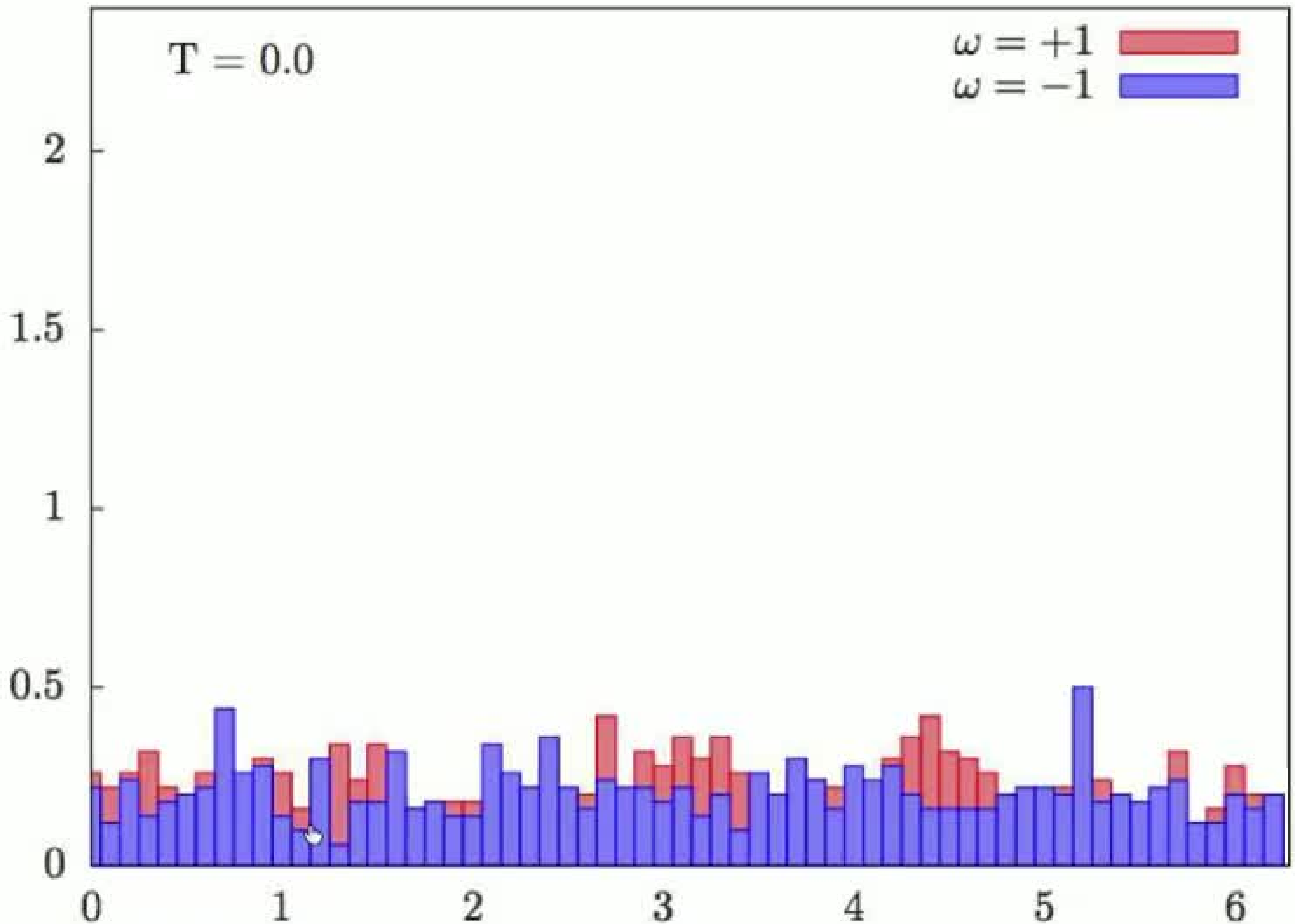

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$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,0} - p_0\|_{-1} \le \varepsilon\right) = 1$$

