A Domain Decomposition Projection Method for the Navier-Stokes Equations Based on the Multiscale Robin Coupled Method

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Introduction

- Simulation of flows modeled by the Navier-Stokes equations in the pore-scale
- The domain for the PDEs will be provided by images from CT-scanners

Outline

- The Navier-Stokes equations
- Projection Method
- The Multiscale Robin Coupled Method
- Results
- Conclusions

Solving the Navier-Stokes Equations

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Navier-Stokes Equations[1] [2]

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{\mu}{\rho} \nabla^2 \mathbf{u} + \mathbf{F}$$

- This system of equations couples the velocity and pressure fields
- Difficulties in numerical simulation: size and non-linearities
- A method that decouples the pressure and velocity fields was proposed by Chorin[3], it is known as the *Projection Method*

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[1] WHITE, F M. McGraw-Hill, 2009

[2] PROSPERETTI, A. TRYGGVASON, G. Cambridge University Press, 2009

[3] CHORIN, A J. Mathematics of computation, 1968

The Projection Method[1]

Non-incremental projection method

- Intermediate velocity field $\mathbf{w}$ considering $\nabla P^{n+1} = 0$

$$\frac{\mathbf{w} - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}^n = \frac{\mu}{\rho} \nabla^2 \mathbf{u}^n + \mathbf{F}^n$$

- $\mathbf{u}^{n+1}$ is calculated by

$$\frac{\mathbf{u}^{n+1} - \mathbf{w}}{\Delta t} = -\frac{1}{\rho} \nabla P^{n+1}$$

- P^{n+1} is calculated so that $\nabla \cdot \mathbf{u}^{n+1} = 0$ is satisfied:

$$\nabla \cdot \left(\frac{1}{\rho} \nabla P^{n+1} \right) = \frac{1}{\Delta t} \nabla \cdot \mathbf{w}, \quad \nabla P \cdot \mathbf{n}|_{\partial \Omega} = 0$$

[1] CHORIN, A J. Mathematics of computation, 1968

Multiscale Methods

Mixed finite element methods

- *Multiscale Mortar Mixed Finite Element Method (MMMFEM)*[1]
- *Multiscale Hybrid-Mixed Finite Element Method (MHM)*[2]
- *Multiscale Mixed Method (MuMM)*[3]
- *Multiscale Robin Coupled Method (MRCM)*[4]
  - Variational formulation of the MuMM with great flexibility for choosing the interface spaces.

[1] ARBOGAST *et al*, SIAM Multiscale Modeling and Simulation, 2007
[2] HARDER *et al*, Journal of Computational Physics, 2013
[3] FRANCISCO *et al*, Mathematics and Computers in Simulation, 2014
[4] GUIRALDELLO, R. T. *et al*, Journal of Computational Physics, 2018

The MRCM

- Pressure and flux continuity ensured by the Robin boundary condition on the interfaces:

$$-\beta_i \mathbf{v}^i \cdot \mathbf{n}^i + P^i = \beta_j \mathbf{v}^j \cdot \mathbf{n}^j + P^j$$

- MMMFEM-like solutions:

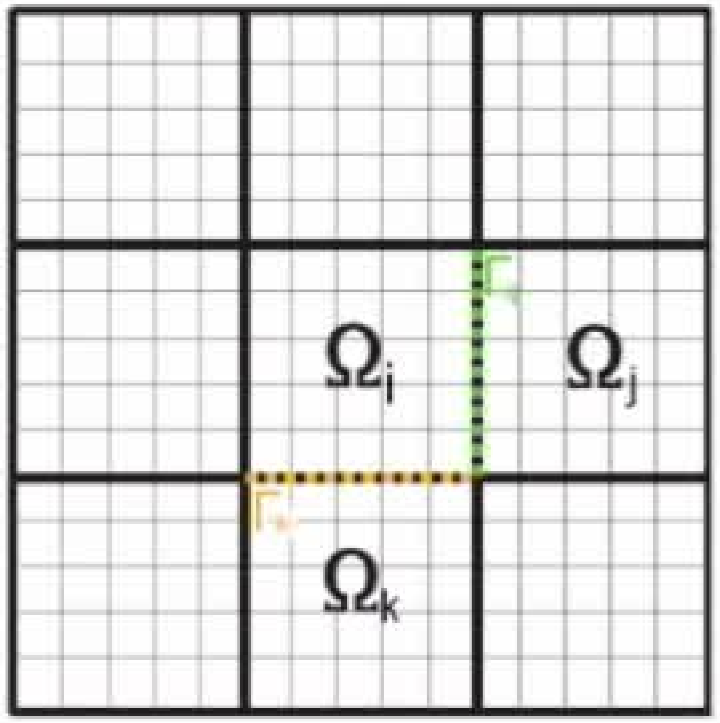
$$\beta_i \rightarrow 0$$

- MHM-like solutions:

