AN ALGEBRO-TOPOLOGICAL PERSPECTIVE ON HIERARCHICAL MODULARITY OF NETWORKS

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Work in progress together with....

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- Ran Levi, Sophie Raynor (University of Aberdeen)
- Paweł Dłotko (INRIA)

OVERVIEW

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THE BLUE BRAIN PROJECT

- Digital reconstruction of the microcircuitry of layers 1 to 6 of the hind-limb somatosensory cortex of a two-week-old rat, based on detailed experimental data from five rat brains
- 42 "columns" of $\sim 31,000$ neurons each, forming $\sim 8.2 \times 10^6$ connections consisting of $\sim 36.7 \times 10^6$ synapses: seven for each rat, and seven average
- Validated against numerous experimental datasets not used in the reconstruction
- Key application: study emergent properties of the microcircuit through simulated activity

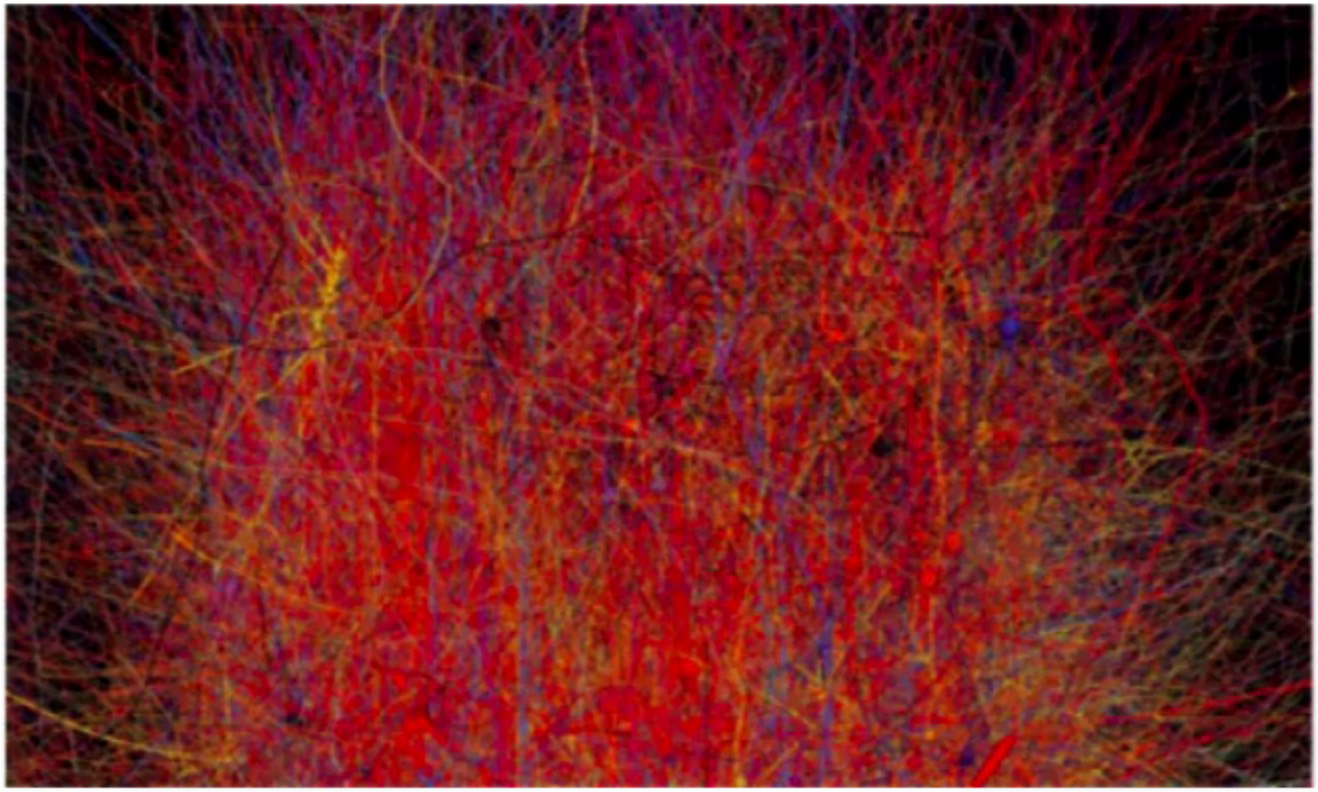

Photo credit: IBM

THE CONNECTOME

The graph describing the brain network, on the level of neurons and synapses or of microcircuits and connections among them or of brain regions and connections among them...or even several levels simultaneously (hierarchically modular graph)

