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Meander of spiral waves and the St. Petersburg paradox

Vadim N. Biktashev and Ian Melbourne

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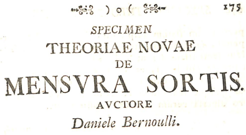
SIAM DS19, Snowbird, May 20, 2019

Daniel Bernoulli and the "St Petersburg paradox"

"Exposition of a new theory on the measurement of risk", *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, **5**:175-192, 1738

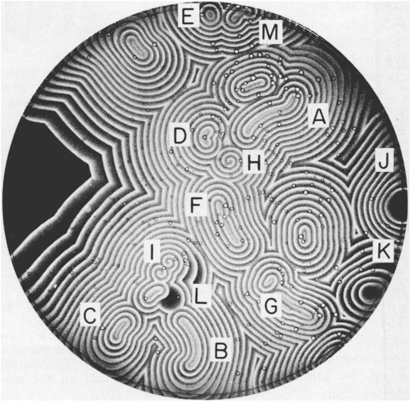
Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. . . . The accepted method of calculation does, indeed, value Paul's prospects at infinity though no one would be willing to purchase it at a moderately high price.


175

SPECIMEN THEORIAE NOVAE DE MENSVRA SORTIS. AVCTORE *Daniele Bernoulli.*

Meander: first experimental evidence

A.T.Winfree "Scroll-Shaped Waves of Chemical Activity in Three Dimensions", *Science* **181**:937–939, 1973



in press). Realization of a stationary scroll axis may require better-controlled experimental conditions than I have yet contrived. The disintegration of elongated λ_0-spiral sources and λ_0-ring sources into shorter segments of the same total parity shows that the scroll axis slowly drifts or writhes about. Even when nearly perpendicular to both interfaces, in thin liquid layers or in a Millipore, it may move about: microscopic observation shows that the interior tip of the involute spiral wave does not propagate quite in circles around a stationary center, but rather meanders in loops of length of the order of λ_0 throughout a core region of diameter about λ_0/π. Thus, no volume element escapes excitation to blue from the orange quiescent state within the span of a few rotations. I do not know whether this symmetry-breaking instability of the scroll axis is due to the interaction of reaction and diffusion, or to local inhomogeneities of temperature, Millipore density, and so forth.

Meander: numerical simulations

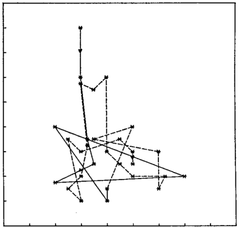


Fig. 6. Meandering of subsequent positions of point **q** (as defined in Fig. 5), from time $t = 5.15$ to time $t = 6.4$. Solid line: time step size $\Delta t = 0.025$; dashed line (starting at $t = 5.4$): $\Delta t = 0.05$. Axes: $0.46 \ldots 0.64$ (horizontal) and $0.32 \ldots 0.5$ (vertical).

O.E. Rössler and C. Kahlert "Winfree Meandering in a 2-Dimensional 2-Variable Excitable Medium", *Z. Naturforsch.* **34a**:565-570, 1979

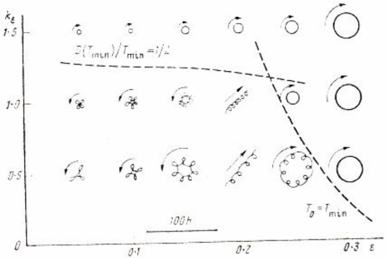
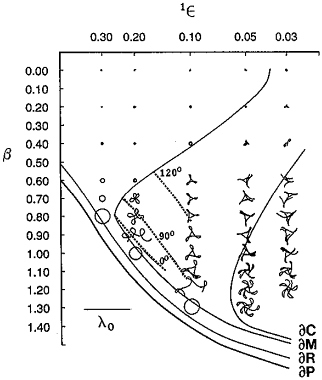


Fig. 3. Trajectory of movement of tip of helical wave obtained in computational experiments for different values of the coefficients ε and k_ε.

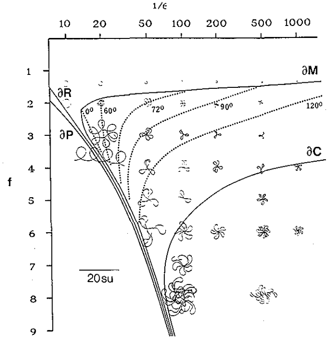
V.S. Zykov "Cycloid circulation of spiral waves in an excitable medium", *Biofizika* **31**(5):862–865, 1986

Hypermeander: "Complex rotation" in simulations

FitzHugh-Nagumo



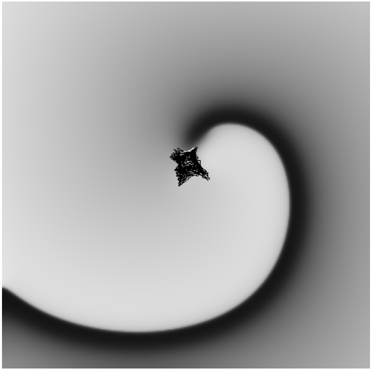
Oregonator



A.T. Winfree "Varieties of spiral wave behaviour: An experimentalist's approach to the theory of excitable media", *Chaos* **1**(3):303–334, 1991

(show the movies?)

Spiral tip trajectories in two variants of the model of guinea pig ventricular tissue



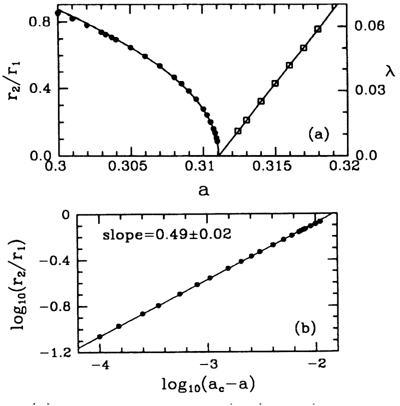
"Zykov" meander in a model with standard parameters (Biktashev and Holden, 1996)

Hypermeander: parameter changed to represent Long QT syndrome (Biktashev and Holden, 1998)

