Customising Image Analysis Using Nonlinear Partial Differential Equations

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Outline

1. Customised Nonlinear PDEs for Image Analysis

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2. Learning the Customised PDE from Image Data

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2. Learning the Customised PDE from Image Data
3. Conclusions and Outlook

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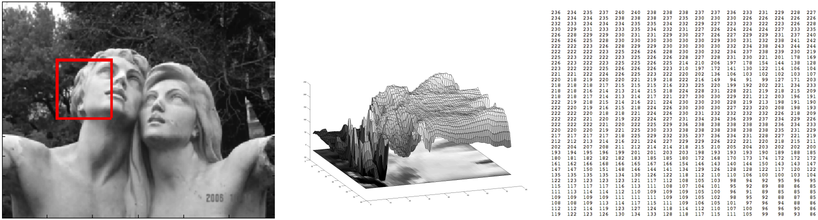
Customising PDEs for image analysis . . .

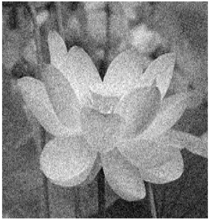
Common approach: taking advantage of the property of solutions of such PDEs and variational problems to be characterised by a few relevant features with certain geometric properties, e.g. curvature, which are recovered by adaptive nonlinear iterations.

Notation

- Ω rectangular image domain
- $f : \Omega \to \mathbb{R}$ given image data
- $u : \Omega \to \mathbb{R}$ computed image (image information)
- T forward operator $u \to f_c$, random noise n,
  $f = f_c + n$



From linear to nonlinear diffusion



References: Perona, Malik, Pattern Analysis and Machine Intelligence '90; Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

From linear to nonlinear diffusion

$$u_t = \Delta u, \; u(x, t = 0) = f(x).$$

Solution $u(x,t) = (G_{\sqrt{2t}} * f)(x), \; t > 0$

References: Perona, Malik, Pattern Analysis and Machine Intelligence '90; Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

From linear to nonlinear diffusion

$$-\Delta t\ \Delta u + u(\Delta t) - u(0) = 0$$

$$\Longleftrightarrow \qquad u_t = \operatorname{div}\left(g(|\nabla u|)\nabla u\right),\ u(x,t=0) = f(x).$$

$$u(\Delta t) = \operatorname{argmin}_v \left\{\Delta t\ \|\nabla v\|_2^2 + \|v - u(0)\|_2^2\right\}$$ e.g. $g(s) = 1/|\nabla u|$, $|\nabla u| \neq 0$.

References: Perona, Malik, Pattern Analysis and Machine Intelligence '90; Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

From linear to nonlinear diffusion

$$-\Delta t\ \Delta u + u(\Delta t) - u(0) = 0$$

$$\iff$$

$$u(\Delta t) = \text{argmin}_v \left\{\Delta t\ \|\nabla v\|_2^2 + \|v - u(0)\|_2^2\right\}$$

$$-\Delta t\ \text{div}\left(\nabla u/|\nabla u|\right) + u(\Delta t) - u(0) = 0$$

$$\iff \quad (|\nabla u| \neq 0)$$

$$\text{argmin}_v \left\{\Delta t \int |\nabla u| + \|v - u(0)\|_2^2\right\}$$

References: Perona, Malik, Pattern Analysis and Machine Intelligence '90; Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

From linear to nonlinear diffusion

$$-\Delta t\ \Delta u + u(\Delta t) - u(0) = 0 \qquad 0 \in -\Delta t\ \partial R(u) + u(\Delta t) - u(0)$$

$$\Longleftrightarrow \qquad\qquad \Longleftrightarrow \ (R \text{ is convex})$$

$$u(\Delta t) = \operatorname{argmin}_v \left\{\Delta t\ \|\nabla v\|_2^2 + \|v - u(0)\|_2^2\right\} \quad \operatorname{argmin}_v \left\{\Delta t\ R(v) + \|v - u(0)\|_2^2\right\}$$

References: Perona, Malik, Pattern Analysis and Machine Intelligence '90; Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

Nonlinear diffusion – the total variation

For $u \in BV(\Omega)$ **(the space of bounded variation functions)**, $\Omega \subset \mathbb{R}^2$,

$$R(u) = |Du|(\Omega) := \sup\left\{ \int_\Omega u \,\mathrm{div}\varphi \, dx : \varphi \in \left[C_c^1(\Omega)\right]^2, \|\varphi\|_\infty \leq 1 \right\}$$

is the **total variation (TV)** of the finite Radon measure Du, the derivative of u in the sense of distributions.

Nonlinear diffusion – the total variation

Properties

- Du is a Radon measure, hence u with finite total variation may be singular, i.e., contains jumps.
- applicable for problems where most of the information is contained in the edge set; penalizes small irregularities/oscillations while respecting instrinsic image features such as edges.
- for a function $u \in C^1(\Omega)$,

$$|Du|(\Omega) = \int_\Omega |\nabla u| \, dx.$$

  (sparsity w.r.t. to the edges)
- …coarea formula

$$|D\chi_E|(\Omega) = Per(\partial E), \qquad \text{for a Borel measurable set } E \subset \Omega,$$

  where χ_E is the indicator function of E.

Other encounters with nonlinear PDEs in image analysis . . .

