Symmetries, Cluster Synchronization, and Isolated Desynchronization in Complex Networks

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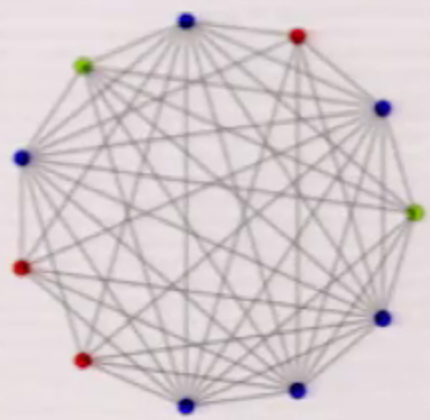
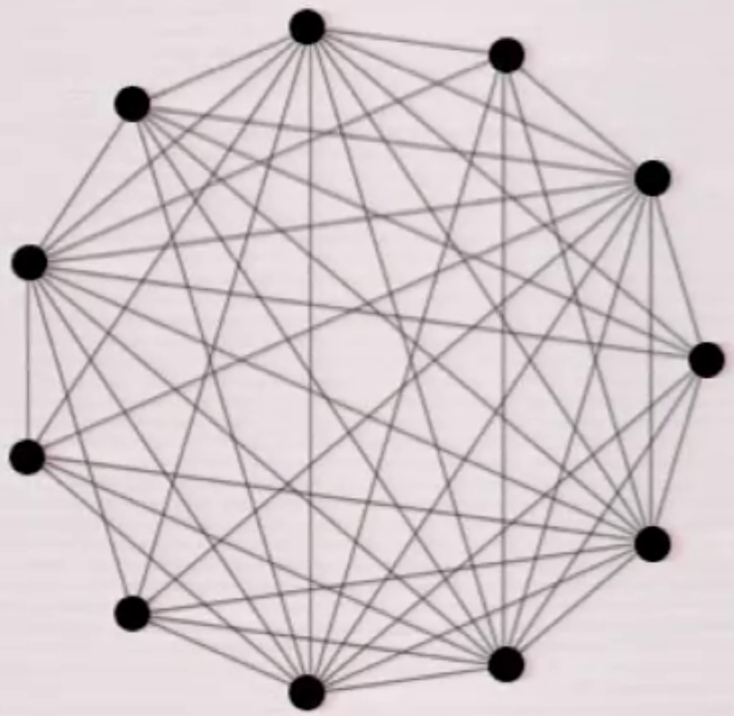
SIAM 18 May 2015 Structure-dynamics Relation in Networks of Coupled Dynamical Systems

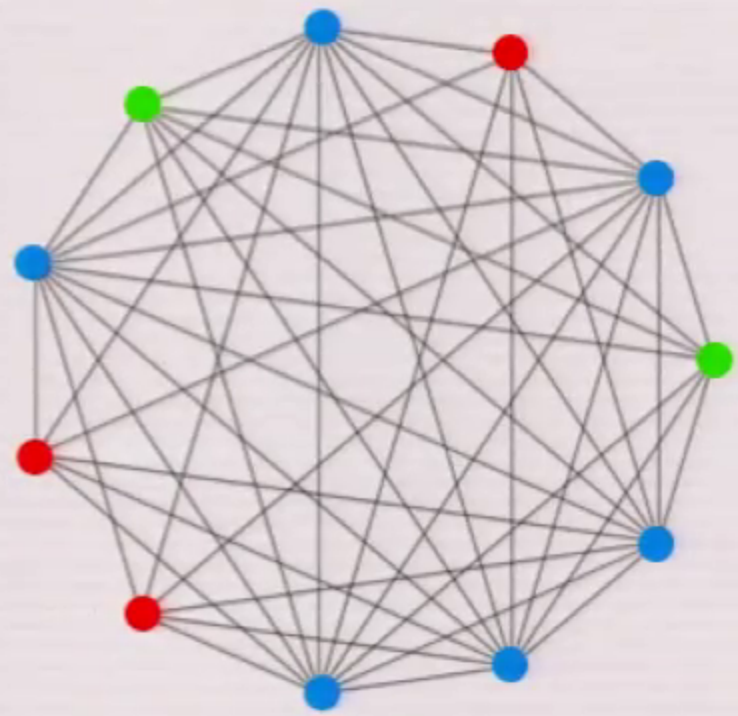


Cluster Synchronization in complex networks

(various versions in the literature)

Identical nodes (oscillators), Identical edges (couplings)





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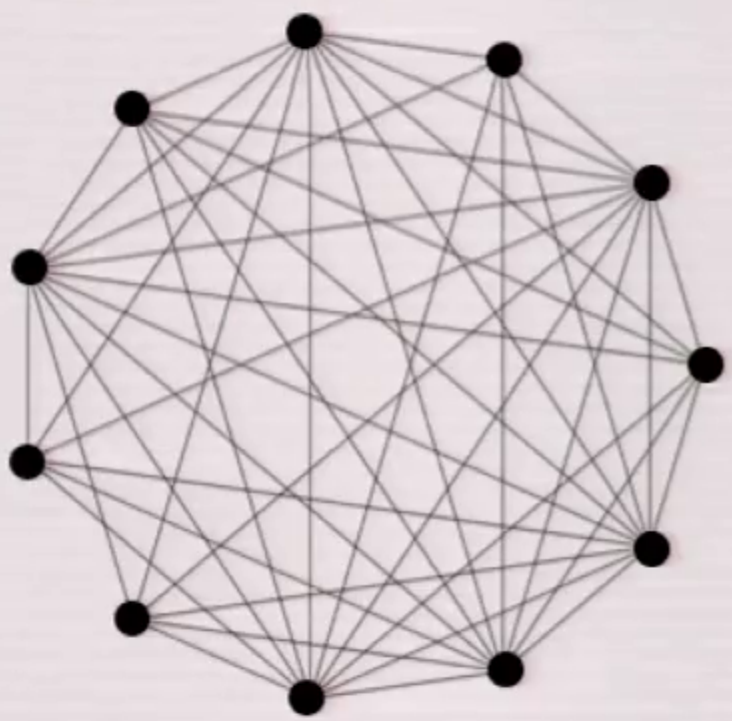
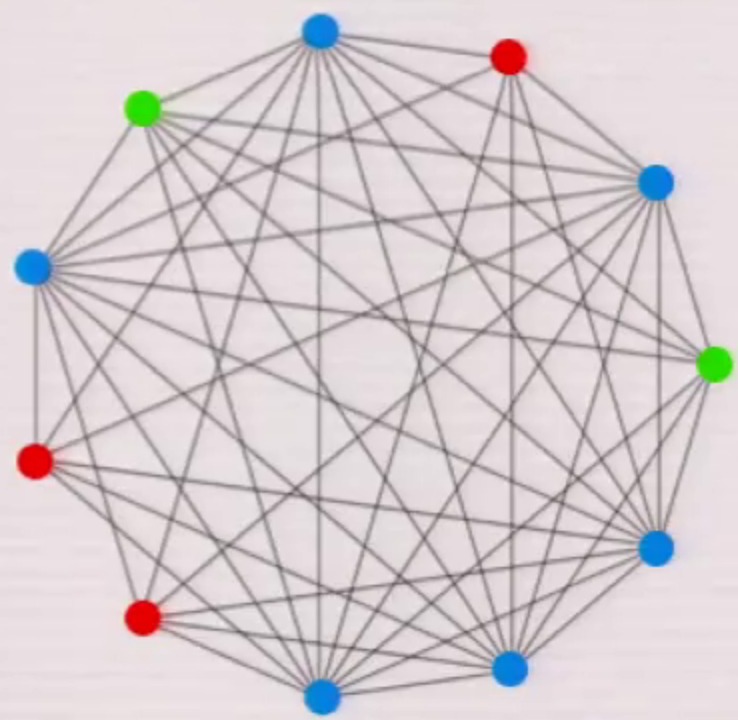




- Identify the clusters?
- Are the clusters stable?
- Desynchronization patterns? (surprise)

} For complex networks

Previous "Cluster" Synchronization work. Mostly special cases.