Transition to meander as a Hopf bifurcation

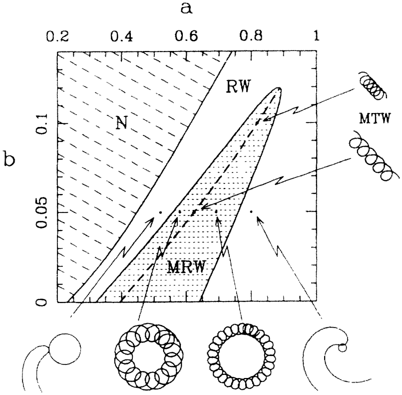
D. Barkley, M. Kness and L.S. Tuckerman "Spiral-wave dynamics in a simple model of excitable media: The transition from simple to compound rotation", *Phys. Rev. A* **42**(4):2489–2492, Aug 1990

independently: A. Karma "Meandering Transition in Two-Dimensional Excitable Media", *Phys. Rev. Lett.* **65**(22):2824–2827, Nov. 1990



Barkley's "normal form"

D. Barkley "Euclidean Symmetry and the Dynamics of Rotating Spiral Waves" *Phys. Rev. Lett.* **72**(1):164–167



Letting $p = x + iy$ and $v = se^{i\phi}$, with "speed" $s \geq 0$, Eqs. (2) become

$$\dot{x} = s\cos\phi, \quad \dot{y} = s\sin\phi, \quad \dot{\phi} = w \cdot h(s^2, w^2), \\ \dot{s} = s \cdot f(s^2, w^2), \quad \dot{w} = w \cdot g(s^2, w^2). \tag{3}$$

We consider the following expansions for f, g, and h:

$$f(s^2, w^2) = \alpha_0 + \alpha_1 s^2 + \alpha_2 w^2 - s^4, \\ g(s^2, w^2) = -1 + \beta_1 s^2 - w^2, \\ h(s^2, w^2) = \gamma_0. \tag{4}$$

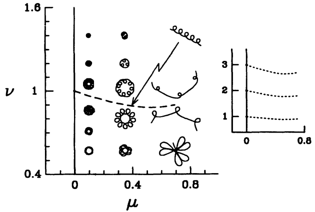


FIG. 3. Phase diagram for the model equations. Numerically obtained plots of $p = x + iy$ over short time intervals are shown centered on corresponding parameter points. The inset shows the three lowest-order resonant bifurcations. Dashed curves show loci of MTW states.

System "reaction-diffusion"

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{u}),$$

where

$$\begin{aligned}
\mathbf{u} &= \left(u^{(1)}, \ldots, u^{(n)}\right)^\top = \mathbf{u}(\vec{r}, t) \in \mathbb{R}^n && \text{(concentrations)};\\
\mathbf{f} &= \mathbf{f}(\mathbf{u}); && \text{(reaction rates)};\\
\mathbf{D} &\in \mathbb{R}^{n\times n}, && \text{(matrix of diffusion coeffs)};\\
n &\geq 2, && \text{(number of components)};\\
\vec{r} &= (x, y) \in \mathbb{R}^2 && \text{(physical space)}.
\end{aligned}$$

This system is equivariant with respect to Euclidean transformations of the spatial coordinates $\vec{r}$.

Reaction-diffusion system as an ODE in functional space

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{u})$$

in a suitable functional space $\mathcal{B}$ can be written as

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{F}(\mathbf{U}),$$

where

$$\mathbf{U} : \mathbb{R} \to \mathcal{B} \qquad \text{represents } \mathbf{u},$$
$$\mathbf{F} : \mathcal{B} \to \mathcal{B} \qquad \text{represents } \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}.$$

An equivariant ODE

Let us suppose that

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{F}(\mathbf{U})$$

is equivariant with respect to a representation T of a Lie group $\mathcal{G}$ in $\mathcal{B}$:

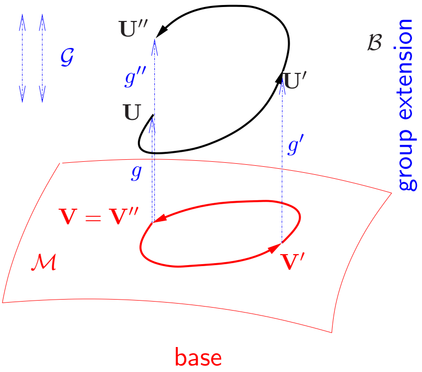
$$\forall g \in \mathcal{G}, \quad \forall \mathbf{U} \in \mathcal{B} : \mathbf{F}(T(g)\mathbf{U}) = T(g)\mathbf{F}(\mathbf{U}).$$

For the reaction-diffusion system, this is the special Euclidean group acting via transformations of $\vec{r}$:

$$\mathcal{G} = SE(2), \qquad \mathcal{G} \ni g : \mathbb{R}^2 \to \mathbb{R}^2,$$
$$T(g)\mathbf{u}(\vec{r}) = \mathbf{u}(g^{-1}\vec{r}), \qquad T(g) : \mathcal{B} \to \mathcal{B}.$$

Decomposition of a trajectory: geometrically

Skew-product decomposition of an equivariant flow using a Representative Manifold $\mathcal{M}$ (RM), which has exactly one transversal intersection with every group orbit $g \in \mathcal{G}$ (GO) and is diffeomorphic to the orbit manifold. Trajectory $(\mathbf{U}, \mathbf{U}', \mathbf{U}'')$ of an equivariant flow in $\mathcal{B}$ is a *relative periodic orbit*: it projects onto the trajectory $(\mathbf{V}, \mathbf{V}', \mathbf{V}'' = \mathbf{V})$ on $\mathcal{M}$ which is periodic. The flow on $\mathcal{M}$ is devoid of symmetry $\mathcal{G}$.



