Nonlinear Stability of Source Defects

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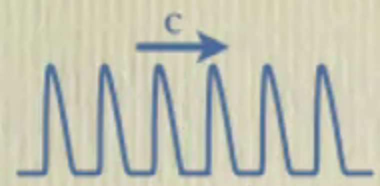
Dynamics of wave trains

$$u_t = Du_{xx} + f(u)$$

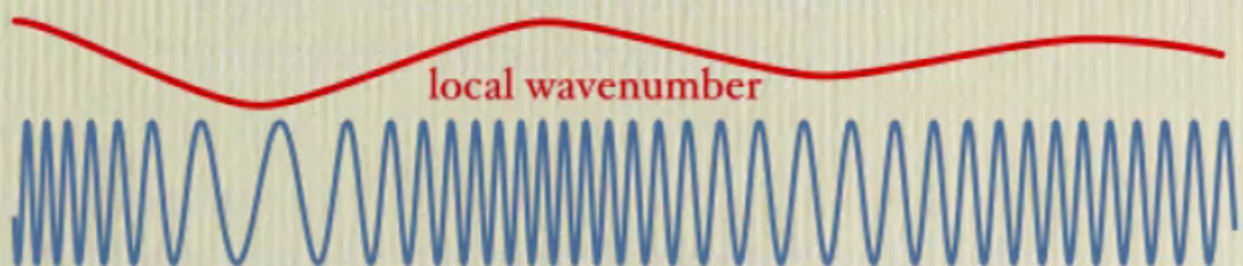
$$u(x,t) = u_{wt}(kx - \omega(k)t)$$



wave train

k wavenumber

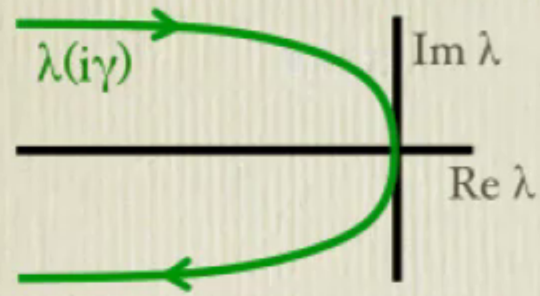
$\omega=\omega(k)$ temporal frequency



slowly varying modulations of wavenumber

$$u_{wv} + \epsilon e^{\lambda t}\cos(\gamma x)u'_{wv}$$
$$\approx \quad u_{wv}\left(x - \omega(k)t + \epsilon\cos(\gamma(x - c_g t))\right)$$

Spectrum of wave trains



$\lambda(i\gamma) = -ic_g\gamma - d\gamma^2 + O(|\gamma|^3)$

Dynamics of wave trains

$$u_t = Du_{xx} + f(u)$$

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wave train

k wavenumber

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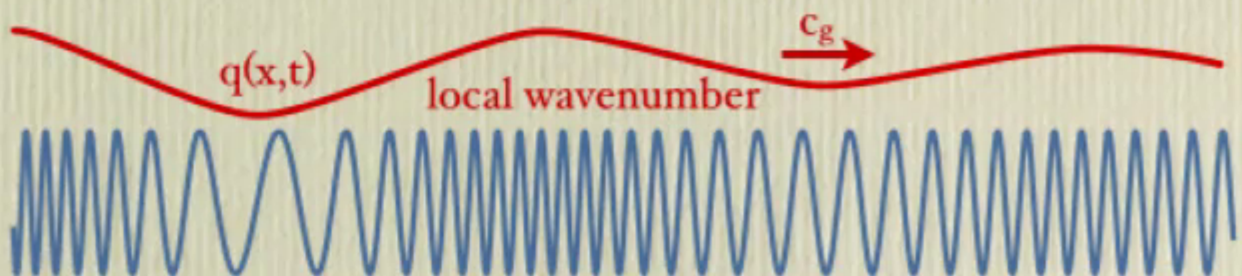