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The graph describing the brain network, on the level of neurons and synapses or of microcircuits and connections among them or of brain regions and connections among them...or even several levels simultaneously (hierarchically modular graph)

- Notion introduced independently by Hagmann and Sporns in 2005
- Experimentally determined to be small world: high ratio of segregation to integration, compared to random graphs
- Various psychiatric illnesses detected and sometimes distinguished by deviation from norm of certain graph-theoretic invariants of the connectome

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Why algebraic topology?

- Natural next step beyond graph theory
- Ideal for studying local-to-global phenomena
- Ideal for characterizing higher-order connectivity
- Tools for studying the embedding of the connectome in space

Topological toolbox

Oriented simplicial complexes

An abstract oriented simplicial complex K consists of

- a set K_0 of vertices, and
- sets K_n of lists $\sigma = (x_0, ..., x_n)$ of $n+1$ elements of K_0 (called n-simplices of K) for $n \geq 1$

such that if $\sigma = (x_0, ..., x_n) \in K_n$, then any sublist $(x_{i_0}, ..., x_{i_k})$ (called a face of σ) is in K_k.

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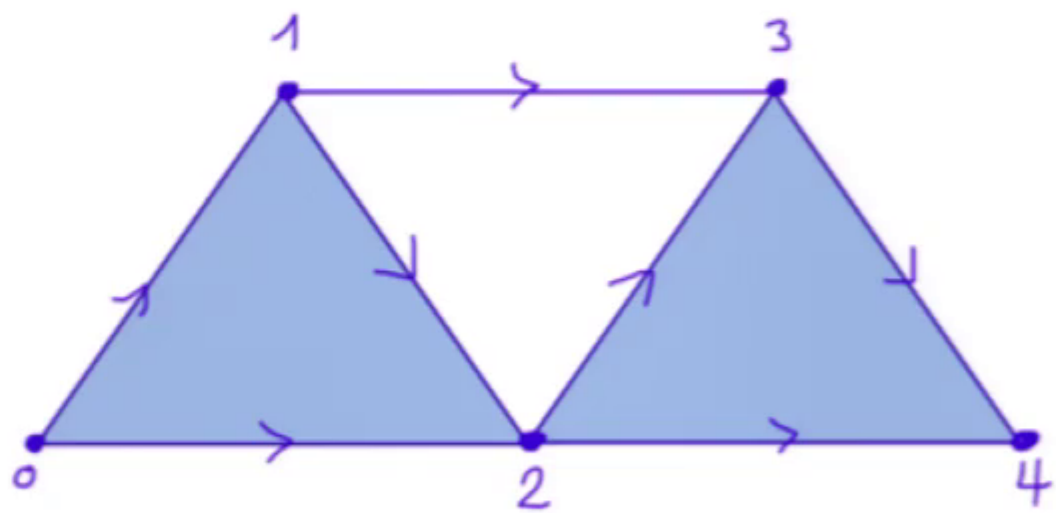
To any such K, one can associate a geometric simplicial complex, by taking one geometric n-simplex for each element of K_n for all n, then gluing them along faces, respecting orientation.

GEOMETRIC REALIZATION: EXAMPLE

Given an abstract simplicial complex K with

- $K_0 = \{0, 1, 2, 3, 4\}$,
- $K_1 = \{(0,1), (1,2), (2,3), (2,4), (3,4), (1,3)\}$,
- $K_2 = \{(0,1,2), (2,3,4)\}$,

its geometric realization is...