then for every $\varepsilon > 0$

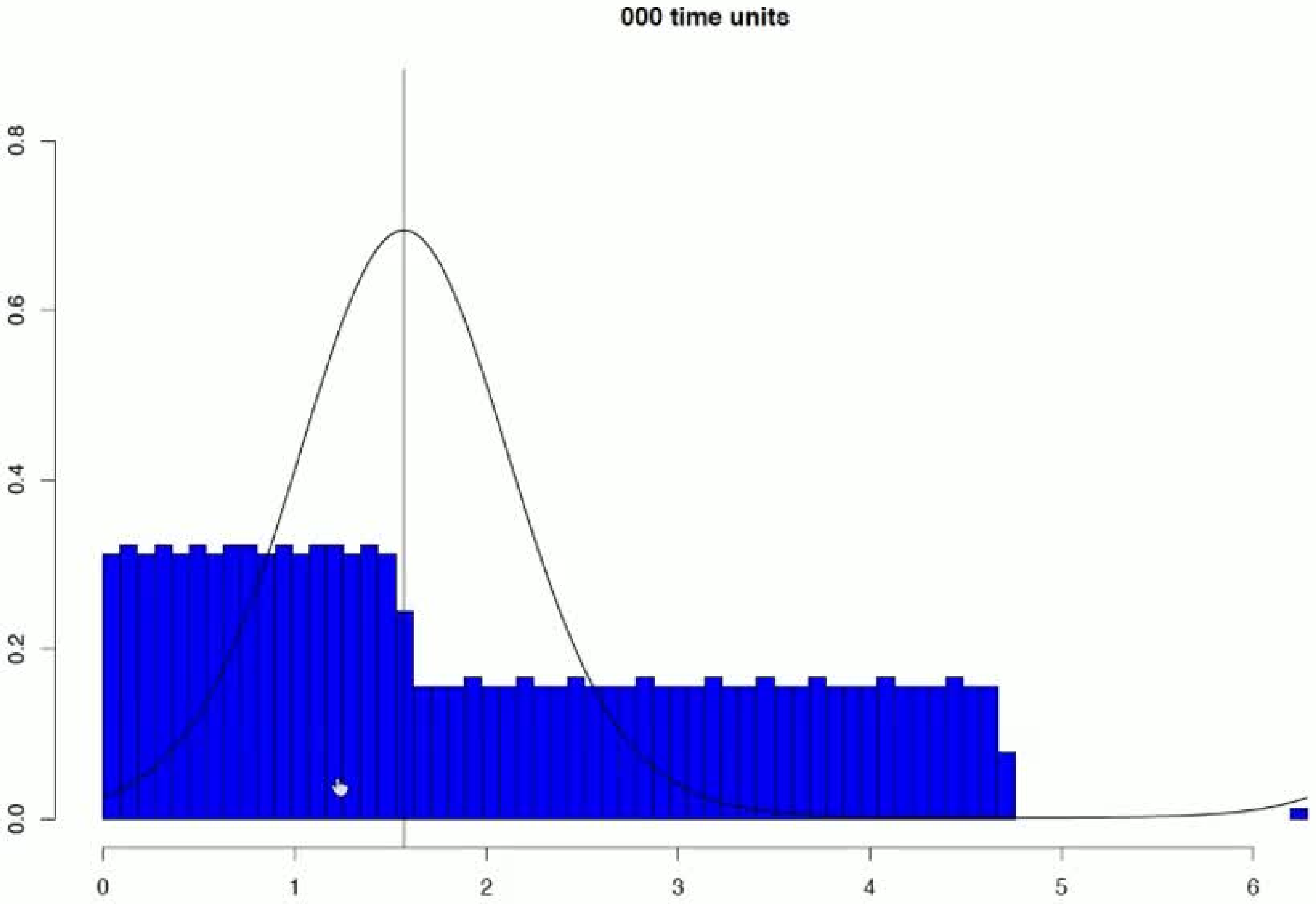
$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{\tau\in[\varepsilon_N,\tau_f]} \left\|\mu_{N,\tau N} - q_{\psi_0 + D_K B^N_\tau}\right\|_{-1} \le \varepsilon\right) = 1$$