Decomposition of a trajectory: analytically

So for all $t \geq 0$, we have

$$\mathbf{U}(t) = T(g)\mathbf{V}(t)$$

where

$$\text{(RM)} \qquad \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F}_{\mathcal{M}}(\mathbf{V}) \qquad \text{base}$$

$$\text{(GO)} \qquad T(g^{-1})\frac{\mathrm{d}\,T(g)}{\mathrm{d}t}\mathbf{V} = \mathbf{F}_{\mathcal{G}}(\mathbf{V}) \qquad \text{extension}$$

Result: skew-product description

Base: reaction-diffusion in the tip frame of reference

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{D}\nabla^2\mathbf{v} + \mathbf{f}(\mathbf{v}) + (\vec{c}\cdot\nabla)\mathbf{v} + \omega\frac{\partial \mathbf{v}}{\partial \theta}, \quad \text{reaction+diffusion+advection}$$

$$v^{(l_1)}(\vec{0},t) = u_*, \quad v^{(l_2)}(\vec{0},t) = v_*, \quad \text{tip position}$$

$$\frac{\partial v^{(l_3)}(\vec{0},t)}{\partial x} = 0, \quad \text{tip orientation}$$

Extension: tip equations of motion

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \omega, \quad \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = e^{\hat{\gamma}\Theta}\vec{c}.$$

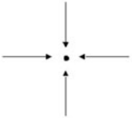
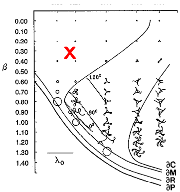
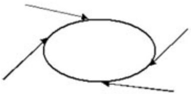
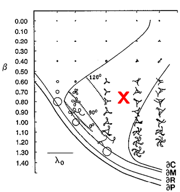
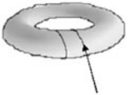
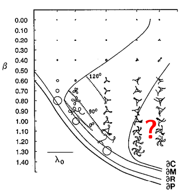
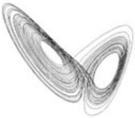
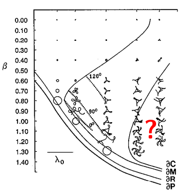
Dynamic variables: $\mathbf{v}(\vec{r},t)$, $\vec{c}(t)$, $\omega(t)$, $\vec{R}(t)$ and $\Theta(t)$.
(NB: can identify $\mathbb{R}^2 \to \mathbb{C}$, $\hat{\gamma} \to i$).

The four types of spiral waves

Base dynamics	**Group extension**
Fixed point	Rigid rotation
Limit cycle	Zykov's "cycloidal"
Quasiperiodic	Hypermeander ?
Chaotic	Hypermeander ?

- Hopf normal point in base system $\Leftrightarrow$ "Barkley normal form" (up to change of variables)
- Chaotic base dynamics $\Rightarrow$ deterministic Brownian motion of tip (Biktashev & Holden 1998)
- Quasiperiodic base dynamics $\Rightarrow$ tip trajectories almost certainly bounded (Nicol, Melbourne & Ashwin 2001)

The four types of spiral waves

Base (generic)	Extension (meander pattern)	Base (generic)	Extension (meander pattern)
Point Attractor (Equilibrium) 	“Pinwheel” 	**Limit Cycle (Periodic Behavior)** 	“Zykov cycloid” 
Limit Torus (Quasi-Periodic Behavior) 	“QP hypermeander” 	**Strange Attractor (Chaotic Behavior)** 	“Ch. hypermeander” 

Drift of “pinwheel” spirals

- Well studied
- Spiral is characterised by the instant centre position and the fiducial phase
- . . . all of which change with the rate proportional to the perturbation
- . . . with the proportionality coefficients defined by the corresponding “response functions”

PHYSICAL REVIEW E **81**, 066202 (2010)

Computation of the drift velocity of spiral waves using response functions

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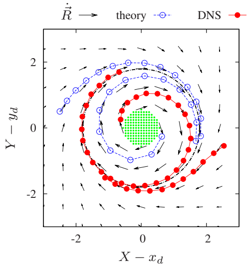


Drift of classically (“Zykov”) meandering spirals

- Theory is nascent
- Spiral is characterized by the instant centre position, the fiducial rotation phase and fiducial meandering phase
- Hence, possibility of locking between the phases

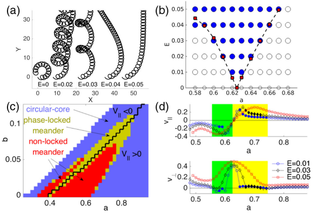


FIG. 4. Phase-locking in Barkley’s model. (a) Drift trajectories with $\vec{E} = E\vec{e}_x$ for parameters as in Fig. 1(a), showing phase-locking when $E > 0.04$. (b) Arnold tongue confirming the theoretical prediction in Eq. (22). (c) Occurrence of phase-locking for $E = 0.03$ in (a),(b) parameter space with $c = 0.02$. (d) Drift components parallel and perpendicular to E, for $b = 0.05$. The colored background indicates meander.

PRL **119,** 258101 (2017) PHYSICAL REVIEW LETTERS week ending 22 DECEMBER 2017

Filament Tension and Phase Locking of Meandering Scroll Waves

Hans Dierckx,[1] I. V. Biktasheva,[2,3] H. Verschelde,[1] A. V. Panfilov,[1] and V. N. Biktashev[3]

Drift of hypermeandering spirals of either kind?

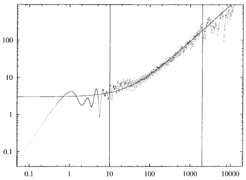
- Not considered so far, to our knowledge
- Even characterization of the unperturbed dynamics is interesting. E.g. how does one tell one from the other?
- Insights can be gained by assuming base dynamics and then solving the simple ODE for the tip motion

Chaotic

ELSEVIER
Physica D 116 (1998) 342–354
PHYSICA D

Deterministic Brownian motion in the hypermeander of spiral waves

V.N. Biktashev [a,b,*], A.V. Holden [b]



$$\langle (\Delta x)^2 \rangle \propto t$$

Quasiperiodic

Institute of Physics Publishing Nonlinearity
Nonlinearity **14** (2001) 275–300 www.iop.org/Journals/no PII: S0951-7715(01)13756-4

Euclidean extensions of dynamical systems

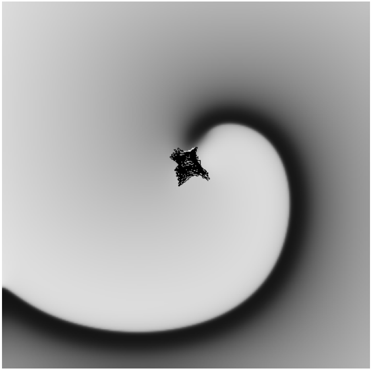
Matthew Nicol[1], Ian Melbourne[2] and Peter Ashwin[3]

Theorem 7.4. *Consider the irrational torus flow $\dot{\theta} = \alpha$ on T^m. Then, for almost every $\alpha \in \mathbb{R}^m$, and for almost every sufficiently smooth* $\mathbf{SE}(2)$*-extension* $(h, k) : T^m \to \mathbf{SE}(2) = \mathbf{SO}(2) \ltimes \mathbb{R}^2$, *the dynamics on* $T^m \times \mathbf{SE}(2)$ *is bounded.*

Plane English: tip trajectories of quasiperiodic hypermeander are almost certainly bounded.