$$\beta_i \rightarrow \infty$$

- The interface spaces will be denoted by $\mathcal{P}_{H,k}$ and $\mathcal{U}_{H,k}$ where k is the polynomials degree

Solving the elliptic equation



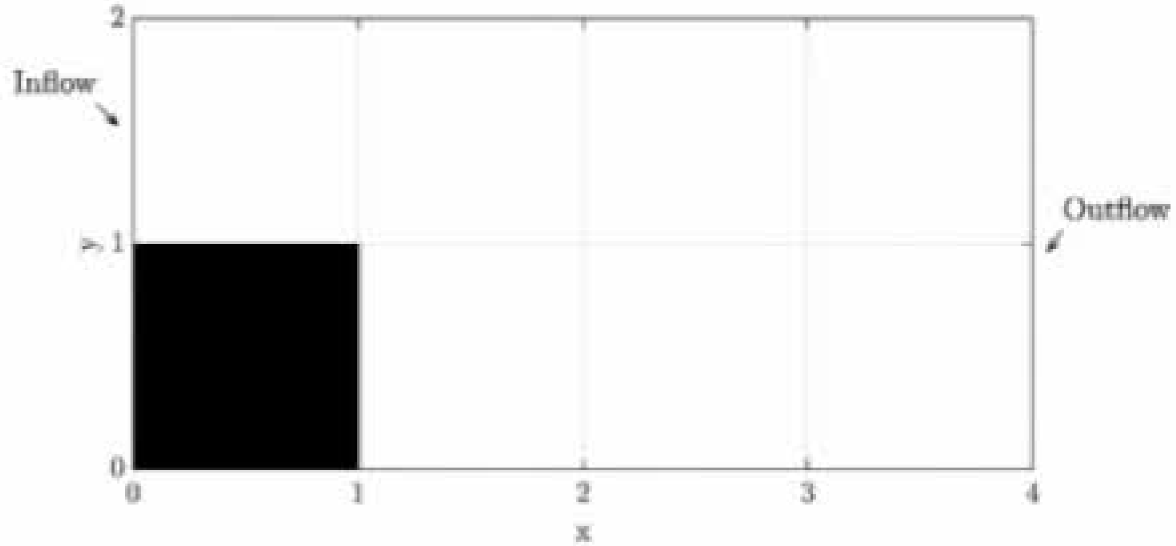
Find $(\mathbf{v}, P)$ so that

$$\begin{cases} \mathbf{v} = \dfrac{1}{\rho}\nabla P & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = \dfrac{1}{\Delta t}\nabla \cdot \mathbf{w} & \text{in } \Omega \\ \mathbf{v} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega_{\mathbf{v}} \end{cases}$$