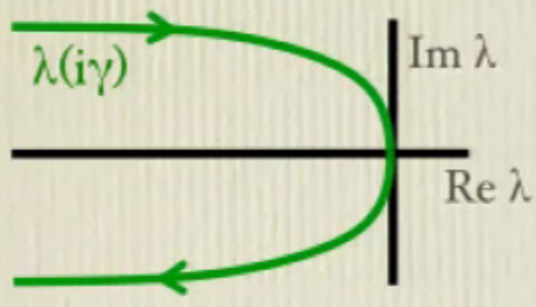
$$q_t = -c_g v_x$$

slowly varying modulations of wavenumber

group velocity:
direction of transport

$$u_{wv} + \epsilon e^{\lambda t}\cos(\gamma x)u'_{wv} \approx u_{wv}\left(x - \omega(k)t + \epsilon\cos(\gamma(x - c_g t))\right)$$

Spectrum of
wave trains

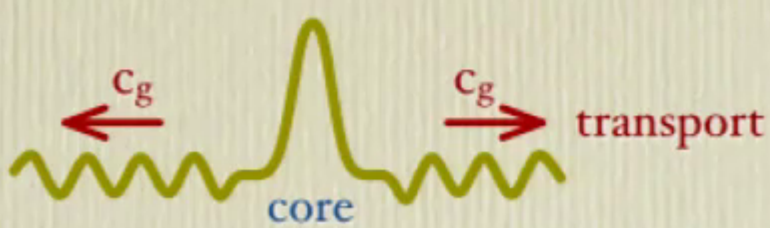


$\lambda(i\gamma) = -ic_g\gamma - d\gamma^2 + O(|\gamma|^3)$

Sources

Sources: outgoing transport

group velocities point away from core

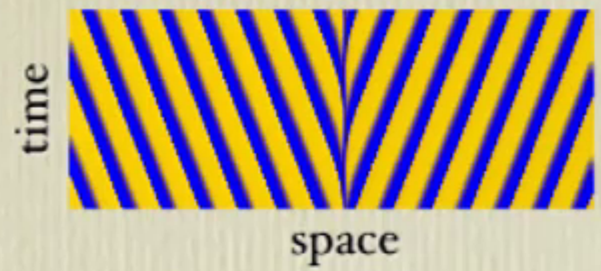


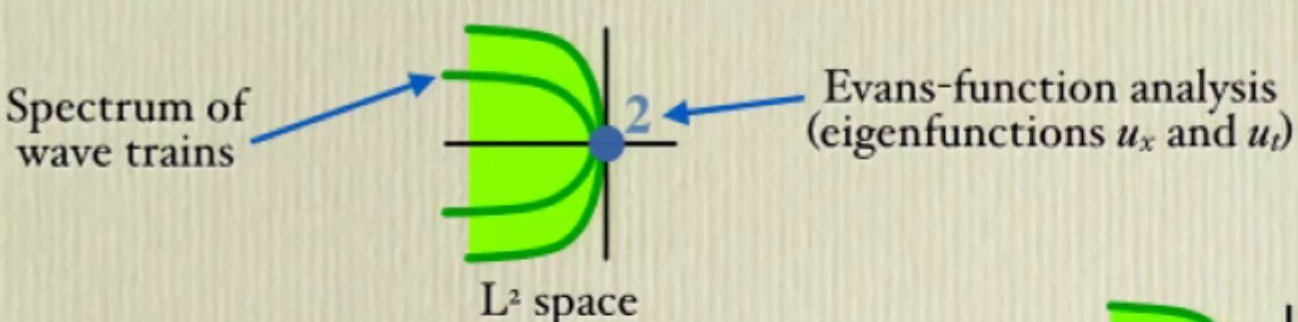
- Spectral and linear stability: linearized equation is time-periodic
- Nonlinear stability: previous methods do not apply

Spectra of sources

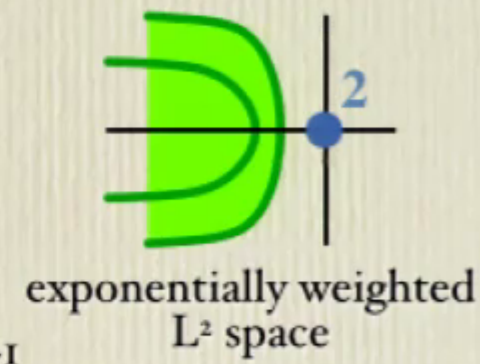
Reaction-diffusion system: $u_t = Du_{xx} + f(u)$