H^1 versus TV regularisation

$$\min_u \left\{\alpha R(u) + \|u-f\|_2^2\right\}$$

(a) original (b) noisy (c) $R(u) = \|\nabla u\|_2^2$

References: Rudin, Osher, Fatemi, Physica D '92; Chambolle, Lions, Numerische Mathematik '97; Vese, Applied Mathematics and Optimization '01, Ambrosio, Belletini, Caselles, March, Novaga, ...

H^1 versus TV regularisation

$$\min_u \left\{ \alpha R(u) + \|u - f\|_2^2 \right\}$$

(d) original

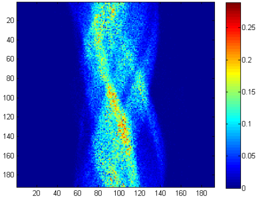
(e) noisy

(f) $R(u) = \int |Du|$

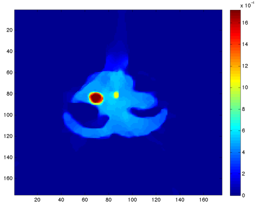
References: Rudin, Osher, Fatemi, Physica D ’92; Chambolle, Lions, Numerische Mathematik ’97; Vese, Applied Mathematics and Optimization ’01, Ambrosio, Belletini, Caselles, March, Novaga, . . .

Image from imperfect data

Positron Emission Tomography (reconstruction from Radon samples)



Measurements $f = K(\mathcal{R}u)$, where $\mathcal{R}$ Radon transform and K models imperfection in measurements.

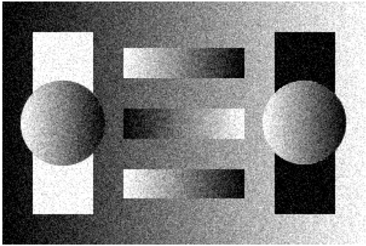


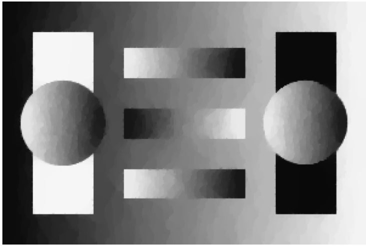
Reconstruct heart of a mouse

$$\alpha_1 \int |Du| + \alpha_2 \int D\mathcal{R}u| + \frac{1}{2}\|(\mathcal{R}u)_{|\Lambda} - f\|^2 \to \min_u$$

References: Barbano, Fokas, CBS, SAMPTA Proceedings ’11; Burger, Müller, Papoutsellis, CBS, Inverse Problems ’14

TV versus 2nd order TV regularisation





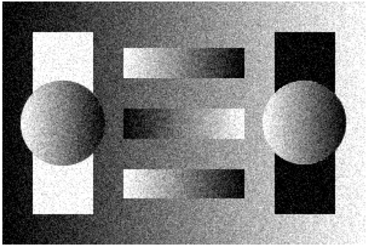
Noisy image TV denoised image

Image courtesy of K. Papafitsoros

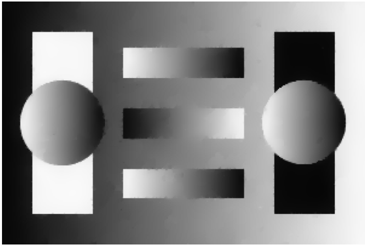
References: Chambolle, Lions, Numerische Mathematik '97; Chan, Marquina, Mulet, SSC '01; Chan, Kang, Shen, SIAM Applied Math '02; Hinterberger, Scherzer, Computing '06; Lysaker, Tai, IJCV '06; Setzer, Steidl, Approximation XII '08; Dal Maso, Fonseca, Leoni, Morini, SIAM Math. Anal. '09; Bergounioux, Piffet, Set Valued and Variational Analysis '10; Bredies, Kunisch, Pock, SIAM Imaging '10; Setzer, Steidl, Teuber, CMS '11; Lefkimmiatis, Bourquard, Unser, '12; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

TV versus 2nd order TV regularisation

$$\min_u \left\{ \min_w \left\{ \alpha_1 \int_\Omega d|Du - w|(x) + \alpha_2 \int_\Omega d|Ew|(x) \right\} + \|u - f\|_2^2 \right\}$$



Noisy image



TGV^2 denoised image

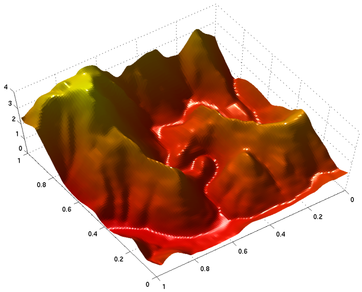
Image courtesy of K. Papafitsoros

References: Chambolle, Lions, Numerische Mathematik '97; Chan, Marquina, Mulet, SSC '01; Chan, Kang, Shen, SIAM Applied Math '02; Hinterberger, Scherzer, Computing '06; Lysaker, Tai, IJCV '06; Setzer, Steidl, Approximation XII '08; Dal Maso, Fonseca, Leoni, Morini, SIAM Math. Anal. '09; Bergounioux, Piffet, Set Valued and Variational Analysis '10; Bredies, Kunisch, Pock, SIAM Imaging '10; Setzer, Steidl, Teuber, CMS '11; Lefkimmiatis, Bourquard, Unser, '12; Papafitsoros, CBS, J. Math. Imaging & Vision, '13 . . .