How big they can be?

Size matters



"Zykov" meander in a model with standard parameters (Biktashev and Holden, 1996)

Hypermeander: parameter changed to represent Long QT syndrome (Biktashev and Holden, 1998)

The $\mathbb{R}^1$-extension of quasiperiodic dynamics

Base

$$\frac{\mathrm{d}q}{\mathrm{d}t} = s(\psi) = \sum_{n\in\mathbb{Z}^k} s_n e^{i(n\cdot\psi)},$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \nu,$$

where $\psi \in \mathbb{T}^k$, $\nu \in \mathbb{R}^k$, $k \geq 2$.

Extension

Termwise integration gives

$$q(t) = q(0) + s_0 t$$

$$+ \sum_{n\in\mathbb{Z}^k\setminus\{0\}}{}' \frac{-is_n}{(n\cdot\nu)} \left(e^{i(n\cdot\nu)t} - 1\right).$$

- For incommensurate ν, we have small denominator problem.
- Nicol et al: for almost all ν, the sum converges.
- However, it diverges for an everywhere dense set of ν.
- Hence the size is an everywhere discontinuos of ν.
- Physically, have to treat it as a random quantity.

The $SE(2)$-extension: size of hypermeandering trajectory

Base

$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(\theta)\, e^{i\varphi},$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = w(\theta),$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \in \mathbb{R}^m$$

Extension

- $\mathbb{R}^1$-extension for φ, then
- $\mathbb{R}^1$-extension for p, leading to
- $$|\Delta_t(\tilde{\omega})|^2 = |p(t) - p(0)|^2$$
  $$= \left| \sum_{n \in \mathbb{Z}^{m+1}}{}' \frac{-iv_n}{(n \cdot \tilde{\omega})} \left(e^{i(n\cdot\tilde{\omega})t} - 1 \right) \right|^2 .$$

Results

Let $\sigma_t^2(\tilde{\omega}) = t^{-1} \int_0^t \left|\Delta_{t'}(\tilde{\omega}) - \mu_t(\tilde{\omega})\right|^2 \, \mathrm{d}t'$, where $\mu_t(\tilde{\omega}) = t^{-1} \int_0^t \Delta_{t'}(\tilde{\omega}) \, \mathrm{d}t'$. Then for continuously distributed $\tilde{\omega}$,

- $$\mathrm{E}\left[\Delta_t^2\right] \approx C_1 t,$$
- $$\mathrm{E}\left[\sigma_t^2\right] \approx C_2 t, \qquad C_1/C_2 = 6,$$
- $$F(x) \equiv \mathbb{P}\left[\sigma_\infty > x\right] \propto x^{-1}, \quad \text{as} \quad x \to +\infty,$$
- $$\mathrm{E}\left[\sigma_\infty\right] = +\infty.$$

The model and its caricature (1/2)

FitzHugh-Nagumo with hypermeandering spirals:

$$u_t = 20(u - u^3/3 - v) + \nabla^2 u,$$
$$v_t = 0.05(u + 1.2 - 0.5v).$$

The trajectory of the tip is emulated by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(\theta)\, e^{i\varphi}, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = w(\theta), \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \in \mathbb{R}^m$$

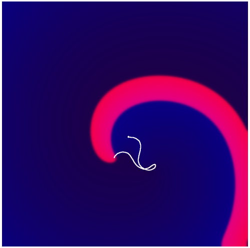
$$m = 2, \quad v(\theta) = (0.6 - 0.2\beta - 0.2\alpha\beta)^{-1} - 1,$$

$$w(\theta) = \big(0.675 + 0.1\alpha + 0.05\beta + 0.5\alpha^2 + 0.5\alpha\beta + 0.2\alpha^3 + 0.6\alpha^2\beta\big)^{-1} - 1,$$

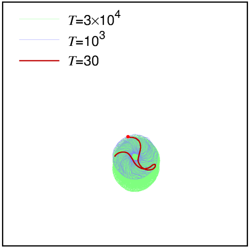
$$\alpha = \cos\theta_1 + 0.05\tanh(30\cos\theta_2), \quad \beta = \sin\theta_1,$$

$$\omega_1 = 0.354, \quad \omega_2 \in [0.475, 0.525].$$

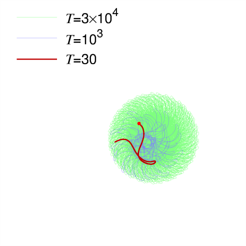
The model and its caricature (2/2)