C. Allefeld, M Muller, and J. Kurths, Eigenvalue decomposition as a generalized synchronizatioun cluster analysis," Int. J. Bif. Chaos 17, 3493{3497 (2007).

P. Ji, T.K. Peron, P.J. Menck, F.A. Rodrigues, and J. Kurths, Cluster explosive synchronization in complex networks," Physical Review Letters 110, 218701 (2013).

C. Zhou and J. Kurths, Hierarchical synchronization in complex networks with heterogeneous degrees," CHAOS 16, 015104 (2006).

A-L.Do, J. Hoefener, and T. Gross, Engineering mesoscale structures with distinct dynamical implications," New Journal of Physics 14, 115022 (2012).

T. Dahms, J. Lehnert, and E. Scholl, Cluster and group synchronization in delay-coupled networks," Physical Review E 86, 016202 (2012).

C. Fu, Z. Deng, L. Huang, and X. Wang, Topological control of synchronous patterns in systems of networked chaotic oscillators," Physical Review E 87, 032909 (2013).

I. Kanter, M. Zigzag, A. Englert, F. Geissler, and W. Kinzel, Synchronization of unidirectional time delay chaotic networks and the greatest common divisor," EPL 93, 6003{1{6 (2011).

D. P. Rosin, D. Rontani, D.J. Gauthier, and E. Scholl, Control of synchronization patterns in neural-like boolean networks," Physical Review Letters 110, 104102 (2013).

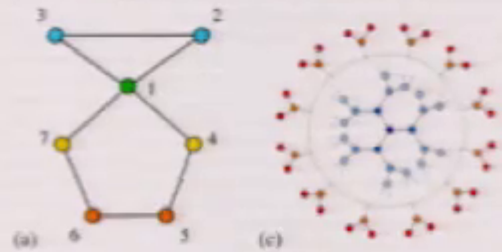
F. Sorrentino and E. Ott, Network synchronization of groups," Physical Review E 76, 056114 (2007).

C. Williams, T. Murphy, R. Roy, F. Sorrentino, T. Dahms, and E. Scholl, Experimental observations of group synchrony in a system of chaotic optoelectronic oscillators," Physical Review Letters 110, 064104 (2013).

V. Belykh, G.V. Osipov, V.S. Petrov, J.K.A. Suykens, and J. Vandewalle, Cluster synchronization in osccillatory networks," CHAOS 18, 037106 (2008).

Previous "Cluster" Synchronization work. Use of group theory.

Nicosia, V., et al. (2013). "Remote Synchronization Reveals Network Symmetries and Functional Modules." <u>Physical Review Letters</u> **110**: 174102.



Previous "Cluster" Synchronization work. Use of groupoid theory.

Golubitsky, M., et al. (2012). "Network periodic solutions: patterns of phase-shift synchrony." <u>Nonlinearity</u> **25**: 1045-1074.

Golubitsky, M., et al. (2005). "Patterns of Synchrony in Coupled Cell Networks with Multiple Arrows." <u>SIAM J. Applied Dynamical Systems</u> **4**(1): 78-100.

Judd, K. (2013). "Networked dynamical systems with linear coupling: Synchronisation patterns, coherence and other behaviours." <u>CHAOS</u> **23**: 043112.

Group theory and control/observability in networks

Whalen, A., et al. (2015). "Observability and Controllability of Nonlinear Networks: The Role of Symmetry." <u>Physical Review X</u> **5**: 011005.

Computational Approach for Large Networks

The Dynamics on a Network

$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} C_{ij} H(\mathbf{x}_j) \quad i=1,\ldots,N$$

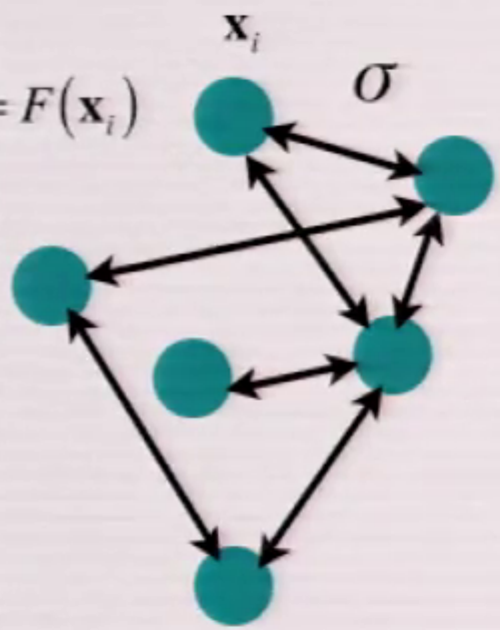
(Adjacency matrix) $C_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$

$F(\mathbf{x}_i)$ can contain self-feedback terms. = constant here.

The Dynamics on a Network

$$\mathbf{x}_i = \begin{pmatrix} x_i^1 \\ x_i^2 \\ \vdots \\ x_i^n \end{pmatrix}$$

$$\frac{d\mathbf{x}_i}{dt} = F(\mathbf{x}_i)$$



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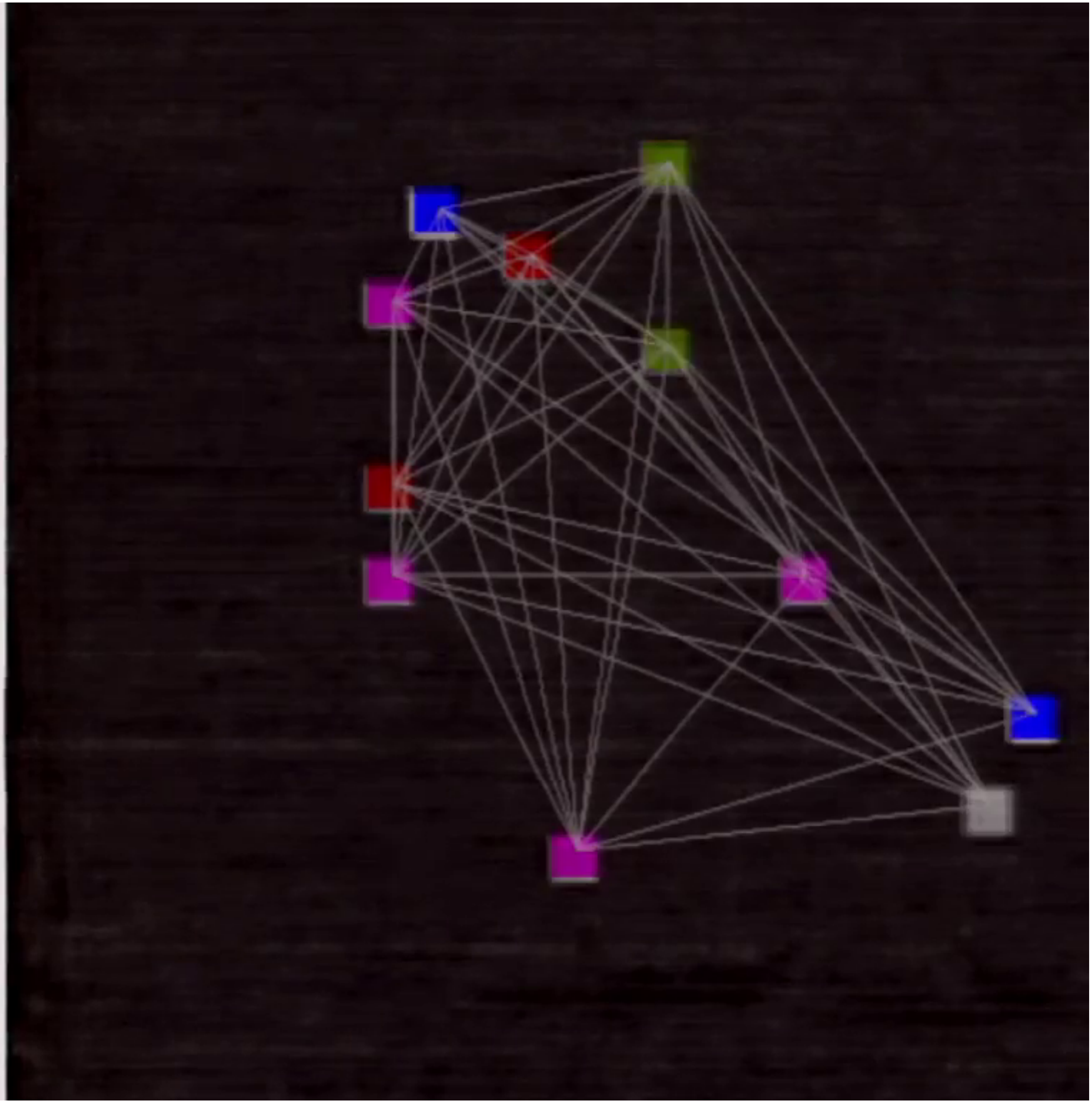
$F(\mathbf{x}_i)$ can contain self-feedback terms. = constant here.