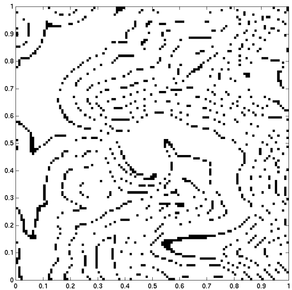
Image inpainting

Mathematics can make you fly! J. Grah, K. Papafitsoros, CBS, EPSRC Science Photo Award '14, Burger, He, CBS, SIAM Imaging Science '09; CBS, CUP '15

Anisotropic TV^3 interpolation



Ground truth

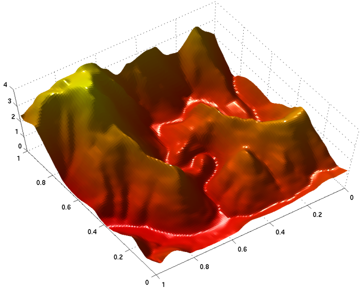


Input contours

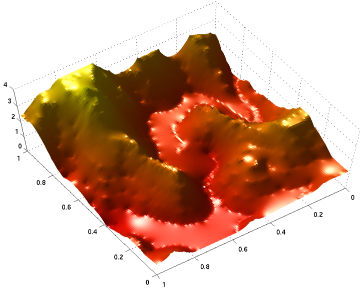
Image courtesy of J. Lellmann

References: Lellmann, Morel, CBS, Scale Space Var. Meth. Comp. Vis. '13; T. Meyer '11; Lellmann, Masnou, Parisotto, CBS, in preparation

Anisotropic TV^3 interpolation



Ground truth



TV^2

Image courtesy of J. Lellmann

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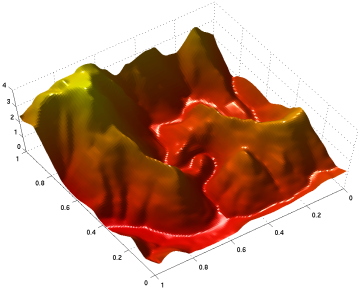
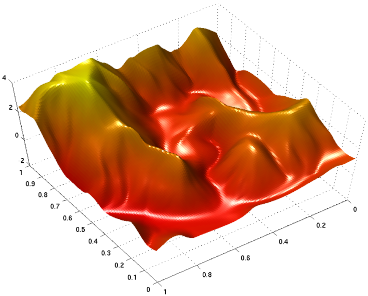
Anisotropic TV^3 interpolation



Ground truth



$\text{TV}_v^3(u) = |D^3 u(v,v,v)|)$

Image courtesy of J. Lellmann

References: Lellmann, Morel, CBS, Scale Space Var. Meth. Comp. Vis. ’13; T. Meyer ’11; Lellmann, Masnou, Parisotto, CBS, in preparation

Anisotropic nonlinear diffusion

Low quality fingerprint

Anisotropic diffusion

$$\begin{cases} u_t = \operatorname{div}\left(D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u\right) & \text{on } \Omega \times (0,\infty) \\ u(x,0) = f(x) & \text{on } \Omega \\ \langle D\left(\mathcal{J}_\rho(\nabla u_\sigma), \mathrm{OF}\right)\nabla u, \vec{n}\rangle = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

where diffusion tensor D is dependent on image structure modelled by tensor $\mathcal{J}_\rho$, and precomputed direction of orientation OF.

References: Weickert, Teubner '98; Gottschlich, CBS '10

Mathematical challenges

PDEs and Energy minimisation to analyse and process images . . .

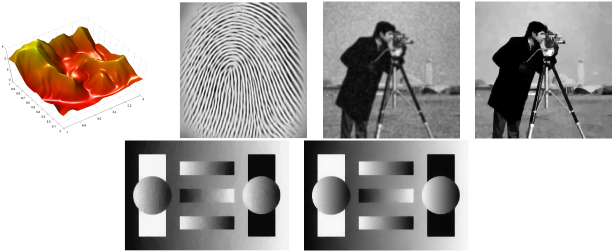
- Non-smoothness
- Non-linearity
- High differential order
- Non-convexity

Modelling $\Rightarrow$ Analysis $\Rightarrow$ Numerical solution

Requires concepts from physics, variational calculus, geometric measure theory, convex analysis, non-linear PDEs, inverse problems, optimisation, . . .

Customisation - next generation?

Which PDE / regularisation to choose?



And how much of it? . . .

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1 Customised Nonlinear PDEs for Image Analysis

2 Learning the Customised PDE from Image Data

3 Conclusions and Outlook

Generic image analysis model

For given data f we seek a regularised image u by minimising

$$\mathcal{J}(u) = \underbrace{R(u)}_{\text{Prior}} + \lambda \underbrace{\int \phi(T(u), f)}_{\text{Data model}} \to \min_{u},$$

where

- $R(u)$ is the prior (regularising) term: modelling a-priori information about the minimiser u in terms of regularity, e.g. $R(u) = \int u^2 \, dx$ which results in $u \in L^2$.
- T linear/nonlinear forward operator, $\phi(T(u), f)$ generic distance function, the data fidelity term of the functional which forces the minimiser u to obey (to a certain extent) the forward model.
- The parameter $\lambda > 0$ balances data model and prior.