where $\varepsilon_N := C_\varepsilon/N$ *and* $B^N_\cdot$ *is adapted to the processes* $\tau \mapsto w_j(\tau N)$*,* $j = 1, \ldots, N$*, and converges to a standard Brownian motion for* $N \to \infty$*. Moreover*

$$D_K := \frac{1}{\sqrt{1 - (I_0(2Kr))^{-2}}}$$

with $I_0(\cdot)$ *the modified Bessel function of first kind and 0 order*

$N = 1000$, $K = 2$, $\sigma = 1$, but much faster



On the Fokker-Planck PDE [Bertini, G., Pakdaman 2010], [G., Pakdaman, Pellegrin 2012]

For $K > 1$ the stable manifold

$$M_0 := \{q(\cdot + \psi) =: q_\psi(\cdot) : \psi \in \mathbb{S}\}$$

attracts everything, except the stationary unstable profile $\frac{1}{2\pi}$ and whatever is attracted to it, that is

$$U := \left\{ p : \int_{\mathbb{S}} p(\theta) \exp(i\theta) \, \mathrm{d}\theta = 0 \right\}$$

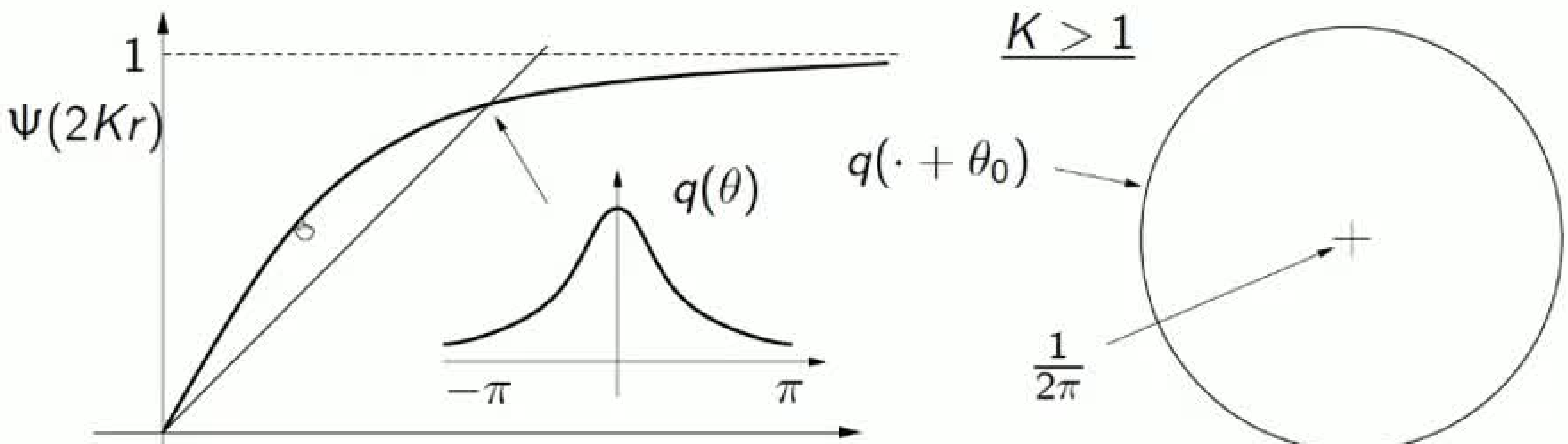
Long time behavior of the PDE

Two important facts for the (FP) PDE ($\sigma > 0$)

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta \left[p_t(\theta)(J * p_t)(\theta)\right] \tag{FP}$$

1. Statmech: (FP) is the gradient flow of a free energy functional
2. All stationary solutions are up to rotation invariance

$$q(\theta) \propto \exp(2Kr\cos(\theta)), \qquad \text{with } r = \Psi(2Kr)$$ and $\Psi(\cdot)$ explicit with $\Psi'(0) = 1/2$