Dynamics can be maps, too

$$\mathbf{x}_i(t+1) = \mathbf{F}(\mathbf{x}_i(t)) + \sigma \sum_{j=1}^{N} C_{ij} \mathbf{H}(\mathbf{x}_j(t))$$

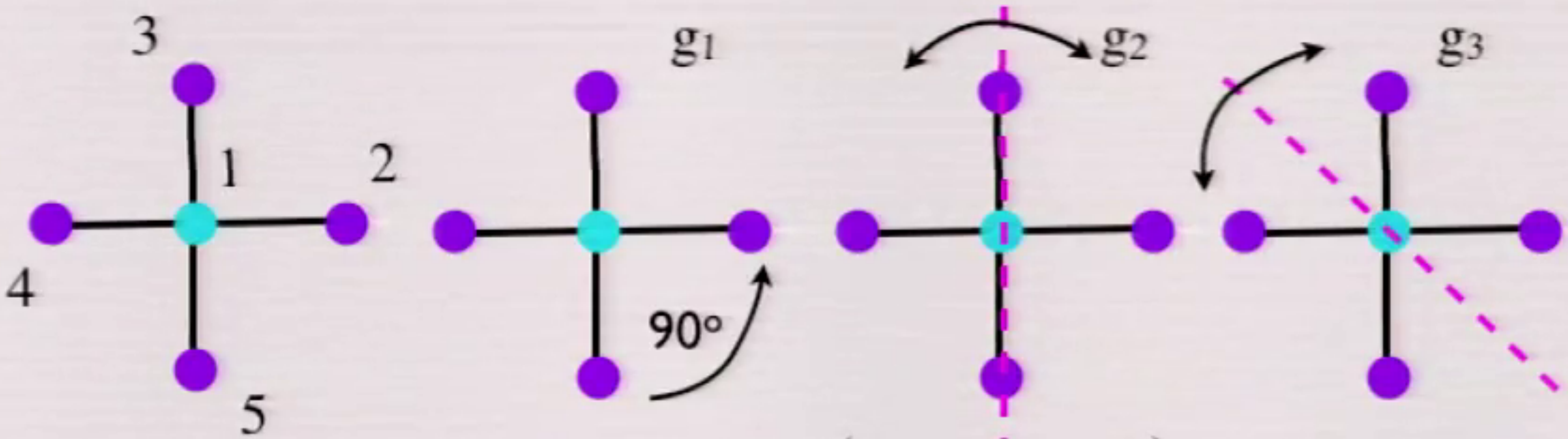
An experimental example…

$\beta = 0.72\,\pi$



Using symmetry to find clusters.

Network with identical nodes (oscillators) and identical edges (each connection has the same weight and is bi-directional)



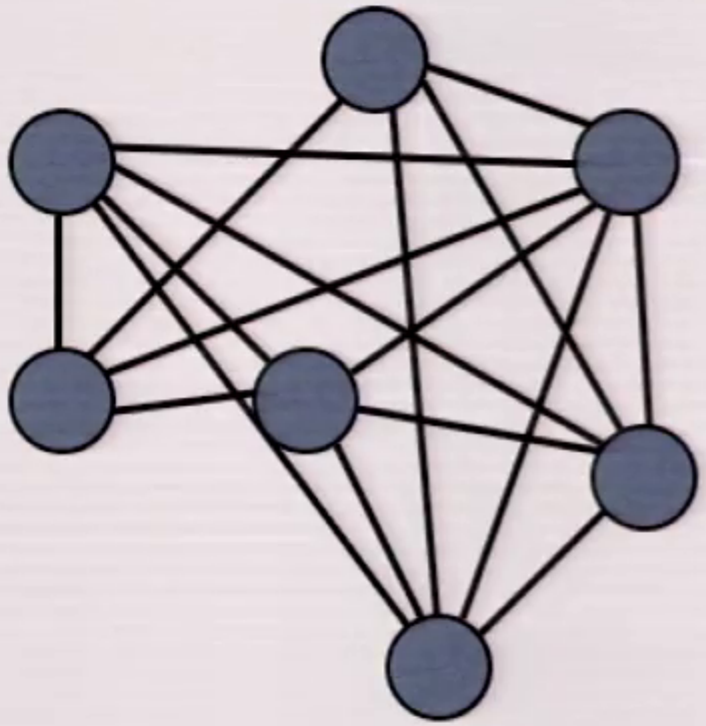
$$R_{g_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad R_{g_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad R_{g_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

group $\mathcal{G} = \{g_i\}$

$\{R_{g_i}\}$ Representation of the group $\mathcal{G}$

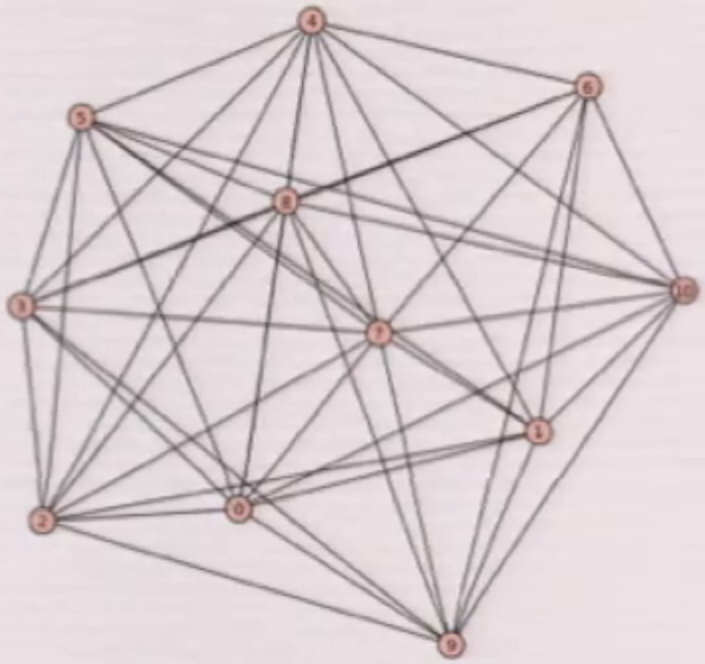
e.g. $g_2 g_1^{-1} = g_3$ => $R_{g_2} R_{g_1}^{-1} = R_{g_3}$