Bilevel optimal reconstruction model

Assumptions

Training set of pairs $(f_k, u_k),\ k = 1, \ldots, N$ with

- f_k imperfect data
- u_k represent the ground truth

Determine optimal regulariser R, data model ϕ, and λ in admissible set $\mathcal{A}$

$$\min_{(R,\phi,\lambda)\in\mathcal{A}} \sum_k \mathrm{Cost}(\bar{u}_k, u_k)$$

subject to

$$\bar{u}_k = \mathrm{argmin}_u \left\{ R(u) + \int_\Omega \lambda\, \phi(Tu, f_k)\, dx \right\}$$

Learning by optimisation in imaging

Some related contributions

- Odone '05–, Tappen et al. '07, '09; Domke '11–: Markov Random Field models; stochastic descent method
- Lui, Lin, Zhang and Su '09: optimal control approach, no analytical justification; promising numerical results.
- Horesh, Tenorio, Haber et al. '03–: optimal design; ℓ_1 minimisation.
- Kunisch and Pock '13, Pock 13' –: results for finite dimensional case; semismooth Newton method; optimal image filters; optimal SVM;
- Chung et al. '14: finite dimensional; bounded operator T.
- Hintermüller et al. '14 – : bilevel optimisation for blind deconvolution, and for adaptive TV denoising.
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Learning by optimisation in imaging

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- ...

No results in function spaces

Learning in function space

Our goal: State and treat a nonsmooth optimization problem in function space (stick to the physical model).

- Infinite dimensional models more amenable to analysis of image features, e.g. edges.
- Lagrange multipliers and optimality condition.
- Compute optimal weights λ_i with a fast derivative-based and mesh independent optimisation method (second-order method), e.g. Hintermüller, Stadler '06; resolution independent imaging Viola, Fitzgibbon, Cipolla '12.

Learning for TV-type regularisation models . . .

Learning TV-type regularisation

Look for $\lambda = (\lambda_1, \ldots, \lambda_M)$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$ solving

$$\min_{(\lambda,\alpha)\in\mathcal{Q}^+} F(u_{\lambda,\alpha})$$

subject to

$$u_{\lambda,\alpha} \in \operatorname{argmin}_{u\in X} \sum_{i=1}^{M} \int_\Omega \lambda_i(x)\phi_i([Tu](x))\, dx + \sum_{j=1}^{N} \int_\Omega \alpha_j(x)\, d|A_j u|(x).$$

Here $T : X \to Y \subset L^1(\Omega; \mathbb{R}^d)$ with X, Y Banach spaces,
$A_j : X \to \mathcal{M}(\Omega; \mathbb{R}^{m_j})$, $(j = 1, \ldots, N)$ are appropriate linear operators,
$|A_j u|$ total variation measure, F is cost function.

Example I: Learning λ's in TV denoising

- Take $\alpha = 1$, $\lambda \in \mathbb{R}_+$.
- Take $X = BV(\Omega) \cup L^2(\Omega)$, $Y = L^2(\Omega)$, and set

$$T(u) = u, \quad A_1 = D$$

$$u_\lambda \in \text{argmin}_{u \in X} \frac{\lambda}{2} \|u - f\|_Y^2 + \int_\Omega d|Du|(x).$$

Example II: Learning (β, α) in $\text{TGV}^2_{\beta,\alpha}$

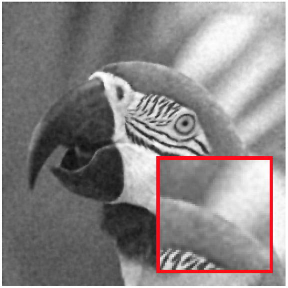
- Take $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$, $\lambda = 1$
- Take $u = (v, w)$ and $\phi(u) = \int_\Omega \phi(u(x))\, dx = \frac{1}{2} \|v - f\|^2_Y$, $Y = L^2(\Omega)$
- Take $X = BV(\Omega) \cup L^2(\Omega) \times BD(\Omega)$ and set

$$T(v, w) = v, \quad A_1 u = Dv - w, \quad \text{and} \quad A_2 u = Ew$$

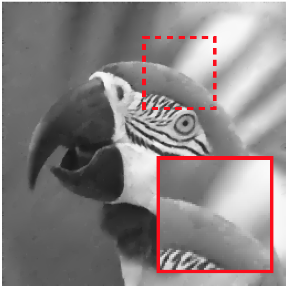
for E the symmetrised differential.

$$u_{\alpha_1,\alpha_2} \in \operatorname{argmin}_{(v,w)\in X} \frac{1}{2} \|v - f\|^2_{L^2(\Omega)} + \alpha_1 \int_\Omega d|Dv - w|(x) + \alpha_2 \int_\Omega d|Ew|(x).$$

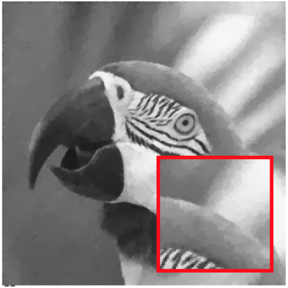
Choice of β in $\mathrm{TGV}^2_{\beta,\alpha}$



(a) Too low β / High oscillation



(b) Optimal β



(c) Too high β / almost TV

Choice of α in $\mathrm{TGV}^2_{\beta,\alpha}$

(a) Too low α, low β. Good match to noisy data

(b) Too low α, optimal β. optimal TV^2-like behaviour

(c) Too high α, high β. Bad TV^2-like behaviour

Cost function: how to measure optimality?