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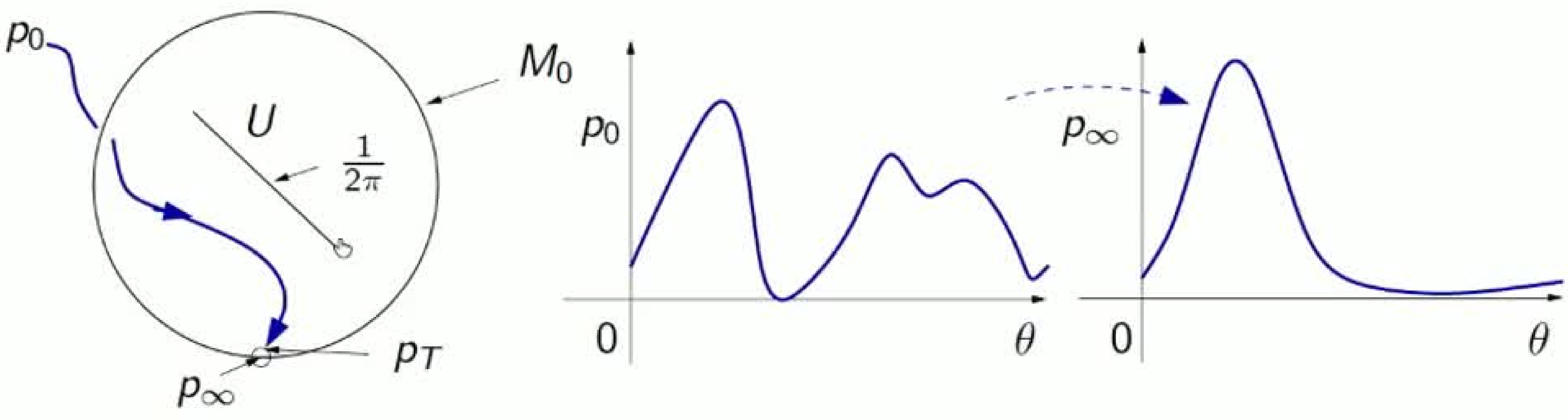
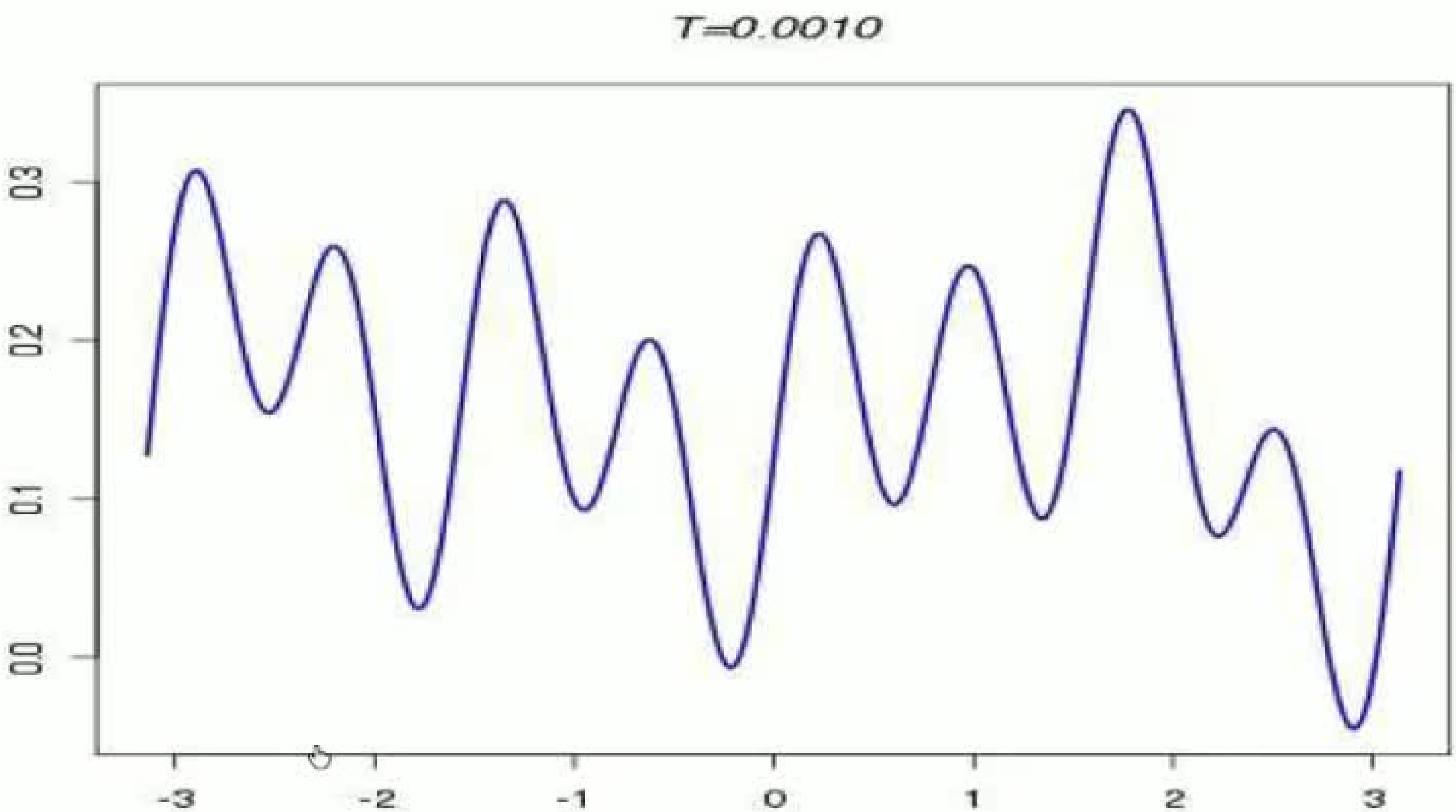