Random Graphs

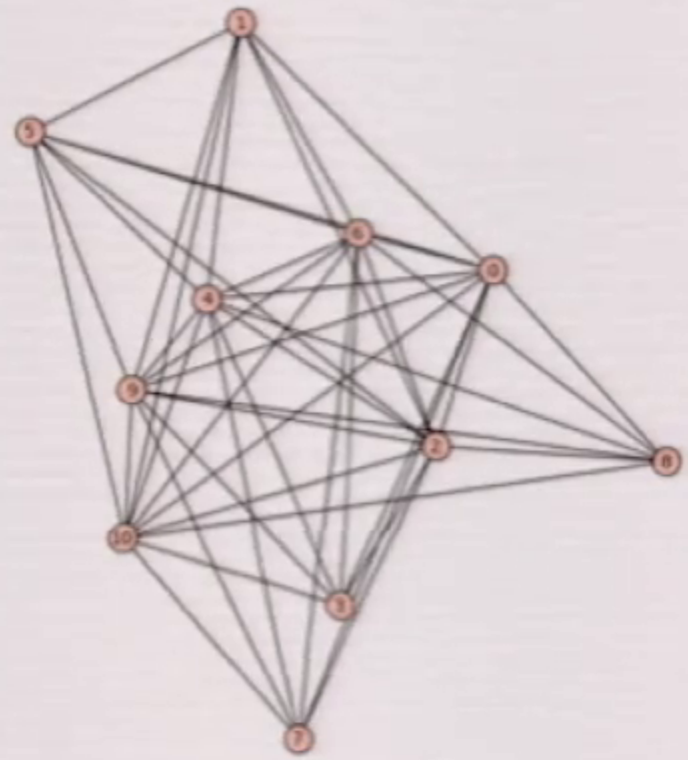


$$N = 7, \quad n_{\text{deleted}} = 4$$

Random networks 11 nodes - 9 random edges



0 symmetries



8640 symmetries

G.gens()= [(7,10), (6,7), (5,6), (4,8), (2,4)(8,9), (1,5), (1,11)]

$\{g_i\}$

Computational Group Theory

GAP

Groups, Algorithms, Programming -
a System for Computational Discrete Algebra

Sage

http://www.sagemath.org/

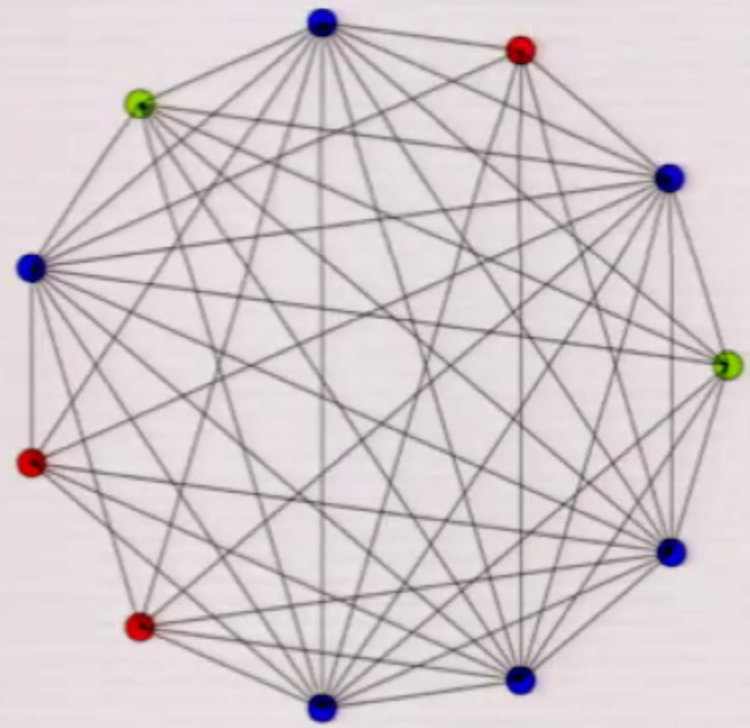
```
sage: x = var('x')
sage: solve(x^2 + 3*x + 2, x)
[x == -2, x == -1]
```

Python

```
for i,row in enumerate(Adjmat):
    rsum= row.sum()
    Cplmat[i,i]= -rsum
print Cplmat
```

Free ! (open source)

```
G.order(), G.gens()= 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
  Node 2

orb= [1, 5]
nodeSyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cycleSyncvec  [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
  Node 1

orb= [2, 4, 11, 6, 9, 10]
nodeSyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cycleSyncvec  [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]
  Node 4

orb= [3, 7, 8]
nodeSyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cycleSyncvec  [0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]
```

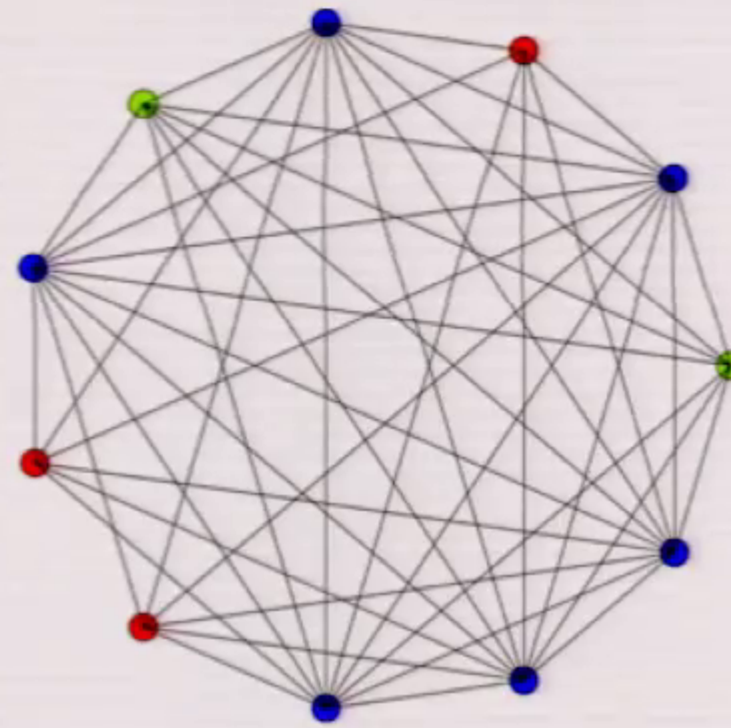


Original coupling Matrix

-7.	1.	0.	1.	1.	1.	0.	0.	1.	1.	1.
1.	-10.	1.	1.	1.	1.	1.	1.	1.	1.	1.
0.	1.	-6.	1.	0.	1.	0.	0.	1.	1.	1.
1.	1.	1.	-10.	1.	1.	1.	1.	1.	1.	1.
1.	1.	0.	1.	-7.	1.	0.	0.	1.	1.	1.
1.	1.	1.	1.	1.	-10.	1.	1.	1.	1.	1.
0.	1.	0.	1.	0.	1.	-6.	0.	1.	1.	1.
0.	1.	0.	1.	0.	1.	0.	-6.	1.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.	1.
1.	1.	1.	1.	1.	1.	1.	1.	1.	1.	-10.

$$\frac{d\,\delta\mathbf{x}_i}{dt} = D\mathbf{F}(\mathbf{x}_i)\delta\mathbf{x}_i + \sigma\sum_{j=1}^{N} C_{ij} D\mathbf{H}(\mathbf{x}_j)\delta\mathbf{x}_j$$

```
G.order(), G.gens()= 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
  Node 2

orb= [1, 5]
nodeSyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cycleSyncvec  [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]
  Node 1

orb= [2, 4, 11, 6, 9, 10]
nodeSyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cycleSyncvec  [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]
  Node 4

orb= [3, 7, 8]
nodeSyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cycleSyncvec  [0, 0, 1, 0, 0, 0, 1, 1, 0, 0, 0]

Variational coupling matrix
```



-6.00	-3.46	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
-3.46	-5.00	-4.24	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	-4.24	-6.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-8.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	-6.00	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	-6.00	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-11.00

$M_g=1$ Trivial Representation

$$\frac{d\xi_\lambda}{dt} = \left(DF(\mathbf{s}) + \Lambda_\lambda DH(\mathbf{s})\right)\cdot \xi_\lambda$$