Spiral wave and a piece of meander tip trajectory in FitzHugh-Nagumo model

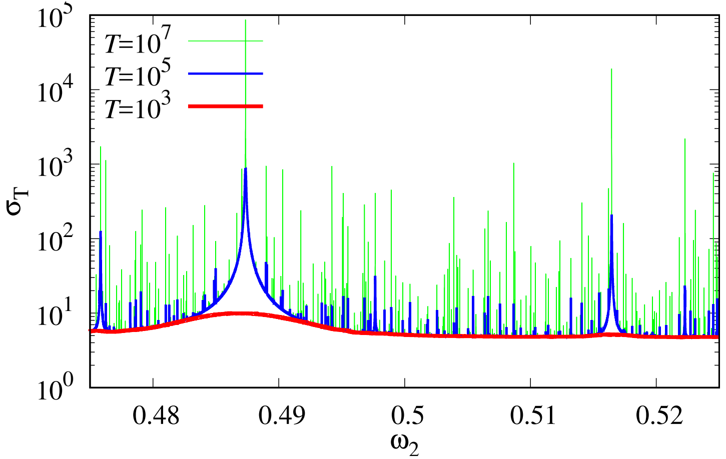


Longer pieces of the same tip trajectory



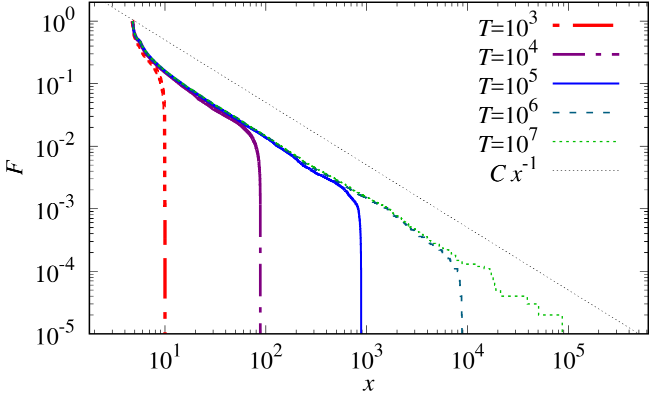
Pieces of trajectory of different lengths generated by the caricature model

Sizes of trajectories of different length, as functions of ω_2



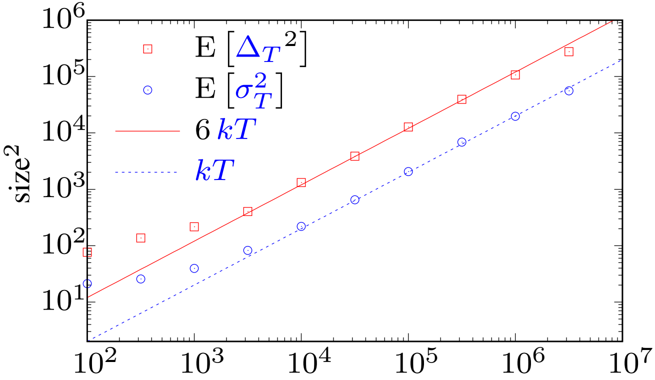
Continuous functions converging to an everywhere discontinuous limit.

Distributions of trajectory sizes



Distribution functions $F(x) = \mathbb{P}\left[\sigma_T(\tilde{\omega}) > x\right]$, for the estimates of σ_T made for different time intervals T in the caricature model. The straight line is the theoretical asymptotic.

Mean square of trajectory size as function of the time interval



Mean square of trajectory size as function of the time interval (log-log plot, two different statistics). The straight lines corresponding to the theoretical predictions.

Conclusions

- Deterministic equations, no chaos involved, but the question allows only probabilistic treatment.
- Trajectory size finite with probability one, but infinite expectation.
- Size of long pieces depends on parameters in an irregular way, everywhere discontinuous in the limit.
- Similar to St Petersburg paradox: infinite expectation, but need infinite time to achieve
- Asymptotic of size $\propto T^{1/2}$ is similar to deterministic Brownian motion of chaotic hypermeander, but in a different sense
- Perturbation theory: still long way away. Here is one small step towards realising how difficult it is even to pose the problem!
- Applicattion: cardiac arrhythmias. Possible: different physics (quasicrystals???)

THE END

System "reaction-diffusion"

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{u}),$$

where

$$\begin{aligned}
\mathbf{u} &= \left(u^{(1)}, \ldots, u^{(n)}\right)^\top = \mathbf{u}(\vec{r}, t) \in \mathbb{R}^n && \text{(concentrations)};\\
\mathbf{f} &= \mathbf{f}(\mathbf{u}); && \text{(reaction rates)};\\
\mathbf{D} &\in \mathbb{R}^{n\times n}, && \text{(matrix of diffusion coeffs)};\\
n &\geq 2, && \text{(number of components)};\\
\vec{r} &= (x, y) \in \mathbb{R}^2 && \text{(physical space)}.
\end{aligned}$$

This system is equivariant with respect to Euclidean transformations of the spatial coordinates $\vec{r}$.

Two popular examples

FitzHugh-Nagumo

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon}\left(u - \frac{u^3}{3} - v\right) + \nabla^2 u,$$
$$\frac{\partial v}{\partial t} = \epsilon(u + \beta - \gamma v),$$

with parameters ϵ, β, γ. Second field can also be diffusive; "cardiac" models only have one diffusive component.

Barkley

$$\frac{\partial u}{\partial t} = \frac{1}{\epsilon}u(1-u)\left(u - \frac{v+b}{a}\right) + \nabla^2 u,$$
$$\frac{\partial v}{\partial t} = u - v,$$

with parameters a, b, ϵ. This is a variation of FHN that is easy to calculate fast, especially if accuracy requirements can be relaxed.

Reaction-diffusion system as an ODE in functional space

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}(\mathbf{u})$$

in a suitable functional space $\mathcal{B}$ can be written as

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{F}(\mathbf{U}),$$

where

$$\mathbf{U} : \mathbb{R} \to \mathcal{B} \qquad \text{represents } \mathbf{u},$$

$$\mathbf{F} : \mathcal{B} \to \mathcal{B} \qquad \text{represents } \mathbf{D}\nabla^2\mathbf{u} + \mathbf{f}.$$

An equivariant ODE

Let us suppose that

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{F}(\mathbf{U})$$

is equivariant with respect to a representation T of a Lie group $\mathcal{G}$ in $\mathcal{B}$:

$$\forall g \in \mathcal{G}, \quad \forall \mathbf{U} \in \mathcal{B} : \mathbf{F}(T(g)\mathbf{U}) = T(g)\mathbf{F}(\mathbf{U}).$$

For the reaction-diffusion system, this is the special Euclidean group acting via transformations of $\vec{r}$:

$$\mathcal{G} = SE(2), \qquad \mathcal{G} \ni g : \mathbb{R}^2 \to \mathbb{R}^2,$$

$$T(g)\mathbf{u}(\vec{r}) = \mathbf{u}(g^{-1}\vec{r}), \qquad T(g) : \mathcal{B} \to \mathcal{B}.$$

Assumption of free action

- Consider a flow-invariant set $\mathcal{B}_0 \subset \mathcal{B}$ such that $\mathcal{G}$ acts freely on on $\mathcal{B}_0$, *i.e.*

$$\forall \mathbf{U} \in \mathcal{B}_0 : T(g)\mathbf{U} = \mathbf{U} \ \Rightarrow \ g = \mathrm{id}$$

  ($\mathcal{B}_0$ is the "principal stratum" of $\mathcal{B}$, corresponding to the trivial isotropy subgroups).
- For RDS: the graph of any function $\mathbf{u}(\vec{r})$ that may be considered as a snapshot of a valid spiral wave solution, is devoid of any rotational or translational symmetry.