$N = \infty$ (Fokker-Planck), $K = 2$, $\sigma = 1$



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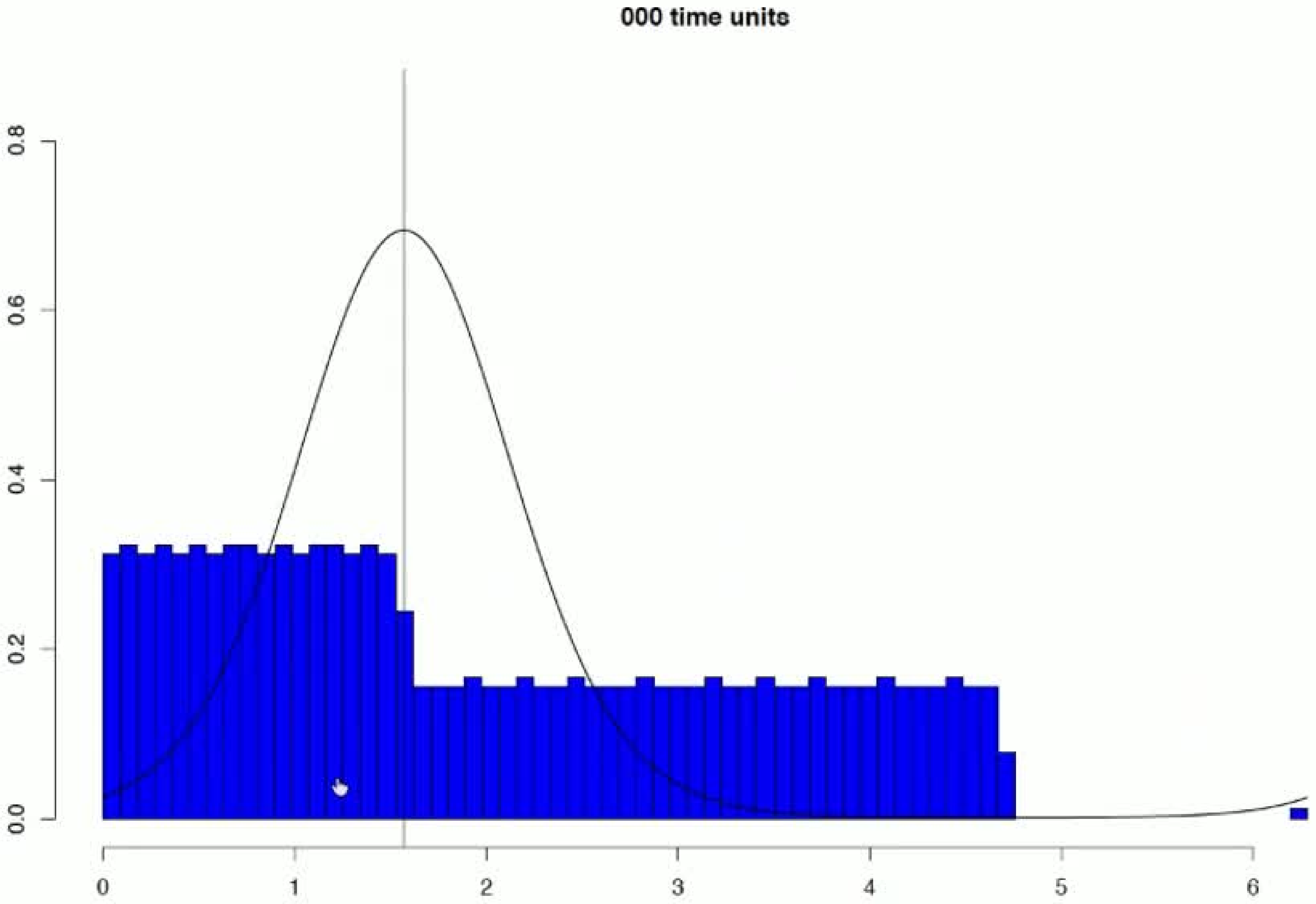