All other Representations

Synchronization Manifold

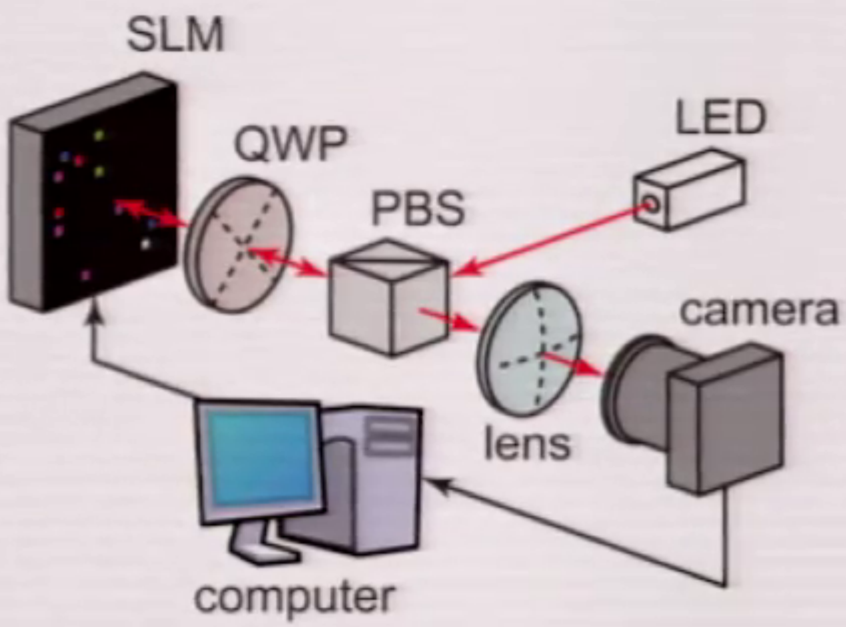
Transverse Manifold (sub-blocks associated with each cluster)

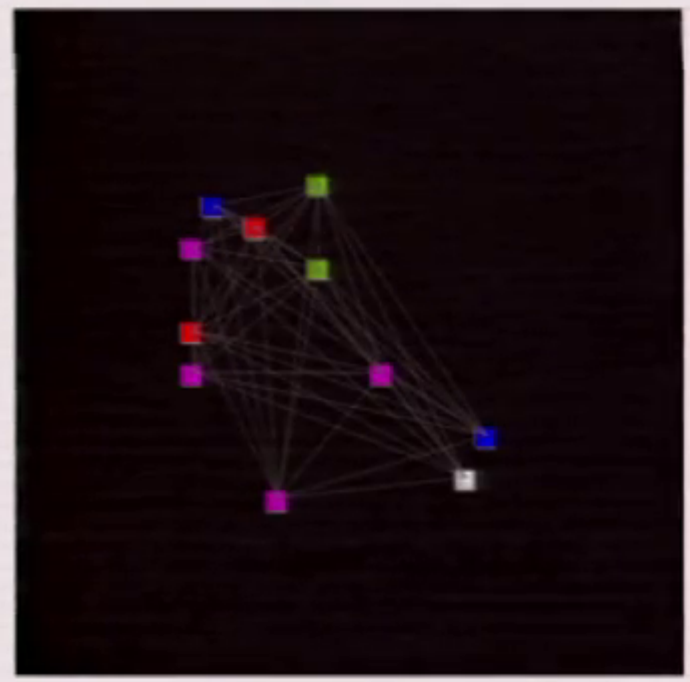
How to implement maps in experiment

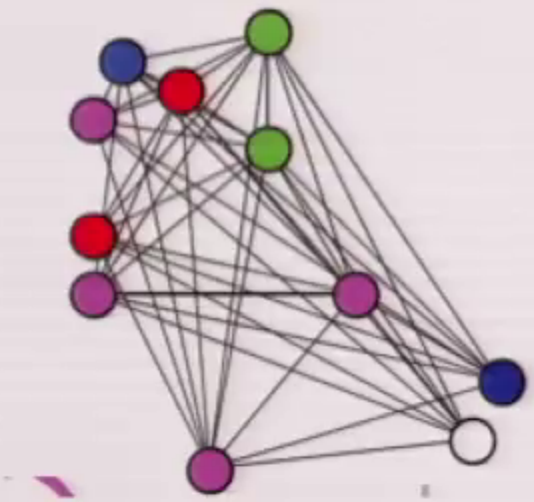
Self feedback strength — Coupling strength — Adjacency matrix — Add phase shift to destabilize fixed point φ=0

$$\phi_i^{t+1} = \beta I(\phi_i^t) + \sigma \sum_j A_{ij} I(\phi_j^t) + \delta$$

Phase written to SLM — **Intensity measured by camera**







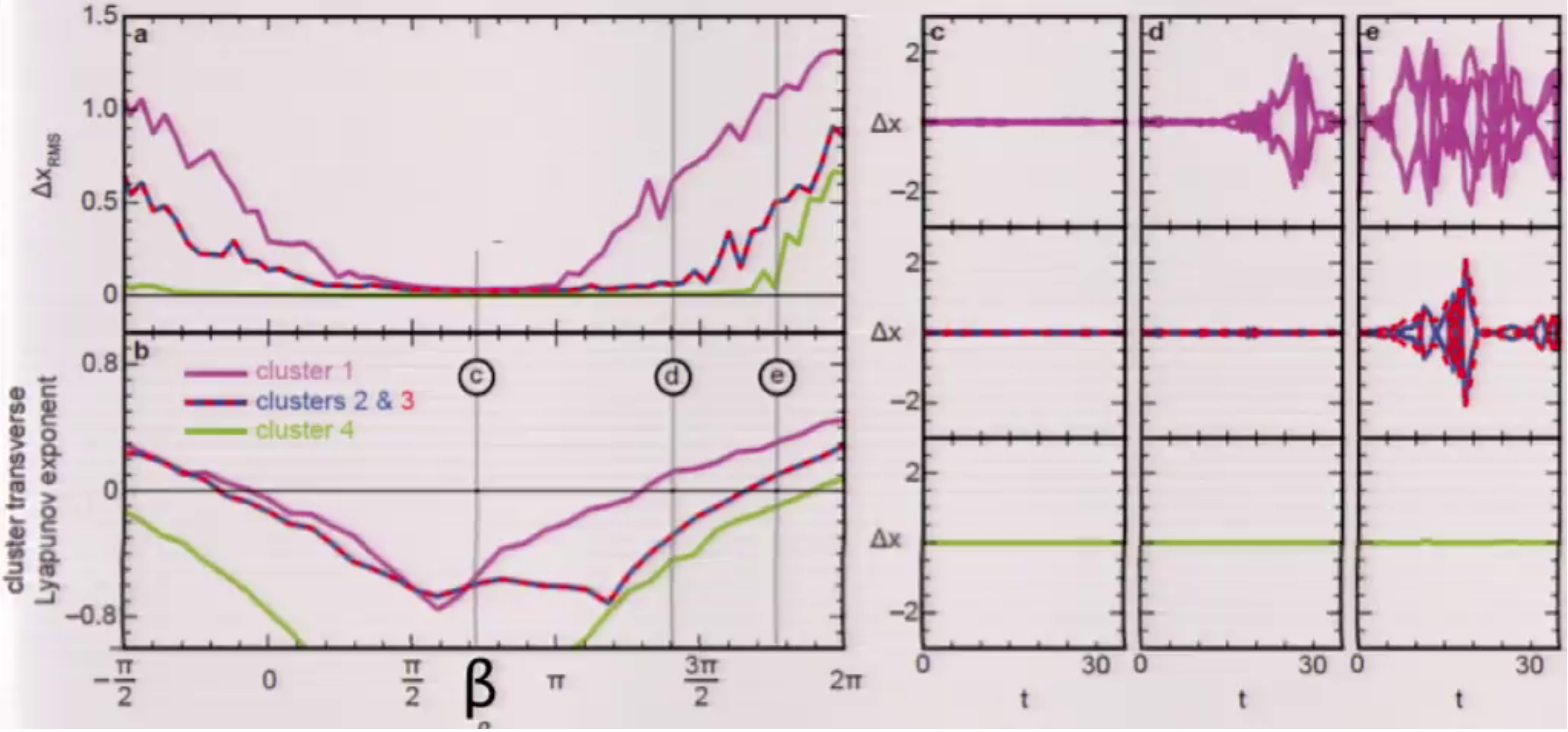
11 node random network (video)