Group orbits foliate the phase space

Definition

A *group orbit* of a given $\mathbf{U}$ is the set $T(\mathcal{G})\mathbf{U} = \{\, T(g)\mathbf{U} \,|\, g \in \mathcal{G}\}$.

- That is, it is a set of all such functions $\mathbf{u}(\vec{r})$ that can be obtained from one another by applying an appropriate Euclidean transformation to $\vec{r}$.
- A group orbit is a manifold in $\mathcal{B}_0$, of a dimensionality equal to $d = \dim \mathcal{G}$ less the dimensionality of the isotropy group. In our case, $\dim SE(2) = 3$, the isotropy group is trivial and the orbits are smooth three-dimensional manifolds.
- $\mathcal{B}_0$ is invariant $\Rightarrow$ is a disjoint union of group orbits ("is foliated").

Assumption of global transversal section

We assume there exists an open subset $\mathcal{S} \subset \mathcal{B}_0$, also flow-invariant and $\mathcal{G}$-invariant, in which the foliation has a global transversal section, *i.e.* we can select one representative from each orbit in $\mathcal{S}$, such that all such representatives form a smooth manifold $\mathcal{M} \subset \mathcal{S}$, which is everywhere transversal to the group orbits. We call this manifold a *Representative Manifold* (RM).

$$\forall\, \mathbf{U} \in \mathcal{S}, \quad \exists'\, (g, \mathbf{V}) \in \mathcal{G} \times \mathcal{M} : \quad \mathbf{U} = T(g)\mathbf{V}.$$

Representative Manifold

- The RM has co-dimensionality equal to the dimensionality of the group orbits, *i.e.* in our case $\operatorname{codim} \mathcal{M} = d = 3$.
- It is assumed to be smooth and we expect that it can locally be described by equations

$$\mu_\ell(\mathbf{V}) = 0, \qquad \ell = 1, \ldots d,$$

where functions $\mu_\ell : \mathcal{B} \to \mathbb{R}$, *i.e.* are functionals when interpreted in terms of the original RDS.

(and possibly some inequalities: see later).

Representative Manifold of Standard Spiral Waves

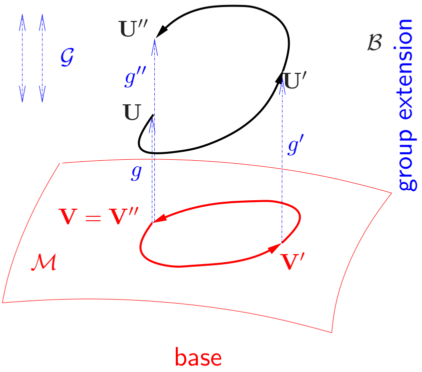
- Let $\mathcal{S}$ consist of spiral waves, in which we can uniquely identify a *tip position* and its *orientation*.
- Then $\mathcal{M}$ can be chosen to consist of those spiral waves that are in a *standard position*: tip at the origin, with a fixed orientation.
- Obviously any spiral wave from $\mathcal{S}$ can be brought to the standard position by a unique Euclidean transformation.

Example:

$$\begin{aligned}
\mu_1[\mathbf{v}(\vec{r})] &= v^{(l_1)}(\vec{0}) - u_* = 0, \\
\mu_2[\mathbf{v}(\vec{r})] &= v^{(l_2)}(\vec{0}) - v_* = 0 \qquad (l_1 \neq l_2), \\
\mu_3[\mathbf{v}(\vec{r})] &= \partial_x v^{(l_3)}(\vec{0}) = 0, \\
\mu_4[\mathbf{v}(\vec{r})] &= \partial_y v^{(l_3)} > 0.
\end{aligned}$$

Decomposition of a trajectory: geometrically

Skew-product decomposition of an equivariant flow using a Representative Manifold $\mathcal{M}$, which has exactly one transversal intersection with every group orbit $g \in \mathcal{G}$ and is diffeomorphic to the orbit manifold. Trajectory $(\mathbf{U}, \mathbf{U}', \mathbf{U}'')$ of an equivariant flow in $\mathcal{B}$ is a *relative periodic orbit*: it projects onto the trajectory $(\mathbf{V}, \mathbf{V}', \mathbf{V}'' = \mathbf{V})$ on $\mathcal{M}$ which is periodic. The flow on $\mathcal{M}$ is devoid of symmetry $\mathcal{G}$.



Decomposition of a trajectory: analytically

So for all $t \geq 0$, we have

$$\mathbf{U}(t) = T(g)\mathbf{V}(t)$$

where

$$\text{(RM)} \qquad \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F}_{\mathcal{M}}(\mathbf{V}) \qquad \text{base}$$

$$\text{(GO)} \qquad T(g^{-1})\frac{\mathrm{d}T(g)}{\mathrm{d}t}\mathbf{V} = \mathbf{F}_{\mathcal{G}}(\mathbf{V}) \qquad \text{extension}$$

A practical approach

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{F} + A, \qquad \text{(base1)}$$

$$\mu_\ell(\mathbf{V}(t)) = 0, \quad \ell = 1, \ldots, d, \qquad \text{(base2)}$$

$$T(g^{-1})\frac{\mathrm{d}T(g)}{\mathrm{d}t}\mathbf{V} = -A, \qquad \text{(extension)}$$

where

$$A = A(\mathbf{V}, g, t) = -\mathbf{F}_{\mathcal{G}}(\mathbf{V}) - \mathbf{H}_{\mathcal{G}}(\mathbf{V}, g, t)$$

is a vector belonging to the three-dimensional tangent space of the group orbit $T(\mathcal{G})(\mathbf{V})$ at $\mathbf{V}$.
Vector A is obtained from (base1) as a condition that $\mathbf{V}$ continues to satisfy (base2). Having thus found A we can proceed with solving (extension).