Cost functions: For noise free data $\tilde{u}$ we take either PSNR

$$F_{L^2}(v) = \frac{1}{2}\|\tilde{u} - v\|_2^2,$$

or Huberised TV cost

$$F_{L^1\nabla_\gamma}(v) = |D(\tilde{u} - v)|_\gamma(\Omega).$$

Existence and beyond . . .

Existence of an optimal solution (under appropriate assumptions optimal parameter lies in the interior!) ✓

Would like to use derivate-based method for the numerical solution of this problem $\Rightarrow$ need gradient of solution map, encoded in adjoint equation of ***optimality system****.*

Smoothed optimization problem

Given one training pair (f, u_{org})

$$\min \; \|\bar{u} - u_{org}\|^2_{L^2(\Omega)}$$

subject to Total Variation denoising

$$\underbrace{\mu \int_\Omega \nabla \bar{u} \cdot \nabla (v - \bar{u})\, dx}_{\text{Elliptic regularisation}} + \underbrace{(h_\gamma(\nabla \bar{u}), \nabla (v - \bar{u}))}_{\text{Huber type smoothing}} =$$

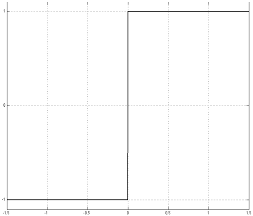
$$-\lambda \int_\Omega (\bar{u} - f)(v - \bar{u})\, dx, \; \forall v \in H^1_0(\Omega),$$

Optimization with PDE constraints

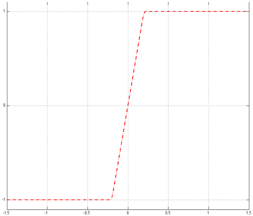
References: Barbu (1984, 1993), Tiba (1990), Bonnans-Tiba (1991), Wenbin-Rubio (1991), Bonnans-Casas (1995), Bergounioux (1998), De Los Reyes (2012)

Huber regularization

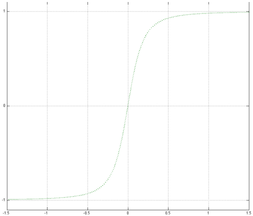
⇒ smoothing of TV measure needed



Subdifferential of $|\cdot|$

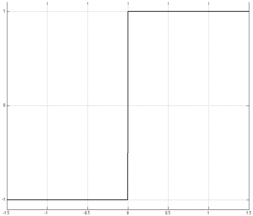


Huber type function



Huber regularization

$\Rightarrow$ smoothing of TV measure needed



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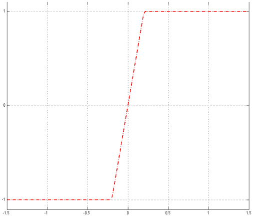
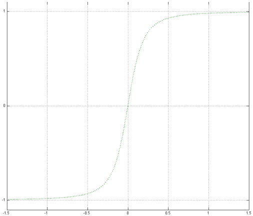


Huber type function

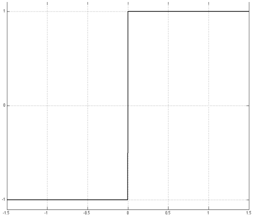


$\frac{x}{\sqrt{x^2+\epsilon^2}}$

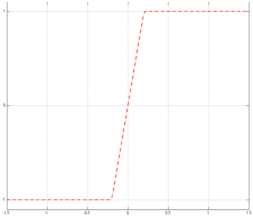
$$h_\gamma(z) = \begin{cases} \frac{z}{|z|} & \text{if } \gamma|z| \geq 1 + \frac{1}{2\gamma} \\ \frac{z}{|z|}(1 - \frac{\gamma}{2}(1 - \gamma|z| + \frac{1}{2\gamma})^2) & \text{if } 1 - \frac{1}{2\gamma} \leq \gamma|z| \leq 1 + \frac{1}{2\gamma} \\ \gamma z & \text{if } \gamma|z| \leq 1 - \frac{1}{2\gamma}, \end{cases}$$

Huber regularization

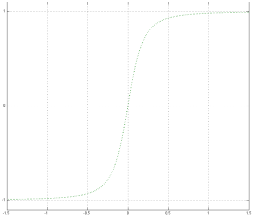
$\Rightarrow$ smoothing of TV measure needed



Subdifferential of $|\cdot|$



Huber type function ✓

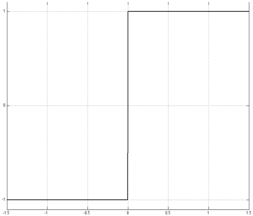


~~$\frac{x}{\sqrt{x^2+\epsilon^2}}$~~

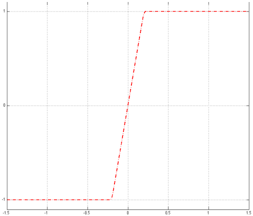
$$h_\gamma(z) = \begin{cases} \frac{z}{|z|} & \text{if } \gamma|z| \geq 1 + \frac{1}{2\gamma} \\ \frac{z}{|z|}(1 - \frac{\gamma}{2}(1 - \gamma|z| + \frac{1}{2\gamma})^2) & \text{if } 1 - \frac{1}{2\gamma} \leq \gamma|z| \leq 1 + \frac{1}{2\gamma} \\ \gamma z & \text{if } \gamma|z| \leq 1 - \frac{1}{2\gamma}, \end{cases}$$