32 Symmetries

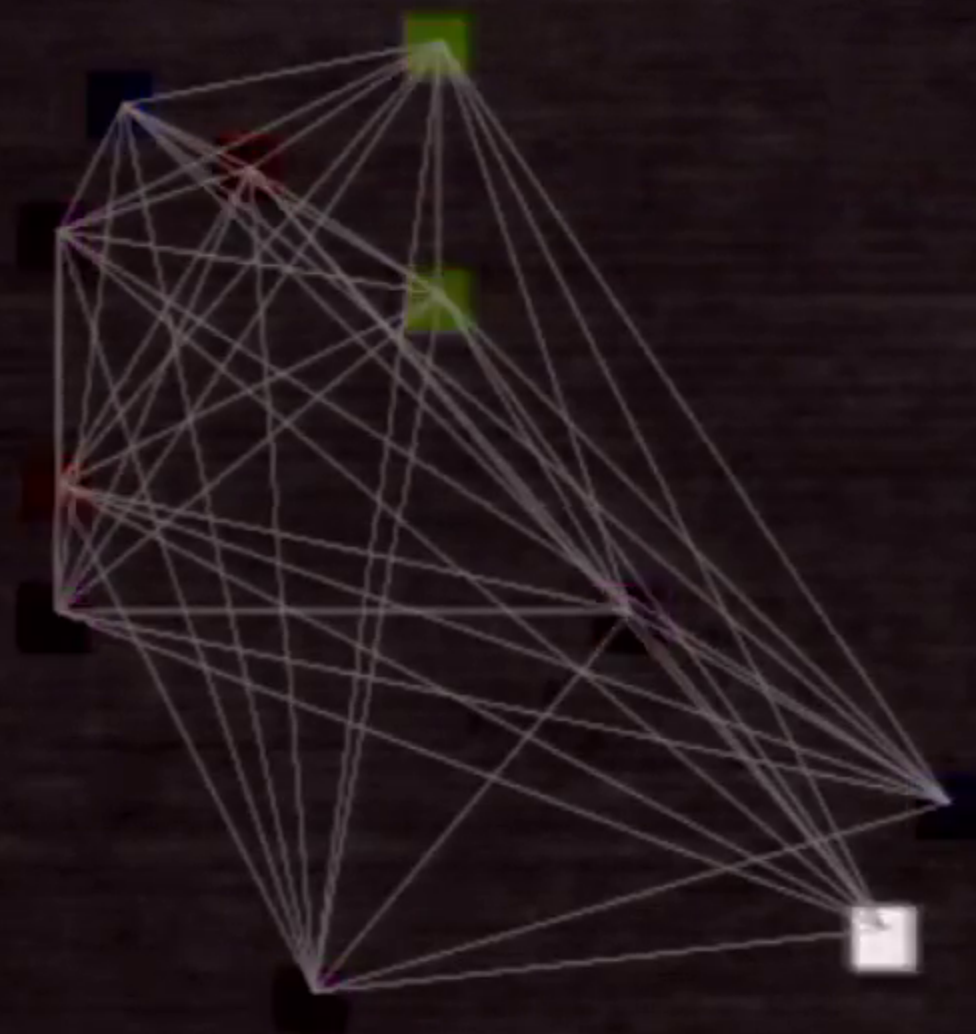
4 nontrivial clusters

1 trivial cluster

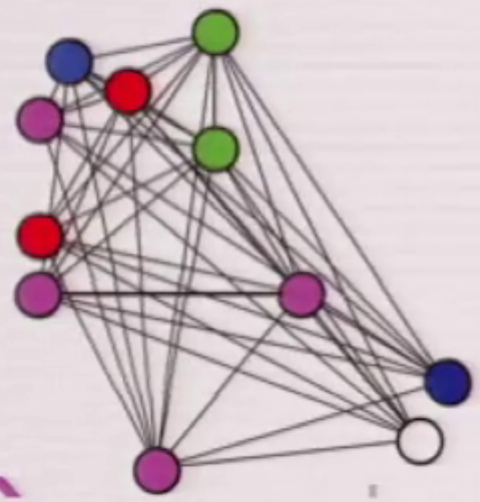
5 x 5 Sync block



$\beta = 1.4\,\pi$



loss of cluster stability
symmetry breaking bifurcation.
Next talk.



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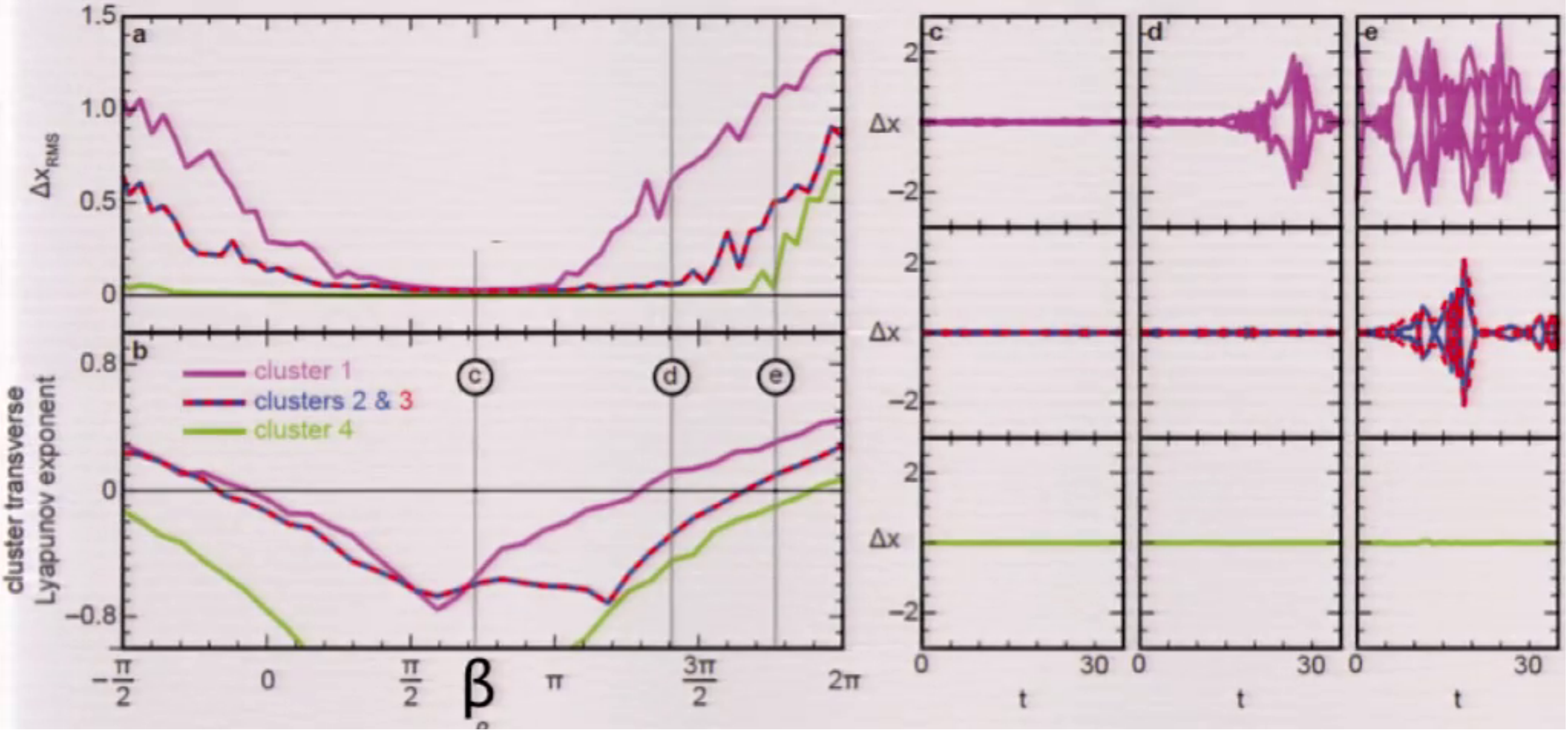
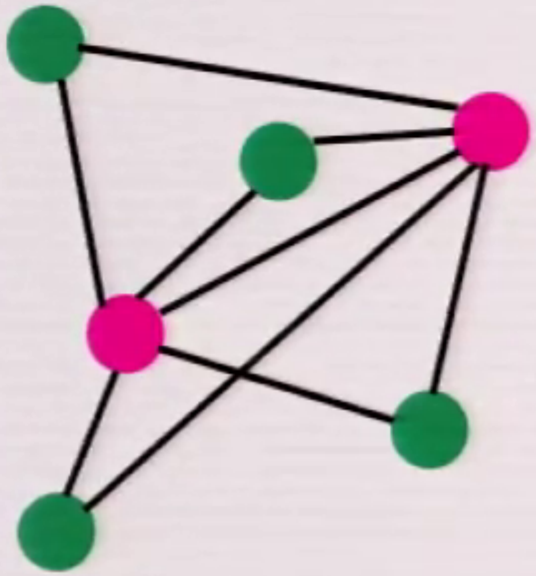