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$$\lim_{N\to\infty} \mathbb{P}\left(\|\mu_{N,0} - p_0\|_{-1} \leq \varepsilon \right) = 1$$

The stochastic Kuramoto model

The Kuramoto model is the mean field XY model with natural frequencies:

$$\mathrm{d}\varphi_j(t) = \omega_j \,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right) \mathrm{d}t + \mathrm{d}w_j(t)$$

Major difference: the natural frequencies break the reversibility.

Nevertheless, the model is mean field and the evolution can be written in terms of the empirical measure

$$\nu_{N,t}(\mathrm{d}\theta, \mathrm{d}\omega) = \frac{1}{N}\sum_{j=1}^{N} \delta_{\varphi_j(t),\omega_j}(\mathrm{d}\theta, \mathrm{d}\omega)$$

and the FP PDE limit follows from the same proceudure as in the reversible case, but statistical mechanics structure is no longer available!

The stochastic Kuramoto model

For every $\theta \in \mathbb{S}$ and every ω in the support of the law μ of ω_1

$$\partial_t p_t(\theta, \omega) = \frac{1}{2}\Delta p_t(\theta, \omega) - \partial_\theta \Big(p_t(\theta, \omega)(\langle J * p_t \rangle_\mu(\theta) + \omega) \Big)$$

$$\langle J * u \rangle_\mu(\theta) = \int_{\mathbb{R}} \int_{\mathbb{S}} J(\theta - \theta') u(\theta', \omega) \, \mathrm{d}\theta \, \mu(\mathrm{d}\omega)$$

[Kuramoto 1975],... , [Dai Pra, den Hollander 1996, Luçon 2011]

PDE is substantially more complex than the reversible case, but all stationary measures can be written if μ is symmetric in a way that is formally equivalent to the reversible case, but the fixed point problem is considerably harder.