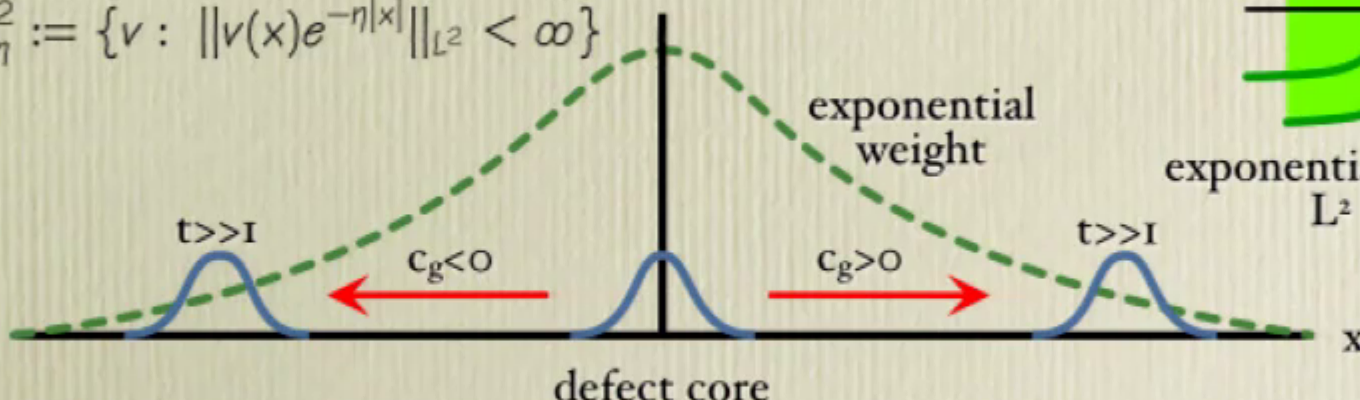
Standing sources are time-periodic:
Floquet spectrum determines spectral stability