Decomposition of network symmetry group



$\{R_i\} = \mathcal{G}$

$\{R_1, R_2, \ldots, R_m\} = \mathcal{H}_1$ permute only

$\{R_{m+1}, R_{m+2}, \ldots, R_{m+n}\} = \mathcal{H}_2$ permute only

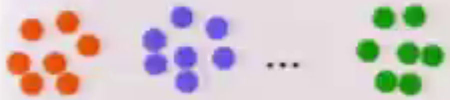
$\mathcal{H}_1$ and $\mathcal{H}_2$ are subgroups of $\mathcal{G}$ which commute

$\mathcal{H}_1$ and $\mathcal{H}_2$ each do not contain all elements of $\mathcal{G}$

$\mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2$ a <u>direct product</u> of all possible combinations of elements of $\mathcal{H}_1$ and $\mathcal{H}_2$

$$\mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2 \times \ldots \times \mathcal{H}_n$$

M clusters $\{\mathcal{K}_l\}_{l=1,\dots,M}$ $\mathbf{x}_i \in \mathcal{K}_l$



1 *2* ... *M*

$\mathbf{x}_i \in \mathcal{K}_q$ $R_g \in \mathcal{H}_n \rightarrow \mathcal{K}_m$

$$\dot{\mathbf{x}}_i(t) = F(\mathbf{x}_i) + \sum_{l=1}^{M} \sum_{j \in \mathcal{K}_l} C_{ij}\mathbf{x}_j = F(\mathbf{x}_i) + \dots + \sum_{j \in \mathcal{K}_m} C_{ij}\mathbf{x}_j$$

Isolated desynchronization

Symmetries and clusters in networks with different topologies

$N = 100$ nodes (oscillators)

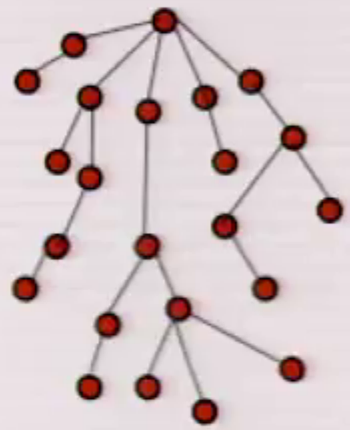
10,000 realizations of each type

Calculate some symmetry statistics

Random

$n_{\text{delete}} = 50$

Scale-free Tree



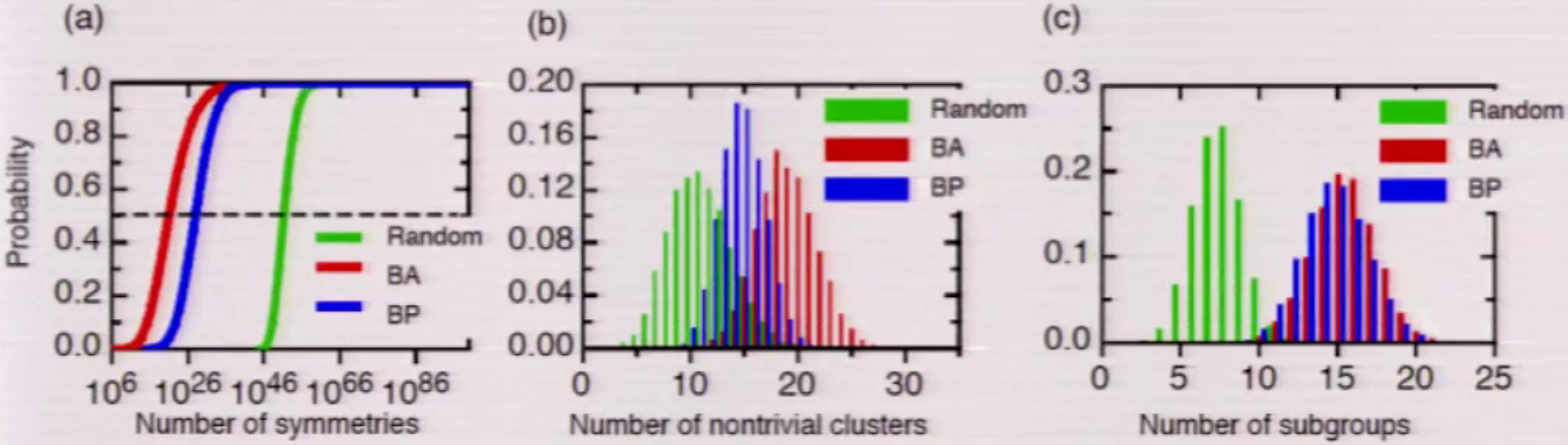
A.-L. Barabasi and R. Albert, \Emergence of scaling in random networks," Science 286, 509{512 (1999).

Random Bipartite

Sage routine RandomBipartite().

Symmetry statistics

10,000 samples of each network type (100 nodes each)



Symmetry statistics

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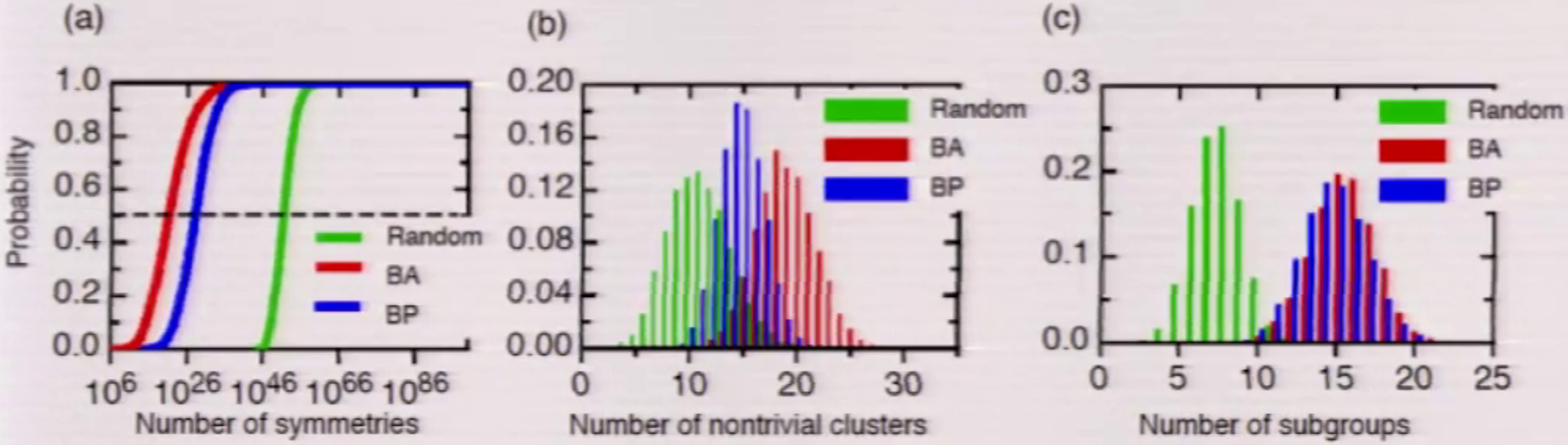