Huber regularization

$\Rightarrow$ smoothing of TV measure needed



Subdifferential of $|\cdot|$



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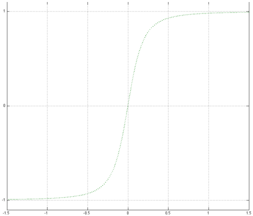

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$$h_\gamma(z) = \begin{cases} \frac{z}{|z|} & \text{if } \gamma|z| \geq 1 + \frac{1}{2\gamma} \\ \frac{z}{|z|}\left(1 - \frac{\gamma}{2}(1 - \gamma|z| + \frac{1}{2\gamma})^2\right) & \text{if } 1 - \frac{1}{2\gamma} \leq \gamma|z| \leq 1 + \frac{1}{2\gamma} \\ \gamma z & \text{if } \gamma|z| \leq 1 - \frac{1}{2\gamma}, \end{cases}$$

Smoothed optimization problem

Given one training pair (f, u_{org})

$$\min \; \|\bar{u} - u_{org}\|^2_{L^2(\Omega)}$$

subject to Total Variation denoising

$$\underbrace{\mu \int_\Omega \nabla \bar{u} \cdot \nabla (v - \bar{u}) \, dx}_{\text{Elliptic regularisation}} + \underbrace{(h_\gamma(\nabla \bar{u}), \nabla(v - \bar{u}))}_{\text{Huber type smoothing}} =$$

$$- \lambda \int_\Omega (\bar{u} - f)(v - \bar{u}) \, dx, \; \forall v \in H^1_0(\Omega),$$

Optimization with PDE constraints

References: Barbu (1984, 1993), Tiba (1990), Bonnans-Tiba (1991), Wenbin-Rubio (1991), Bonnans-Casas (1995), Bergounioux (1998), De Los Reyes (2012)

In this setting we can prove . . .

- existence of an optimal solution, also in the case $\mu = 0$ (under appropriate assumptions optimal parameter lies in the interior!) ✓
- convergence of optimal parameters and corresponding reconstructions to solution of original, non-smooth optimisation problem as $\gamma \to +\infty$ and $\mu \to 0$ ✓
- differentiability of solution operator and derivation of sharp optimality system ✓ De Los Reyes '12
- convergence of optimality system as Huber regularisation $\gamma \to +\infty$ to sharp optimality system for non-smooth problem De Los Reyes '12.

. . . and in the numerics the parameters $0 < \mu \ll 1$ and $\gamma \gg 1$.

. . . open: limit of optimality system for $\mu \to 0$???

References: De Los Reyes 2012; CBS, De Los Reyes 2013; Calatroni, CBS, De Los Reyes 2014; **De Los Reyes, CBS, Valkonen 2015**

Numerical strategy

Solve

$$\min_{(\lambda,\alpha)\in\mathcal{Q}^+} F(u_{\lambda,\alpha})$$

subject to

$$u_{\lambda,\alpha} = \text{argmin}_u$$
$$\frac{\mu}{2}\|\nabla u\|_2^2 + \sum_{i=1}^{M}\int_\Omega \lambda_i(x)\phi_i(x,[Tu](x))\,dx + \sum_{j=1}^{N}\int_\Omega \alpha_j(x)\,d|A_j u|_\gamma(x).$$

by quasi-Newton method (BFGS)

- state equation is solved by Newton type algorithm (varies with ϕ)
- evaluation of the gradient of the cost functional with adjoint information
- Armijo line search with curvature verification.
- For $M, N \gg 1$ we use dynamic sampling technique for constraints á la Byrd et al.

Parameters: we typically choose $10^{-10} \approx \mu \ll 1$, $100 \approx \gamma \gg 1$.

Some examples

Optimal parameter for TV

$$\min_{\lambda \geq 0} \|u - u_k\|_{L^2}^2$$

subject to:

$$\min_{u \geq 0} \left\{ \frac{\mu}{2} \|Du\|_{L^2}^2 + \|Du\|_\gamma + \frac{\lambda}{2} \|u - f_k\|_{L^2}^2 \right\}$$

Noise $n \in N(0, 0.002)$ (optimal parameter $\lambda^* = 2980$)

Optimal parameter for TV

$$\min_{\lambda \geq 0} \ \|u - u_k\|_{L^2}^2$$

subject to:

$$\min_{u \geq 0} \left\{ \frac{\mu}{2} \|Du\|_{L^2}^2 + \|Du\|_\gamma + \frac{\lambda}{2} \|u - f_k\|_{L^2}^2 \right\}$$

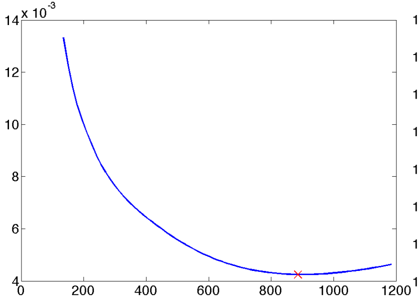
Noise $n \in N(0, 0.02)$ (optimal parameter $\lambda^* = 1770.9$)

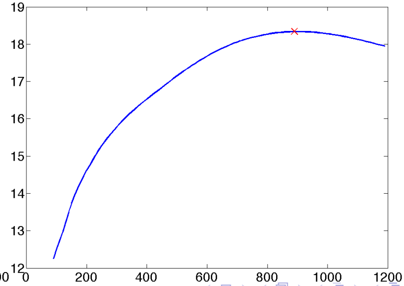
Optimality?