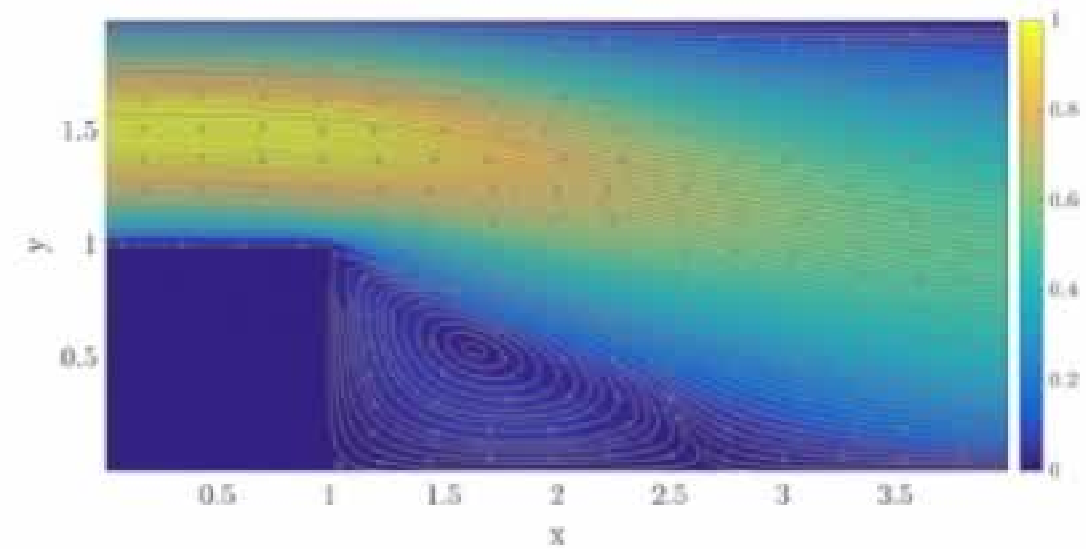
- *Offline stage*:
  - Construction of multiscale base functions, with Robin boundary conditions: **local problems**
- *Online stage*:
  - One local problem in each subdomain: **changes in the source term**
  - Solution of the global interface problem: **coupling of the sub-domains**

Channel flow behind a backward-facing step

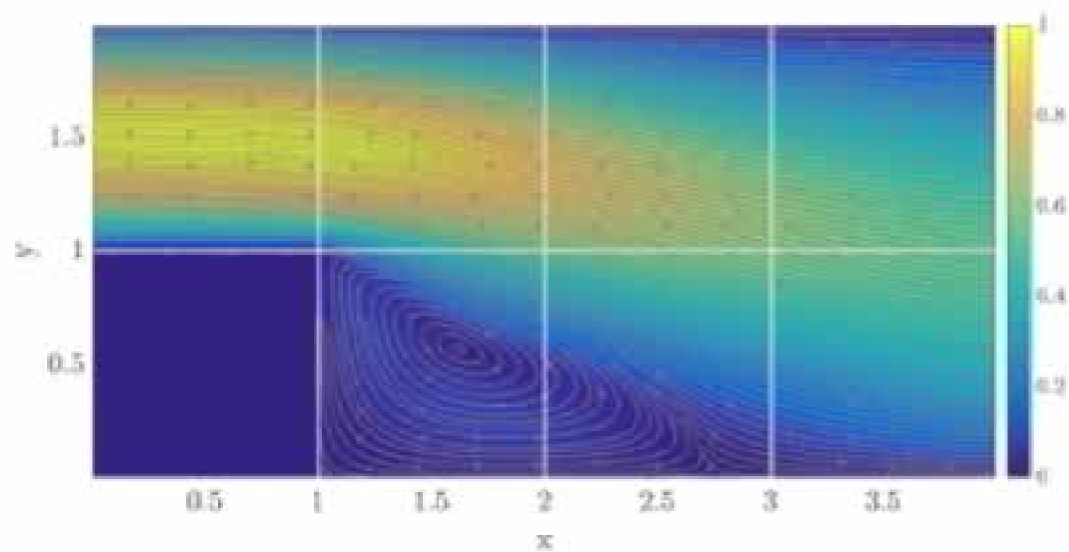
Domain[1]