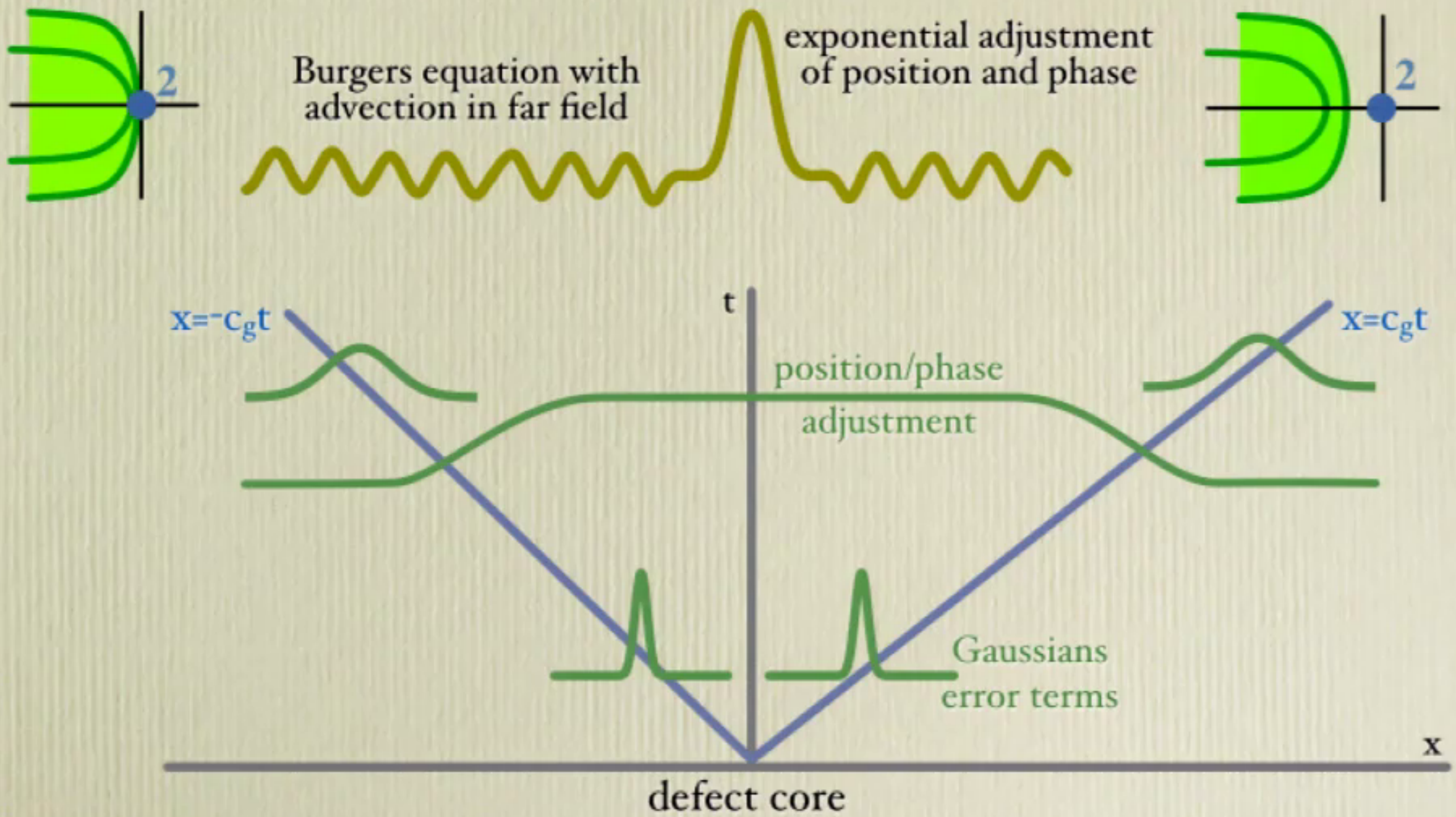


$L^2_\eta := \{v : \|v(x)e^{-\eta|x|}\|_{L^2} < \infty\}$





Expected dynamics

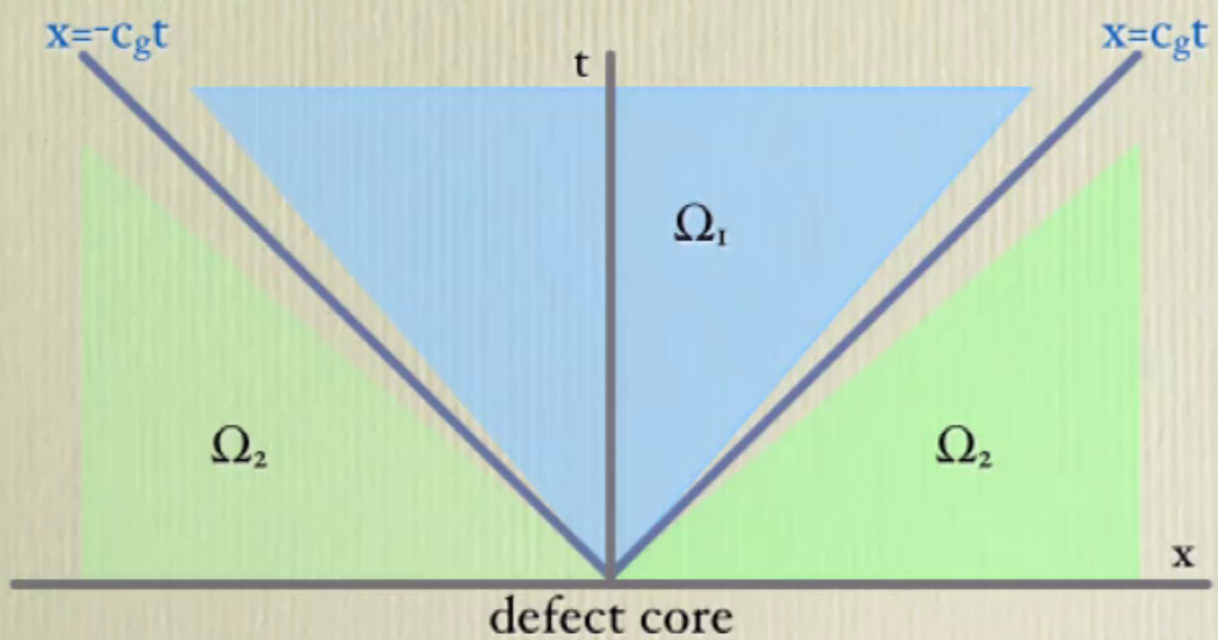


Nonlinear stability

Theorem [Beck, Nguyen, S., Zumbrun]: Assume $u^*(x,t)$ is a spectrally stable source and let $u(x,0)=u^*(x,0)+v_0(x)$ where $\|v_0(x)\exp(x^2/M)\|<\varepsilon$ is sufficiently small. Then there are constants $|p_\infty|$, $|\phi_\infty|<\varepsilon$ such that

$$|u(x,t)-u^*(x-p_\infty,t-\phi_\infty)| < \varepsilon C \exp(-\eta t) \text{ for } (x,t) \text{ in } \Omega_1 \text{ and}$$

$$|u(x,t)-u^*(x,t)| < \varepsilon C \exp(-\eta t) \text{ for } (x,t) \text{ in } \Omega_2.$$



Nonlinear stability proofs

- Define appropriate offset from source:

$$u(x,t) = u^*(x,t) + v(x,t)$$

- Derive equation for offset:

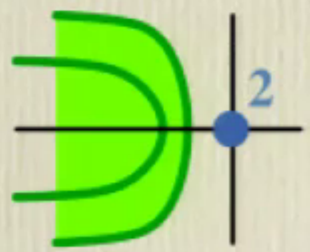
$$v_t = Lv + O(|v|^2)$$

- Variation-of-constants formula:

$$v(t) = e^{Lt}v_0 + \int_0^t e^{L(t-s)} O(|v(s)|^2)\, ds$$