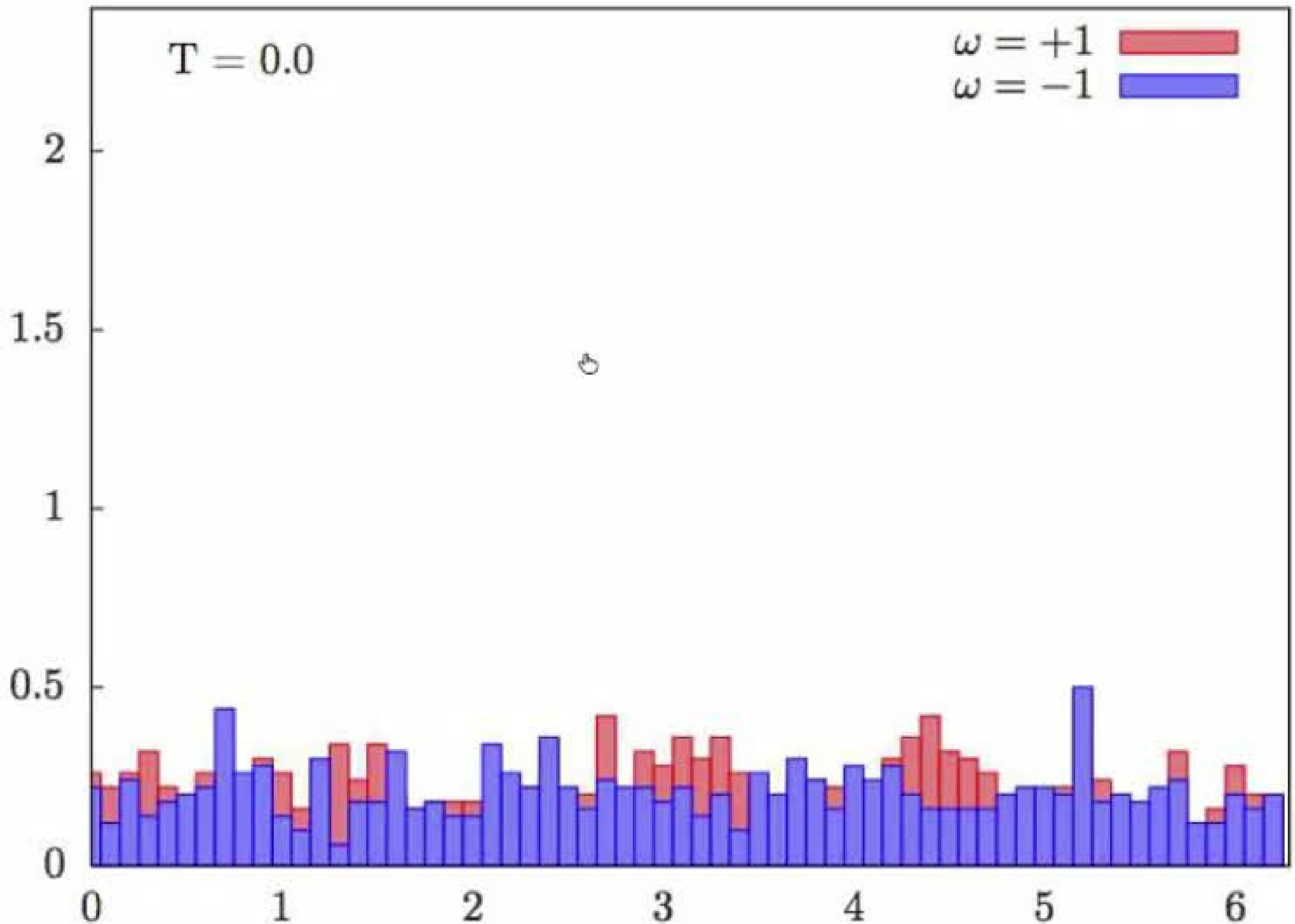
Nevertheless one can go very far when the support of μ is discrete and in $[-\delta, \delta]$, for δ small [Luçon and Poquet AIHP 2017]: PDE validity breaks down on time scale $\sqrt{N}$

The inhomogeneous graph case

Just one example from [Delattre, G., Luçon JSP2016]

$$\mathrm{d}\varphi_j(t) = \cancel{\omega_j \mathrm{d}t} - \frac{1}{N}\sum_{i=1}^{N} K_{i,j} \sin\left(\varphi_j(t) - \varphi_i(t)\right) \mathrm{d}t + \sigma \, \mathrm{d}w_j(t)$$

$N = 700$ ($\sqrt{700} = 26.45\ldots$, $K = 5$, $\sigma = 1$



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