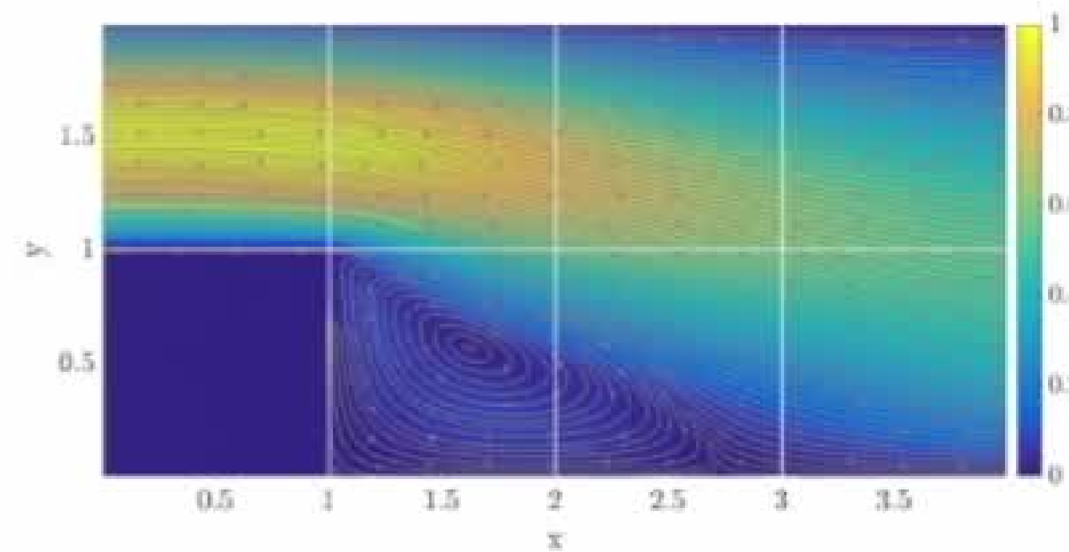
Fine mesh solution



MRCM with $\mathcal{P}_{H,0}, \mathcal{U}_{H,1}$



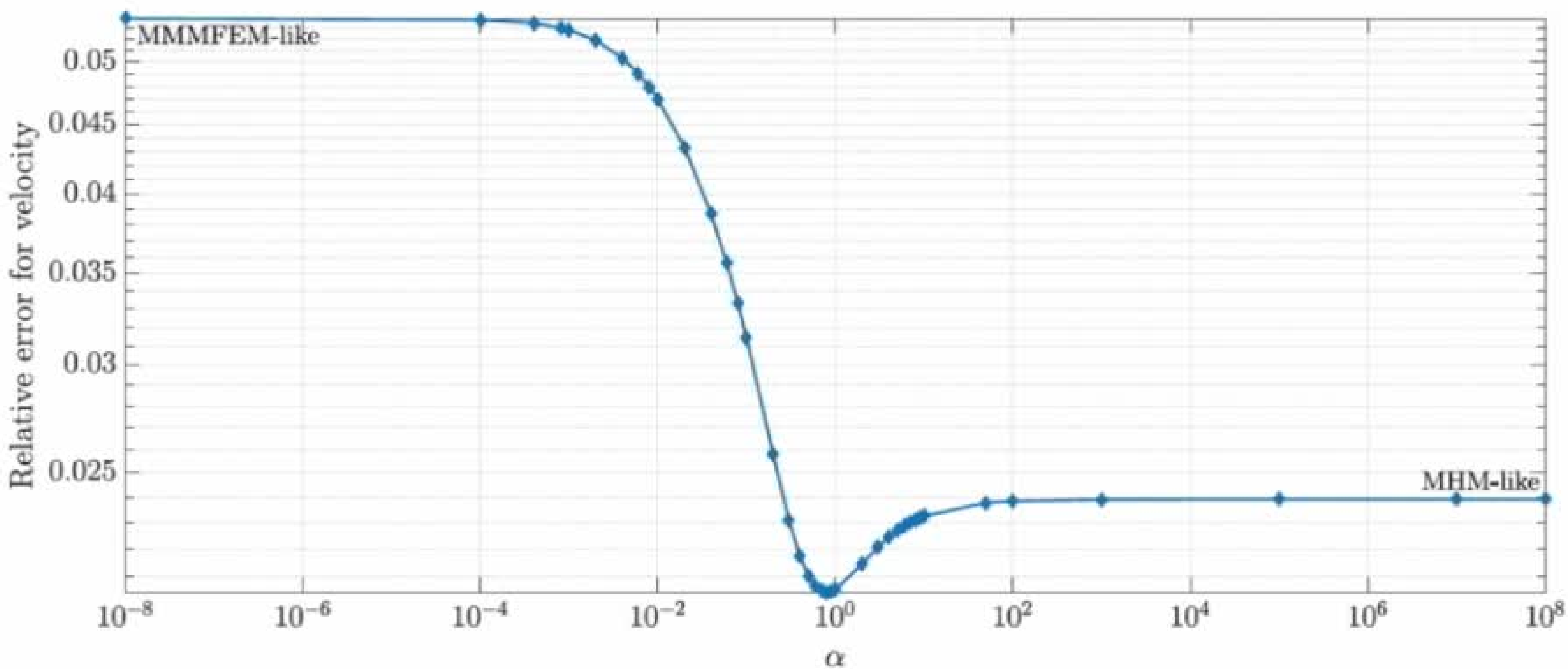
MRCM with $\mathcal{P}_{H,1}, \mathcal{U}_{H,1}$