Quality measure

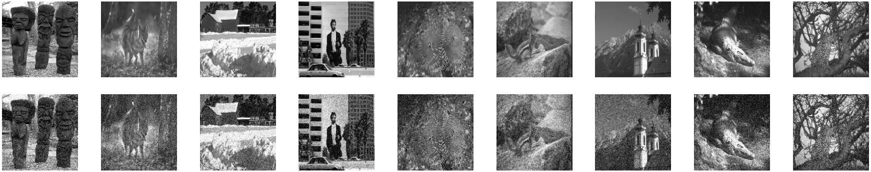
- Original cost functional (left figure) $\|u - u_k\|_{L^2}^2$
- Signal to noise ratio (right figure)

$$SNR = 20 \times \log_{10}\left(\frac{\|u_k\|_{L^2}}{\|u - u_k\|_{L^2}}\right),$$





Robustness and efficiency



N	10	20	30	40
λ^*	2732.15	2766.32	2170.23	2292.51

Learning (β, α) in $\mathrm{TGV}^2_{\beta,\alpha}$

(a) Original image

(b) Noisy image

Optimal $\mathrm{TGV}^2_{\beta,\alpha}$

(c) TGV^2 denoising, $L_1\nabla_\gamma$ cost functional

(d) TGV^2 denoising, L_2^2 cost functional

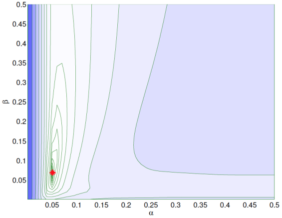
$L^1\nabla_\gamma$ cost

$$(\alpha, \beta) = (0.069/n^2, 0.051/n)$$

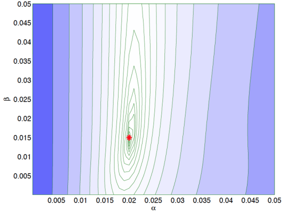
L^2 cost

$$(\alpha, \beta) = (0.058/n^2, 0.041/n)$$

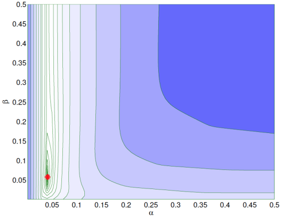
Optimal (β, α) in $\text{TGV}^2_{\beta,\alpha}$?



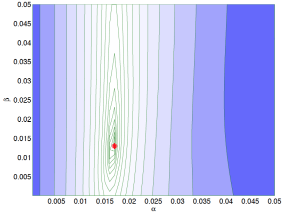
(a) Parrot, TGV^2, $L_1\nabla_\gamma$ cost functional



(b) Uplands, TGV^2, $L_1\nabla_\gamma$ cost functional



(c) Parrot, TGV^2, L_2^2 cost functional



(d) Uplands, TGV^2, L_2^2 cost functional

For TGV a good initialisation is important!

TV versus TGV^2 versus ICTV

(g) TV denoising, $L^1_\eta\nabla$ cost

(e) ICTV denoising, $L^1_\eta\nabla$ cost

(c) TGV^2 denoising, $L^1_\eta\nabla$ cost

TV versus TGV^2 versus ICTV

Test performance of TV versus TGV versus ICTV regularisation on 200 images from Berkeley image database; for noise levels $\sigma^2 = 2, 10$ and 20; and for the two cost functionals.

Evaluate performance wrt PSNR, SIIM, and cost functional (L^2 and L$^1\nabla$).

Perform statistical 95% one-tailed paired t-test on each of criteria, and pair of regularisers, to see whether any pair of regularisers can be ordered.

TV versus TGV^2 versus ICTV

Noise level $\sigma^2 = 20$; cost functional is $L^1\nabla$.

Colour coding: TV best; ICTV best; TGV best

TV versus TGV^2 versus ICTV

Overall result:

- Overall, studying the t-test and other data, the ordering of the regularisers appears to be

$$ICTV > TGV^2 > TV$$

- For high noise TGV^2 and ICTV performance are comparable.
- TGV^2 is better than ICTV for images with large smooth areas.
- L^2 cost corresponds to high PSNR; L$^1\nabla$ cost seems to relate to high SIIM.