FLAG COMPLEXES

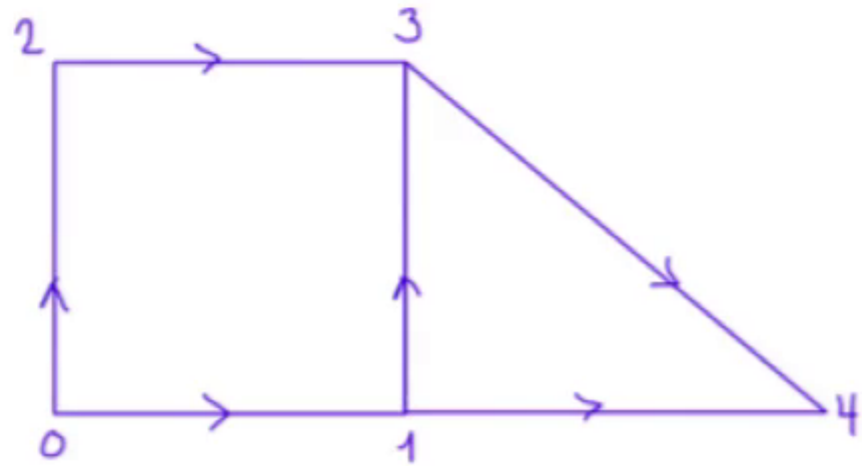
Let G be an oriented graph with vertex set V and edge set E.

The flag complex $K(G)$ of G is the oriented simplicial complex with

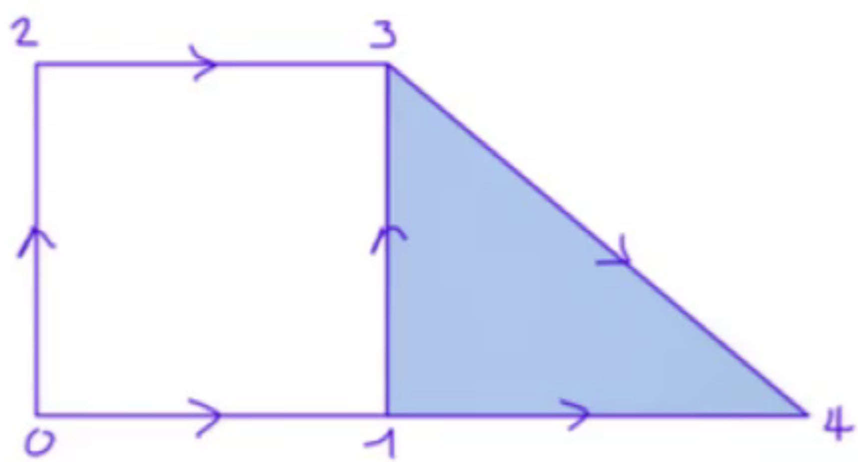
- $K(G)_0 = V$, and
- $K(G)_n = \{(v_0, ..., v_n) \mid (v_i, v_j) \in E \text{ for all } i < j\}$.

FLAG COMPLEX: EXAMPLE

A graph...



...and its flag complex.



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EULER CHARACTERISTIC

Let K be a simplicial complex with $n+1$ vertices.

Let $|K_k|$ denote the number of k-simplices in K.

The Euler characteristic of K is

$$\chi(K) = \sum_{k=0}^{n} (-1)^k |K_k|.$$

Betti numbers: definition

Let K be a simplicial complex, and let n be a natural number.

The n^{th} mod 2 Betti number of K is

$$\beta_n(K) = \dim_{\mathbb{F}_2} H_n(K;\mathbb{F}_2) = \dim_{\mathbb{F}_2} \ker \partial_n - \dim_{\mathbb{F}_2} \operatorname{Im} \partial_{n+1},$$

where

$$\partial_k : \mathbb{F}_2\langle K_k \rangle \to \mathbb{F}_2\langle K_{k-1} \rangle$$

is the linear transformation that sends a k-simplex to the sum of its $(k-1)$-dimensional faces.

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Betti numbers: properties

- Two vertices of a simplicial compex are in the same connected component if there is a path of 1-simplices connecting them. $\beta_0(K)$ is equal to the number of connected components of K.
- In general, $\beta_p(K)$ counts the number of "p-dimensional holes" in K.
- If K has $n+1$ vertices, then $\chi(K) = \sum_{k=0}^{n}(-1)^k \beta_k(K)$.

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First-order connectivity

Let K be a simplicial complex with n vertices.

The degree of connectivity of a vertex $x \in K_0$ is

$c(x) =$ maximal dimension of a simplex of K to which x belongs.

The segregation coefficient of K is

$$s(K) = \frac{1}{n} \sum_{x \in K_0} c(x),$$

the average degree of connectivity.

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FIRST-ORDER CONNECTIVITY

Let K be a simplicial complex with n vertices.