Differentiation of the Euclidean group

- To make this into a working algorithm, we need to translate it from abstract language to the terms of the original PDE.
- Vector A is a result of action of a linear combination of the generators of the Lie group $T(\mathcal{G})$ as linear operators on $\mathbf{V}$.
- Let us introduce coordinates $(\vec{R}, \Theta)$ on $\mathcal{G} = SE(2)$:

$$g = (\vec{R}, \Theta) \ : \ \vec{r} \mapsto \vec{R} + e^{\hat{\gamma}\Theta} \vec{r},$$

where $\hat{\gamma} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, so $\exp(\hat{\gamma}\Theta)$ is mx of rotation by angle Θ. Then

$$A = \omega \partial_\theta \mathbf{v} + (\vec{c} \cdot \nabla) \mathbf{v},$$

where

$$\omega = \dot{\Theta}, \qquad \vec{c} = e^{-\hat{\gamma}\Theta} \dot{\vec{R}}, \qquad \partial_\theta = x\partial_y - y\partial_x$$

(θ is the polar angle in the (x, y) plane)

Result: skew-product description

Base: reaction-diffusion in the tip frame of reference

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{D}\nabla^2\mathbf{v} + \mathbf{f}(\mathbf{v}) + (\vec{c}\cdot\nabla)\mathbf{v} + \omega\frac{\partial \mathbf{v}}{\partial \theta}, \quad \text{reaction+diffusion+advection}$$

$$v^{(l_1)}(\vec{0},t) = u_*, \quad v^{(l_2)}(\vec{0},t) = v_*, \quad \text{tip position}$$

$$\frac{\partial v^{(l_3)}(\vec{0},t)}{\partial x} = 0, \quad \text{tip orientation}$$

Extension: tip equations of motion

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \omega, \quad \frac{\mathrm{d}\vec{R}}{\mathrm{d}t} = e^{\hat{\gamma}\Theta}\vec{c}.$$

Dynamic variables: $\mathbf{v}(\vec{r},t)$, $\vec{c}(t)$, $\omega(t)$, $\vec{R}(t)$ and $\Theta(t)$.
(NB: can identify $\mathbb{R}^2 \to \mathbb{C}$, $\hat{\gamma} \to i$).

Problem setting

Equations along the group (tip EoM):

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \Omega(t), \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = V(t)\, e^{i\varphi}$$

where $p = p_x + ip_y \in \mathbb{C}$ is tip's complex coordinate and φ is its orientation angle, and $\Omega(t)$ and $V(t)$ are defined by the quasiperiodic base dynamics,

$$\Omega(t) = w(\theta(t)), \qquad V = v(\theta(t)),$$

where $\theta \in \mathbb{T}^m = (\mathbb{R}/2\pi\mathbb{Z})^m$ are coordinates on the invariant m-torus, $m \geq 2$, so that

$$\dot{\theta} = \omega,$$

and $\omega \in \mathbb{R}^m$ is a set of (typically incommensurate) frequencies.

The $\mathbb{R}^1$-extension of the quasiperiodic dynamics

First step: point with coordinate $q \in \mathbb{R}^1$ moving according to

$$\frac{\mathrm{d}q}{\mathrm{d}t} = s(\psi) = \sum_{n\in\mathbb{Z}^k} s_n e^{i(n\cdot\psi)}, \qquad \frac{\mathrm{d}\psi}{\mathrm{d}t} = \nu,$$

where $\psi \in \mathbb{T}^k$, $\nu \in \mathbb{R}^k$, $k \geq 2$. Termwise integration gives

$$q(t) = q(0) + s_0 t + \sum_{n\in\mathbb{Z}^k}{}' \frac{-is_n}{(n\cdot\nu)} \left(e^{i(n\cdot\nu)t} - 1\right).$$

Here and later, $\sum'$ is the sum over $n \neq 0$.

The $\mathbb{R}^1$-extension of the quasiperiodic dynamics

Consider

$$\Delta(t;\nu) = q(t) - q(0) - s_0 t = \sum_{n\in\mathbb{Z}^k}{}' \frac{-is_n}{(n\cdot\nu)}.$$

- For a typical ν, its components are incommensurate.
- The denominators in the infinite sum are nonzero, but many of them are very small.
- However if $s(\psi)$ is sufficiently smooth, its Fourier coefficients s_n quickly decay with $|n|$.
- Therefore, the infinite sum remains bounded for $t \geq 0$, for typical $s(\psi)$ and almost all ν (Nicol *etal* 2001).
- But: bounded by what? How big the trajectories may be?

The $\mathbb{R}^1$-extension of the quasiperiodic dynamics

E.g.: let $p(t) = q(t) - s_0 t$, $\Delta_t(\nu) = p(t) - p(0)$,
$\mu_T(\nu) = T^{-1} \int_0^T \Delta_t(\nu)\,\mathrm{d}t$, $\sigma_T^2(\nu) = T^{-1} \int_0^T |\Delta_t(\nu) - \mu_T(\nu)|^2\,\mathrm{d}t$,

$$\sigma_\infty^2(\nu) = \sum_{n\in\mathbb{Z}^k}{}' \frac{|s_n|^2}{(n\cdot\nu)^2}.$$

- For almost any vector ν, this $\sigma_\infty^2(\nu)$ is finite.
- Typically all s_n are nonzero, so $\sigma_\infty^2(\nu)$ is infinite for all ν such that $(n\cdot\nu) = 0$, and this is an everywhere dense set.
- Function $\sigma_\infty(\nu)$ is almost everywhere defined and finite, but is everywhere discontinuous, and discontinuities are not removable.
- For any physical purpose, question of the value of the function at a particular point is meaningless.
- A deterministic view on function $\sigma_\infty(\nu)$ is inadequate, and we are forced to adopt a probabilistic view

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The $\mathbb{R}^1$-extension of the quasiperiodic dynamics

Suppose we know ν approximately, say, its probability density is uniformly distributed in $B = B_\delta(\nu_0)$, a ball of radius δ centered at ν_0. The expectation of the trajectory size is then

$$\mathrm{E}\left[\sigma_\infty(\nu)\right] = \frac{1}{\mathrm{mes}(B)} \int_B \sigma_\infty(\nu)\, \mathrm{d}\nu = \frac{1}{\mathrm{mes}(B)} \int_B \left({\sum_{n\in\mathbb{Z}^k}}' \frac{|s_n(\nu)|^2}{(n,\nu)^2} \right)^{1/2} \mathrm{d}\nu.$$