Impulse noise

$$\min \ \frac{1}{2}\|u - u_{org}\|_{L^2}^2$$

subject to:

$$\min_{u\geq 0}\left\{\frac{\mu}{2}\|Du\|_{L^2}^2 + \|Du\|_\gamma + \lambda\|u-f\|_\gamma\right\}$$

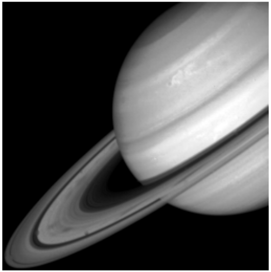
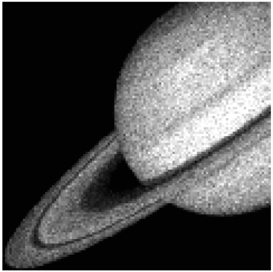
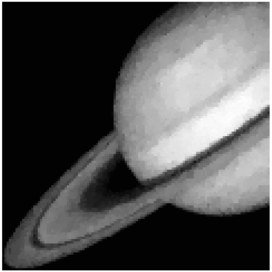
Impulse noise with 5% corrupted pixels; optimal parameter $\lambda^* = 45.88$

Poisson noise

$$\min_{\lambda \geq 0} \frac{1}{2} \|u - u_{org}\|_{L^2}^2$$

subject to:

$$\min_{u \geq 0} \left\{ \frac{\mu}{2} \|Du\|_{L^2}^2 + \|Du\|_\gamma + \lambda \int_\Omega (u - f \log u) \, dx \right\}.$$

Optimal parameter λ^*= 1013.76.

Spatial dependent noise

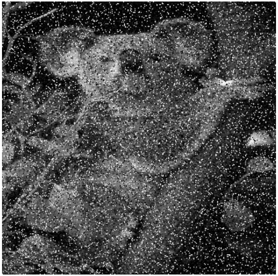
Gaussian noise with $\sigma = 0.04$ outside of region outlined in red and $\sigma = 0.06$ inside.

Mixed Impulse & Gaussian noise

$$\min_{\lambda_1,\lambda_2\geq 0} \frac{1}{2}\|u-u_{org}\|_{L^2}^2$$

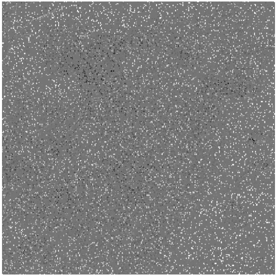
where u is the solution of the optimisation problem:

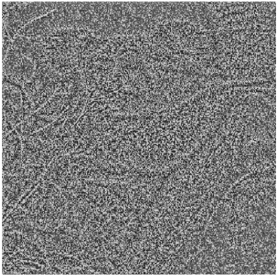
$$\min_{\substack{v\in BV\\ n\in L^2}} \left\{\frac{\mu}{2}\|\nabla v\|_{L^2}^2 + \|Dv\|_\gamma + \lambda_1\|n\|_\gamma + \lambda_2\|f-v-n\|_{L^2}^2\right\},$$



Original image (left) and noisy image (right) corrupted by impulse

Mixed Impulse & Gaussian noise





From left to right: Denoised image, impulse noise residuum and Gaussian noise residuum. Optimal parameters: $\lambda_1^* = 351.23$ and $\lambda_2^* = 5200.1$.

Outline

1. Customised Nonlinear PDEs for Image Analysis
2. Learning the Customised PDE from Image Data
3. Conclusions and Outlook

Data learning versus physical modelling?

Data learning versus physical modelling?

Physical modelling	Classical data learning
physical model	non-physical
non-adaptive to data	adaptive to data
insight in structure of problem	blackbox
reconstruction guarantees stability, error analysis, . . .	in general no guarantees guarantees optimality?
heavily relies on a-priori model	learns the model from the data.

Data learning versus physical modelling?

Happy marriage of physical modelling and data learning?

Data learning versus physical modelling?

Happy marriage of physical modelling and data learning?

Hope: adaptive physical models.

Conclusions and outlook

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- Nonlinear PDEs for customised image analysis
- Customise PDE to image data by bilevel optimisation
- Analysis of bilevel model in function space
- Example: Choice of optimal TV-based regularisation

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- Example: Choice of optimal TV-based regularisation

Outlook:

- Alternative cost functionals. How to measure optimality? Non-reference quality measures.
- Model learning for inverse problems: general linear/nonlinear operator T (MRI, PET, ET, ...)
- Learning other elements in the model, e.g. acquisition (sampling), inpainting procedure, segmentation (Leaci, Tomarelli, CBS) ...
- Combine model learning with Bayesian statistics (Peyrera)

Thank you very much for your attention!

- J. C. De Los Reyes, and C.-B. Schönlieb, *Image denoising: Learning noise distribution via PDE-constrained optimisation*, Inverse Problems and Imaging, Vol. 7, 1183-1214, 2013.
- L. Calatroni, J. C. De Los Reyes, and C.-B. Schönlieb, *Dynamic sampling schemes for optimal noise learning under multiple nonsmooth constraints*, IFIP TC7-2013 proceedings.
- L. Calatroni, C. Cao, J. C. De Los Reyes, C.-B. Schönlieb, and T. Valkonen, *Bilevel approaches for learning of variational imaging models*, to appear in Radon book series 2015.
- J. C. De Los Reyes, and C.-B. Schönlieb, and T. Valkonen, *The structure of optimal parameters for image restoration problems*, Journal of Mathematical Analysis and Applications 434 (2016), 464-500.
- J. C. De Los Reyes, and C.-B. Schönlieb, and T. Valkonen, *Optimal parameter learning for higher-order total variation regularisation models*, submitted.

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