The degree of connectivity of a vertex $x \in K_0$ is

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The cumulative connectivity distribution $p_K : \mathbb{N} \to [0, 1]$ is the function defined by

$$p_K(m) = \frac{1}{n} \cdot \#\{x \in K_0 \mid c(x) \geq m\},$$

the fraction of vertices with degree of connectivity at least m.

SECOND-ORDER CONNECTIVITY

Let K be a simplicial complex, and let d be a natural number.
Let $x, y \in K_0$.

Second-order connectivity

Let K be a simplicial complex, and let d be a natural number. Let $x, y \in K_0$.

A *d-dimensional highway* from x to y consists either of a simplex $\sigma = (x, x_1, \ldots, x_{d-1}, y)$ in K_d or of a sequence

$$\sigma_0, \ldots, \sigma_m$$

of simplices such that $\sigma_i \cap \sigma_{i+1}$ is a front face of σ_{i+1} of dimension at least d, for all $i \geq 0$ and such that $v \in \sigma_0$ and $w \in \sigma_m$.

The *highway dimension* of a pair (x, y) of distinct vertices is

$$h(x, y) = \text{maximal dimension of a highway from } x \text{ to } y.$$

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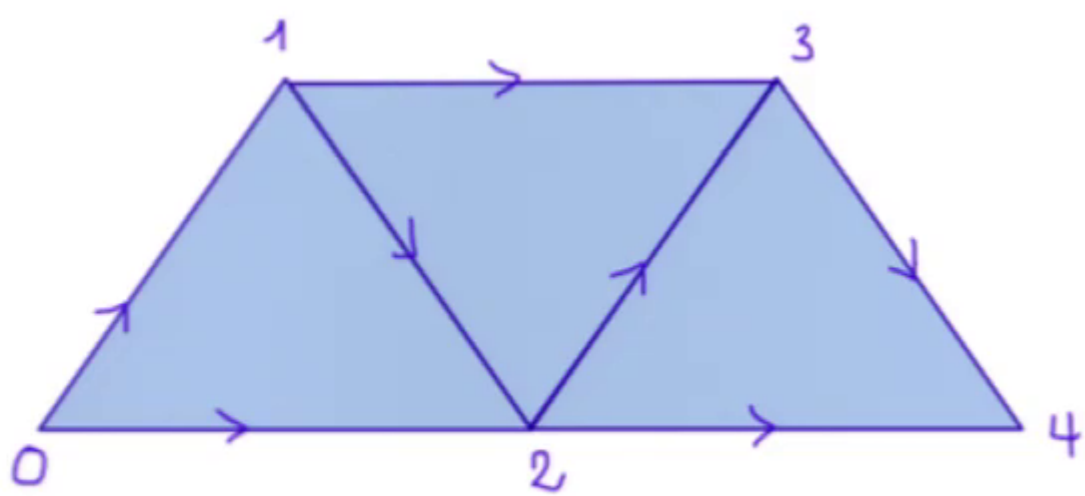
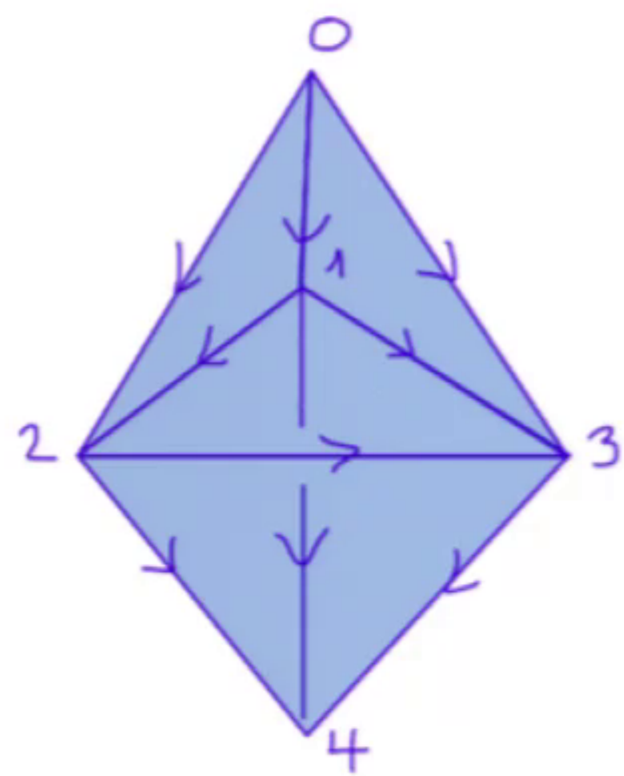
HIGHWAYS

A 1-dimensional and a 2-dimensional highway from 0 to 4.