- Fixed-point argument in appropriate function space:

No decay in
L^2 spaces



Decay in weighted L^2 spaces,
but nonlinearity not well defined

$$L^2_\eta := \{v : \ \|v(x)e^{-\eta|x|}\|_{L^2} < \infty\}$$

Nonlinear stability proofs

- Define appropriate offset from source:

$$u(x+p(x,t),t+\phi(x,t)) = u^*(x,t) + v(x,t)$$

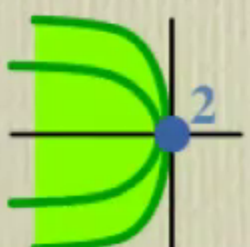
- Derive equation for offset:

$$\underbrace{\partial_t(v - pu^*_x - \phi u^*_t) = \left(D\partial_x^2 + c\partial_x + f_u(u^*)\right)(v - pu^*_x - \phi u^*_t)}_{\text{linearization about } u^*} + O(|v|^2+|p_{x,t}|^2+|\phi_{x,t}|^2)$$

- Variation-of-constants formula:

$$(v-pu^*_x-\phi u^*_t)|_{(x,t)} = \int_{\mathbb{R}} G(x,t;y,0)(v-pu^*_x-\phi u^*_t)|_{(y,0)}\,dy + \int_0^t\int_{\mathbb{R}} G(x,t;y,s)N(v,p,\phi)|_{(y,s)}\,dy\,ds$$

- Fixed-point argument in appropriate function space:



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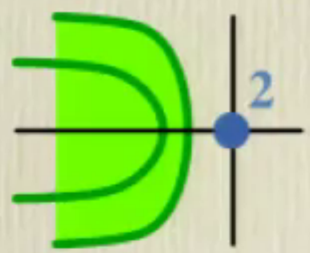