The set of hyperplanes $(n \cdot \nu)$, $n \in \mathbb{Z}^k$ is everywhere dense. For any n such that $\{\nu : (n \cdot \nu) = 0\} \cap B \neq \emptyset$, we have

$$\mathrm{E}\left[\sigma_\infty(\nu)\right] \geq \frac{1}{\mathrm{mes}(B)} \int_B \left| \frac{s_n(\nu)}{(n,\nu)} \right| \mathrm{d}\nu \geq A \int_{-\epsilon}^{\epsilon} \frac{\mathrm{d}z}{|z|} = +\infty.$$

That is, the deviation from steady motion is almost certainly finite, but its average expected value is infinite.

$SE(2)$ extension: quasiperiodic hypermeander

$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(\theta)\, e^{i\varphi}, \qquad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = w(\theta), \qquad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega \in \mathbb{R}^m$$

where tip position $p \in \mathbb{C}$, tip orientation $\varphi \in \mathbb{T}^1$, coordinates on the invariant torus $\theta \in \mathbb{T}^m$.

1. The system for φ and θ, with unfolding $\varphi \in \mathbb{R}$, makes a $\mathbb{R}^1$ extension, so $\varphi = \varphi_0 + w_0 t + \Phi(\theta)$, where $\Phi(\theta)$ is, according to the above results, typically a bounded function of $\theta = \omega t$.
2. Then
$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(\tilde{\theta}), \qquad \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}t} = \tilde{\omega} = (\omega, \omega_{m+1}) \in \mathbb{R}^{m+1},$$
where $\omega_{m+1} = w_0$, $\tilde{\theta} = (\theta, \theta_{m+1}) \in \mathbb{T}^{m+1}$ and $v(\tilde{\theta}) = V(\theta)\, e^{i\Phi(\theta)}\, e^{i\varphi_0}\, e^{i\theta_{m+1}}$ is in turn a pair of $\mathbb{R}^1$ extension (with a special feature: $v_0 = 0$).

$SE(2)$ extension: quasiperiodic hypermeander

The moral:

- The expectation of $\sigma_\infty(\tilde{\omega})$, the size of the trajectory, defined as the root mean square of the distance of the tip from the centroid of the trajectory, is infinite.
- Similar conclusions can be made for the expectation of other statistics, such as average displacement from the initial point, or for the suprema of the distance from the centroid or of the displacement from the initial point.

Asymptotic distribution of the trajectory size

- The trajectory size

$$\sigma_\infty(\tilde{\omega}) = \left(\sum_{n \in \mathbb{Z}^{m+1}}{}' \frac{|v_n(\tilde{\omega})|^2}{(n, \tilde{\omega})^2} \right)^{1/2}$$

  is large if at least one of the terms in the infinite sum is large.
- It is most likely that the largest term by far exceeds all the others.
- So, the distribution of σ_∞ can be understood via the distribution of individual terms $M_n(\tilde{\omega}) = |v_n(\tilde{\omega})|^2/(n, \tilde{\omega})^2$.
- Clearly, $\mathbb{P}\left[M_n > x^2\right] \propto x^{-1}$ as $x \to +\infty$ as long as $\{(n, \tilde{\omega}) = 0\} \cap B \neq \emptyset$, and the distribution of σ_∞ corresponds to the distribution of the square root of the largest of such terms.
- Hence, for a typical continuous distribution of $\tilde{\omega}$, we expect

$$\mathbb{P}\left[\sigma_\infty > x\right] \propto x^{-1}, \quad \text{as} \quad x \to +\infty.$$

Growth rate of the trajectory size.

In practice we can observe the trajectory only for a finite, even if large, time interval T. Let us see how the expectation of the trajectory size grows with T. E.g.

$$\left|\Delta(T,\tilde{\omega})\right|^2 = \sum_{n'',n'\in\mathbb{Z}^{m+1}}{}' \frac{\overline{v}_{n''}\, v_{n'}}{(n''\cdot\tilde{\omega})\,(n'\cdot\tilde{\omega})} \left(e^{-i(n''\cdot\tilde{\omega})T}-1\right)\left(e^{i\left(n'\cdot\tilde{\omega}\right)T}-1\right).$$

For large T, the principal contribution is provided by terms with $n' = n''$ (long story, but true) which gives an approximation

$$\left|\Delta_T\right|^2 \approx \sum_{n\in\mathbb{Z}^{m+1}}{}' \frac{4\left|v_n\right|^2}{(n\cdot\tilde{\omega})^2}\sin^2\left((n\cdot\tilde{\omega})\,T/2\right).$$

Growth rate of the trajectory size.

The corresponding expectation is

$$\mathrm{E}\left[|\Delta_T|^2\right] \approx \frac{4}{\mathrm{mes}\, B} \sum_{n\in\mathbb{Z}^{m+1}}{}' |v_n|^2 \int_{\tilde\omega\in B} \frac{\sin^2\left((n\cdot\tilde\omega)\, T/2\right)}{(n\cdot\tilde\omega)^2}\, \mathrm{d}\tilde\omega.$$

Define $\chi_n = \mathrm{mes}\left(\{\tilde\omega \mid (n\cdot\tilde\omega) = 0\} \cap B\right)$, and $\|n\| = \left(n_1^2 + \cdots + n_{m+1}^2\right)^{1/2}$. Then

$$\mathrm{E}\left[\Delta_T^2\right] \approx C_1 T, \quad C_1 = \frac{2\pi}{\mathrm{mes}\, B} \sum_{n\in\mathbb{Z}^{m+1}}{}' \frac{|v_n|^2}{\|n\|}\chi_n.$$

Similarly,

$$\mathrm{E}\left[\sigma_T^2\right] \approx C_2 T, \quad C_2 = \frac{\pi}{3\,\mathrm{mes}\, B} \sum_{n\in\mathbb{Z}^{m+1}}{}' \frac{|v_n|^2}{\|n\|}\chi_n.$$

THE FINAL END