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SECOND-ORDER CONNECTIVITY

Let K be a connected simplicial complex with n vertices, and let d be a natural number.

The highway dimension of a pair (x, y) of distinct vertices is

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The integration coefficient of K is

$$i(K) = \frac{1}{n(n-1)} \sum_{x \neq y \in K_0} h(x, y).$$

Results

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Structural network

- The flag complex of the average column is of dimension 6 and has

  $\sim 10^4$ maximal 1-simplices
  $\sim 10^7$ maximal 2-simplices
  $\sim 10^7$ maximal 3-simplices
  $\sim 10^6$ maximal 4-simplices
  $\sim 10^5$ maximal 5-simplices
  $\sim 10^3$ maximal 6-simplices.

- Computing all the Betti numbers of this flag complex requires too much memory even of a large mainframe, but we know that it has nonzero Betti numbers through dimension 5, with $\beta_5 \sim 50$.

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MEASURING ACTIVITY

Blue Brain simulation:

- Five trials of each of four different stimuli applied to resting-state cortical column.
- State of column sampled every 25 ms for 2,2 seconds: subgraph of structural graph, encoded in a binary connectivity matrix.
- Resulting sequence of connectivity matrices compared with sequence of randomly shuffled matrices.

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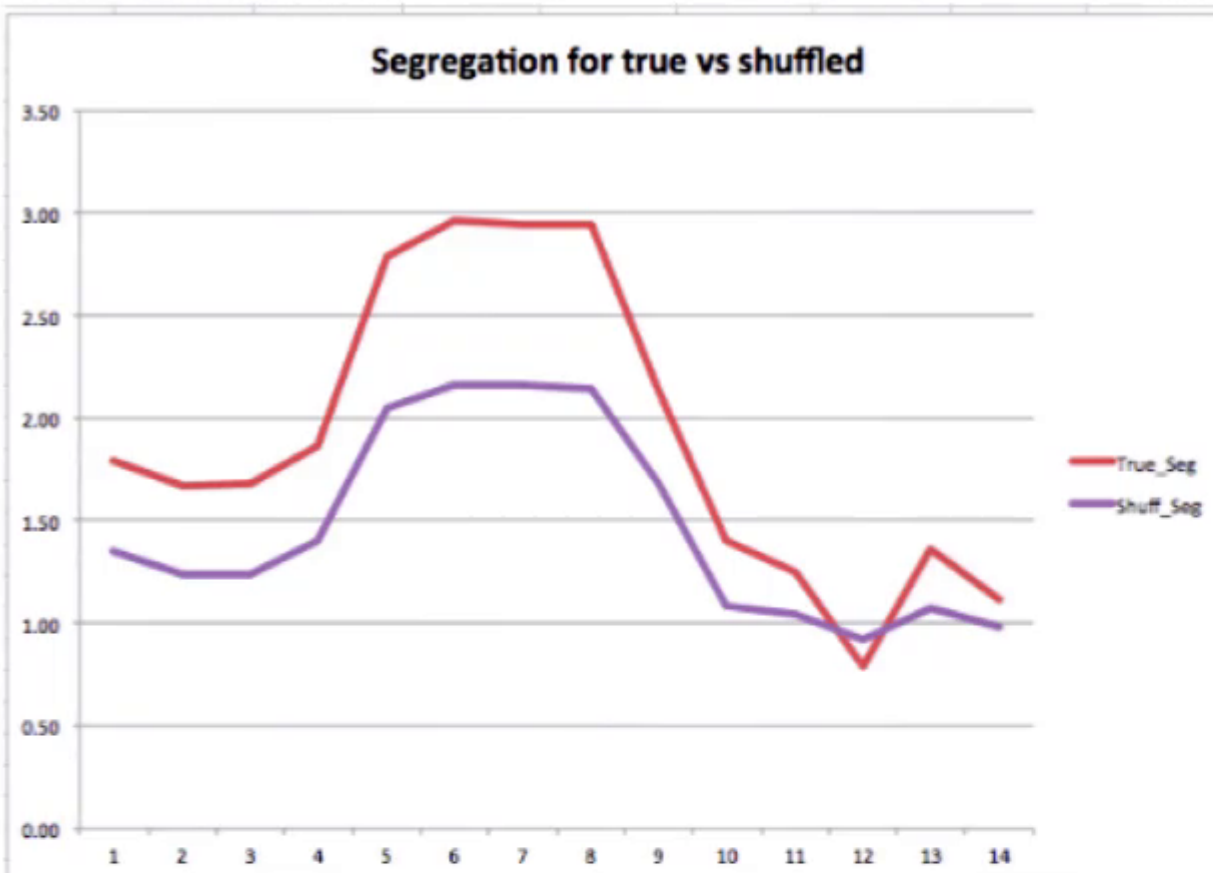
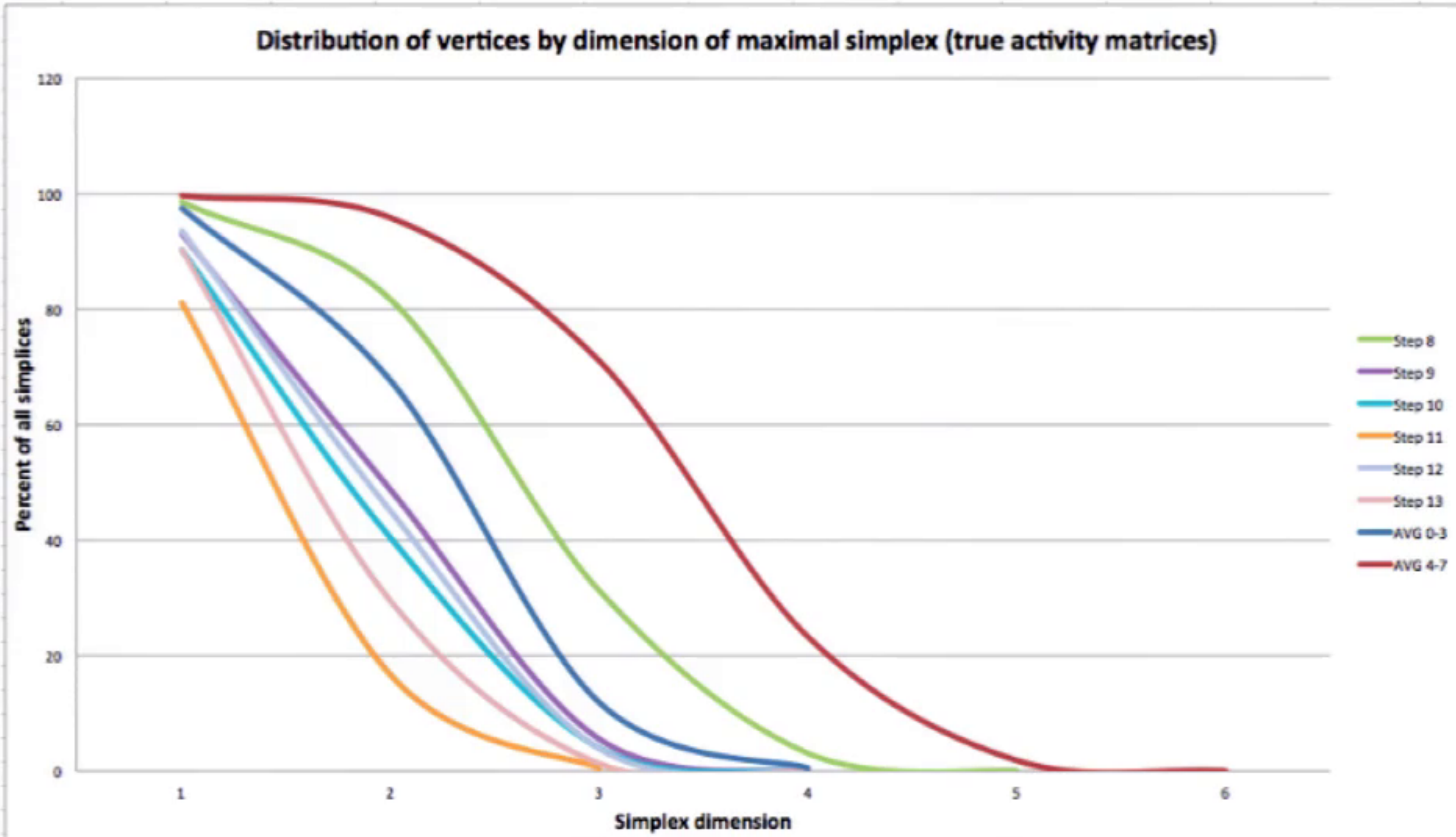
SEGREGATION