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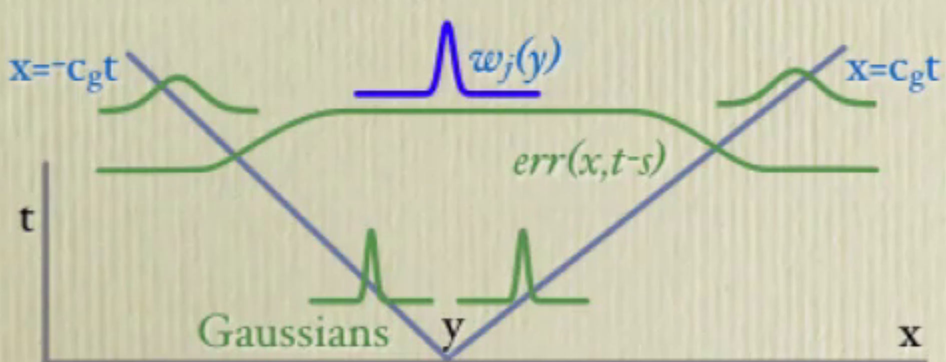
- Fixed-point argument in appropriate function space:

$$G(x,t;y,s) = a_1(x,t;s,y)u^*_x(x,t)+a_2(x,t;s,y)u^*_t(x,t)+G_R(x,t;y,s)$$

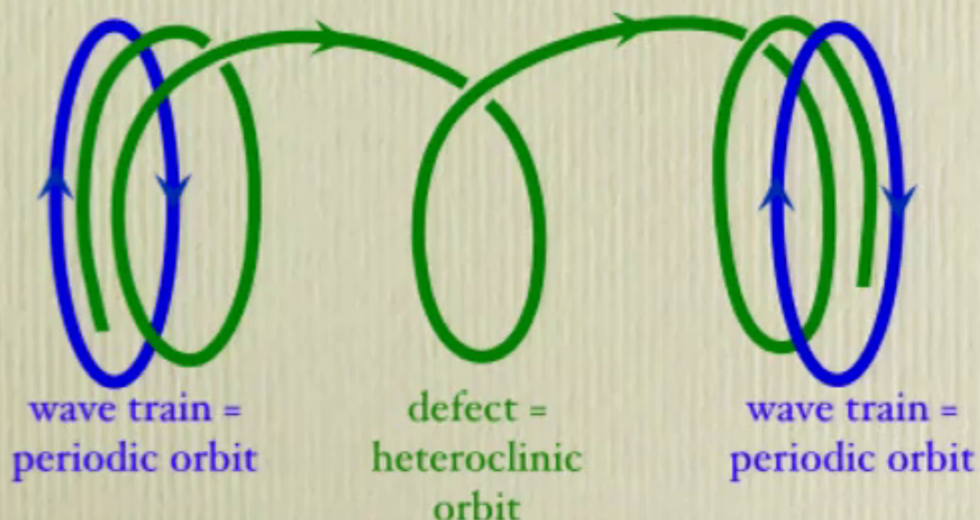


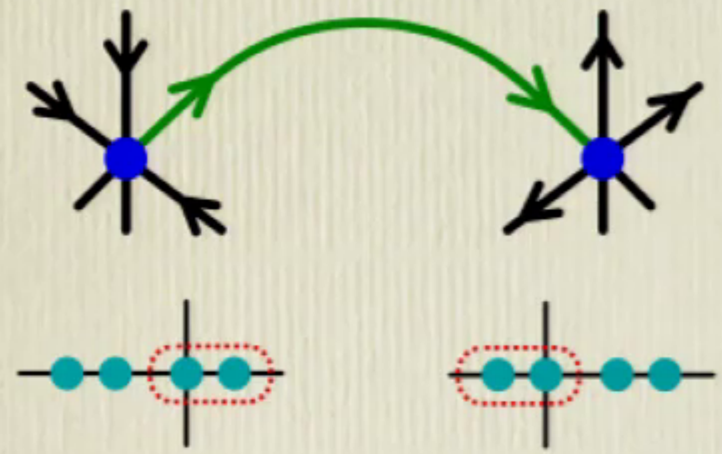
- $a_j(x,t;y,s)$ = $err(x,t\text{-}s)\ w_j(y)$ + Gaussians
- $G_R(x,t;y,s)$ ≈ "differentiated" Gaussians

Spatial dynamics

$$\begin{aligned} u_x &= v \\ v_x &= D^{-1}[u_t - c_d v - f(u)] \end{aligned}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \in H^{\frac{1}{2}}(S^1) \times L^2(S^1)$$





[S. & Scheel]