[1] BISWAS, G. *et al*, Journal of fluids engineering, 2004

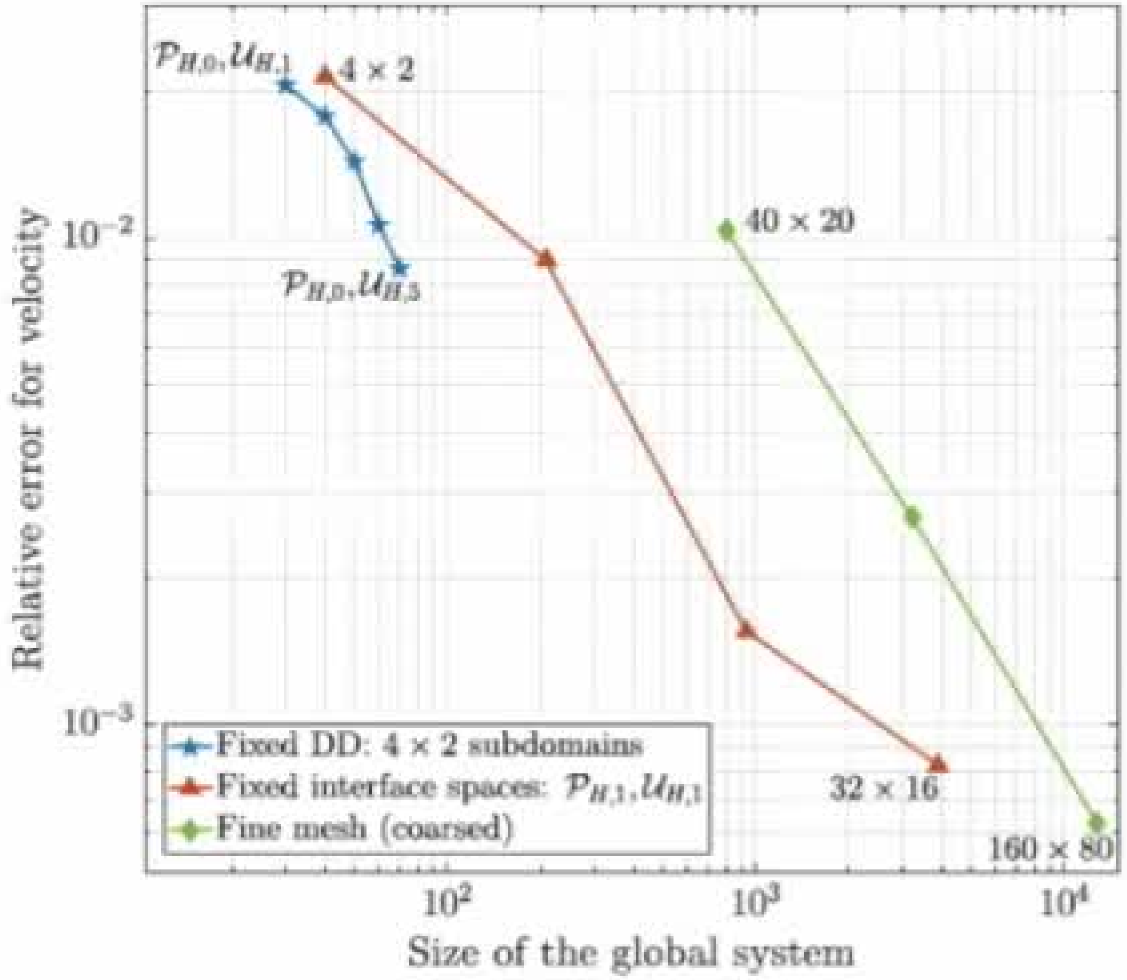
Channel flow behind a backward-facing step

- Relative errors calculated with different values of $\beta_i = \dfrac{\alpha\rho}{H}$
- $\mathcal{P}_{H,0}, \mathcal{U}_{H,1}$