DISTRIBUTION OF CONNECTIVITY



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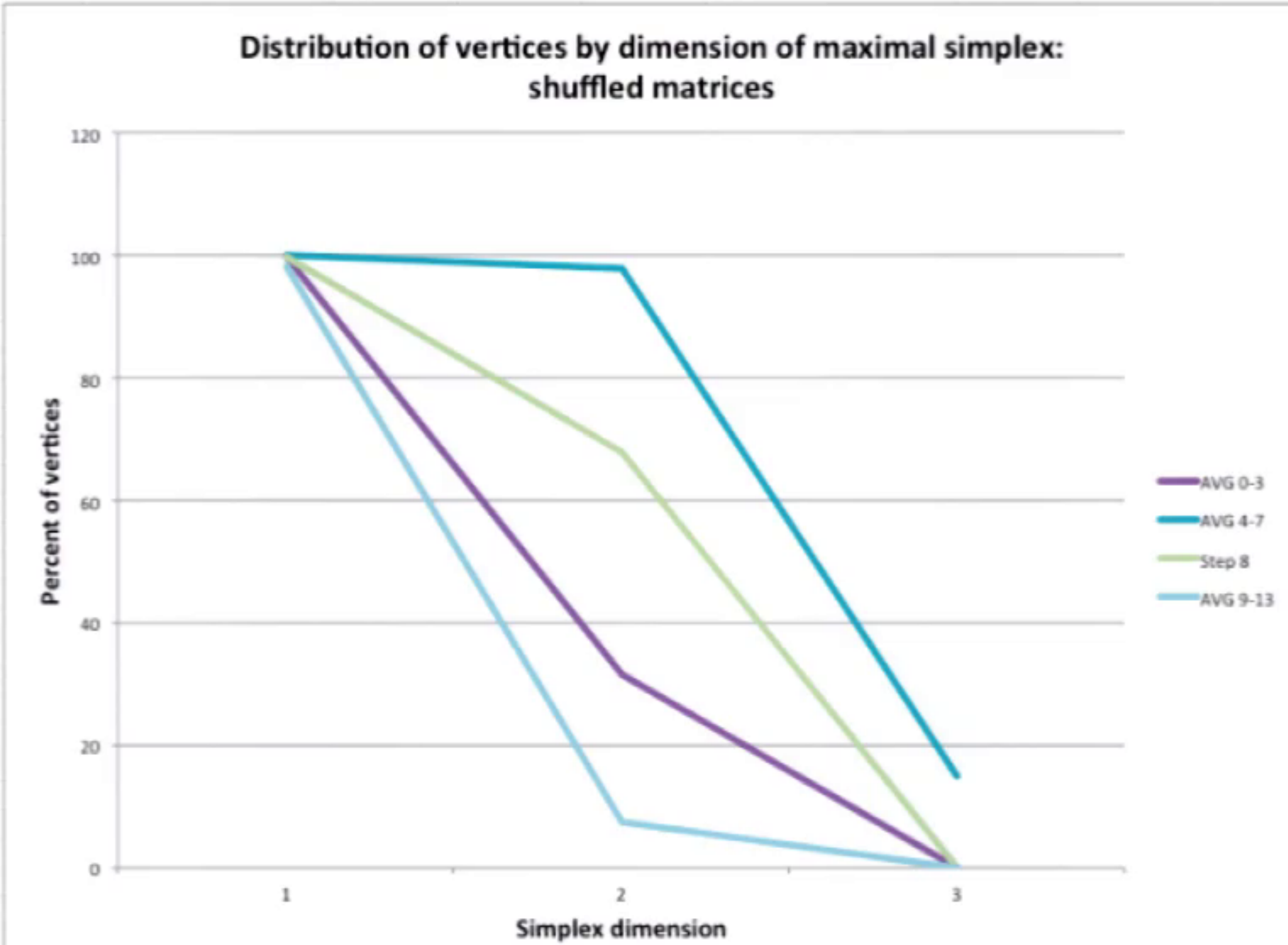

HIGHWAY DIMENSION