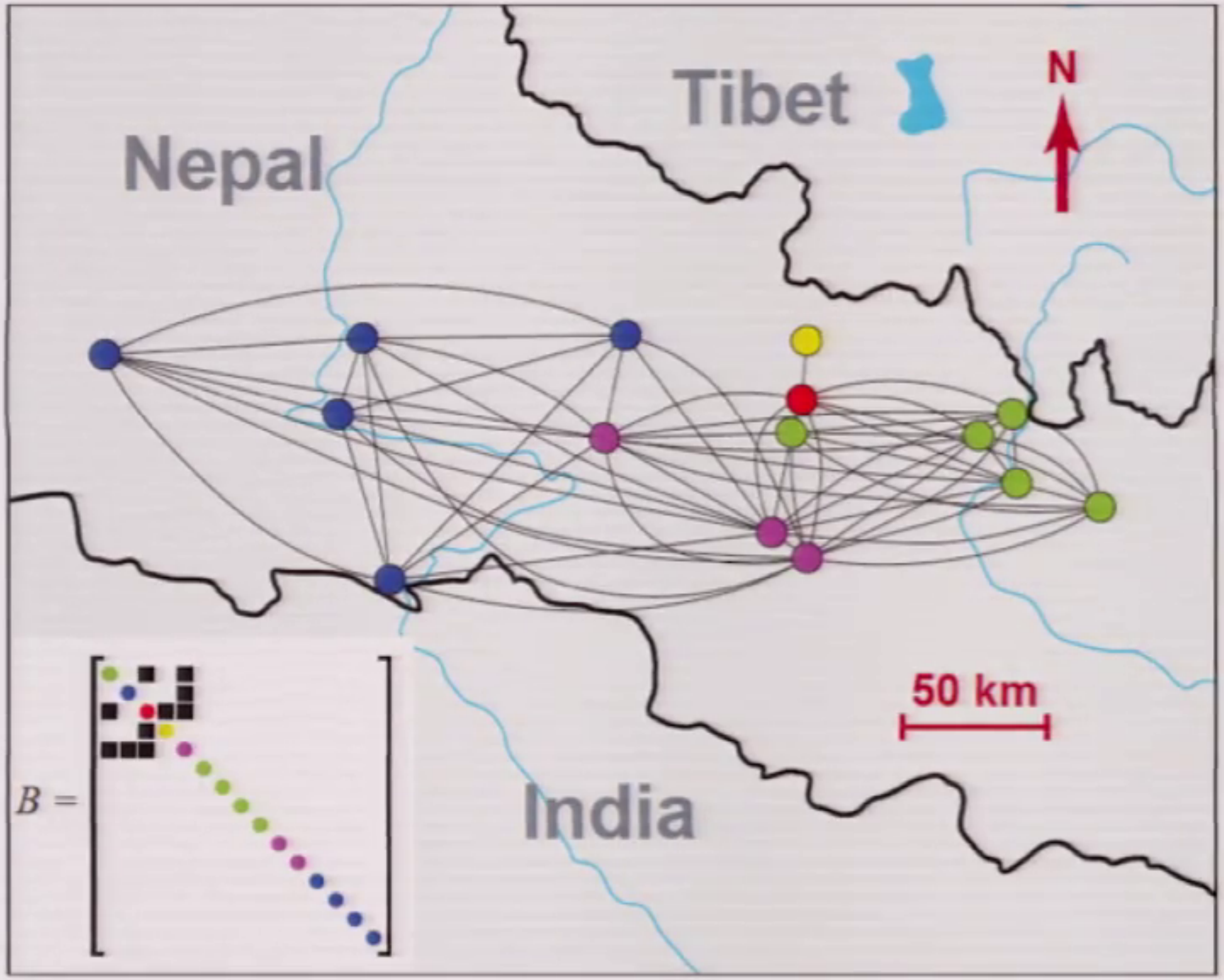
Many common models of networks have symmetries, clusters, and subgroup decompositions.

Smallworld networks, too

Isolated desynchronization

Electric power grid of Nepal N=15

Cluster synchronization?



86,400 symmetries
15 Nodes
3 clusters
2 trivial clusters
3 subgroups

B.D. MacArthur, R.J. Sanchez-Garcia, and J.W. Anderson, \On automorphism groups of networks," Discrete Appl. Math. 156, 3525 (2008).

Geometric decomposition into subgroups.

Network	Number of Nodes $N_{\mathcal{G}}$	Number of Edges $M_{\mathcal{G}}$	Number of Symmetries $a_{\mathcal{G}}$
Human B Cell Genetic Interactions[3]	5,930	64,645	5.9374×10^{13}
C. elegans Genetic Interactions[26]	2,060	18,000	6.9985×10^{161}
BioGRID datasets[23]:			
Human	7,013	20,587	1.2607×10^{485}
S. cerevisiae	5,295	50,723	6.8622×10^{64}
Drosophila	7,371	25,043	3.0687×10^{493}
Mus musculus	209	393	5.3481×10^{125}
Internet (Autonomous Systems Level)[12]	22,332	45,392	$1.2822 \times 10^{11,298}$
US Power Grid[25]	4,941	6,594	5.1851×10^{152}

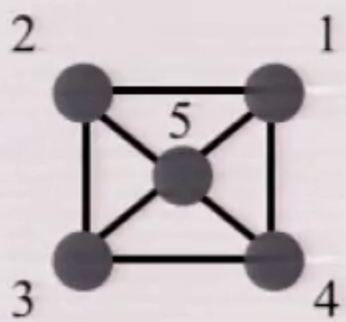
> 88% of nodes are in clusters in all above networks

Conclusions and remarks

- Encompasses or overlaps other "phenomena"

  Cluster sync, Partial sync, Remote sync, Some Chimera states

- Directed edges/couplings
- Weighted edges/couplings
- Different oscillators
- Laplacian coupling - diag = – row sum



$$\begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

Conclusions and remarks

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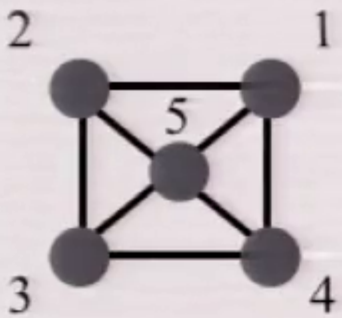
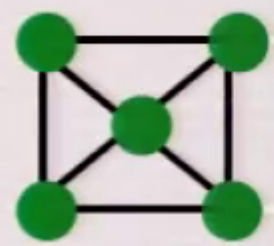
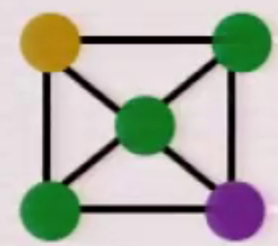


$$\begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$





Next talk: Francesco Sorrentino, nonsymmetry clusters in networks using group theory.

Symmetry Breaking and Synchronization Patterns in Networks of Coupled Oscillators

Conclusions and remarks (cont.)

- Bifurcation form

  Normal forms & symmetry

  M. Golubitsky, I. Stewart, and D.G. Schaeffer, Singularities and groups in bifurcation theory, Vols. I & II (Springer-Verlag, New York, NY, 1985).

Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries, Pecora, Sorrentino, Hagerstrom, Murphy, and Roy, *Nature Communications*, 5, 4079 (13 June 2014)

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Conclusions and remarks (cont.)

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Thanks to:

B.D. MacArthur and R.J. Sanchez-Garcia, Spectral characteristics of network redundancy,"
Physical Review E 80, 026117 (2009).

B.D. MacArthur, R.J. Sanchez-Garcia, and J.W. Anderson, \On automorphism groups of networks,"
Discrete Appl. Math. 156, 3525 (2008).