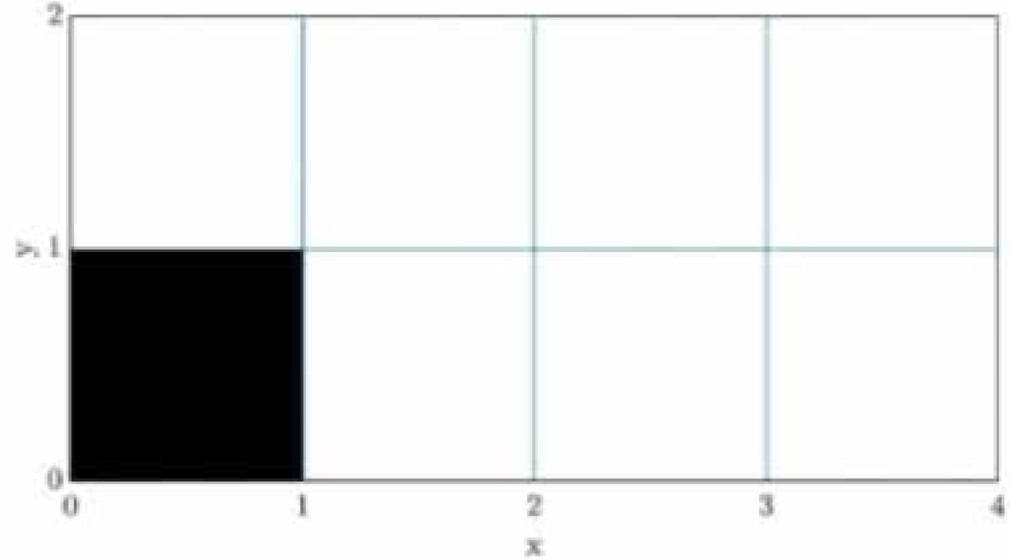


Channel flow behind a backward-facing step

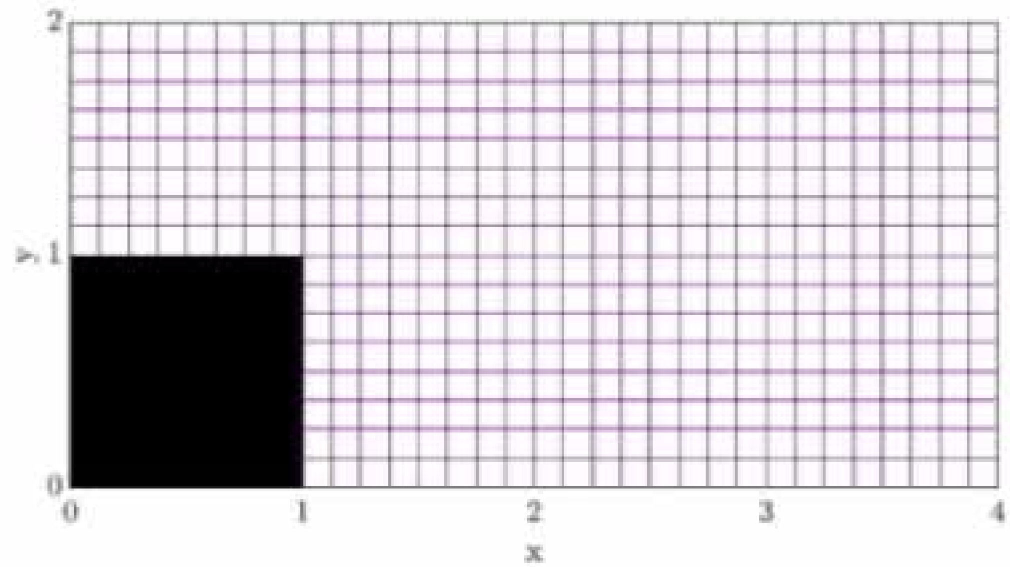
- Fixed interface spaces and refining the domain decomposition,
- Fixed domain decomposition enhancing the flux interface space,
- Fine mesh reference: 320×160 cells,
- $\alpha = 1$.



Coarsest domain decomposition



Finest domain decomposition



Conclusions

- These results provide a strong support for the benefits of using the *offline* and *online* stages when solving the Navier-Stokes equations
- We presented an appropriate method for multicore devices: both *offline* and *online* stages are naturally parallelizable
- Best strategy: to fix the interface spaces and refine the domain decomposition

Thank you!

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