Stability of the solutions for scalar conservation laws with moving flux constraints

Thibault Liard, Benedetto Piccoli

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A PDE-ODE system

PDE : We consider the Lighthill-Whitham-Richards model which describes the global traffic evolution :

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{LWR}$$

Above, $\rho = \rho(t,x) \in [0,1]$ is the mean traffic density. The flux f is defined by

$$f(\rho) = \rho v(\rho) \quad \text{with} \quad v(\rho) = 1 - \rho.$$

ODE : We consider the following ODE which describes the trajectory of a vehicle :

$$\begin{cases} \dot{y}(t) = \omega(\rho(t, y(t)+))), & t \in \mathbb{R}^+, \\ y(0) = y_0, & x \in \mathbb{R}. \end{cases} \tag{ODE}$$

The variable y denotes the bus position and ω is the velocity of the vehicle.

A constraint on the flux f

We assume that the vehicle is a bus and the velocity of the bus is described by :

$$\omega(\rho) = \begin{cases} V_b & \text{if } \rho \leq \rho^* := 1 - V_b, \\ v(\rho) & \text{otherwise,} \end{cases} \tag{1}$$

with $V_b \in (0,1)$ denotes the maximal speed of the bus.

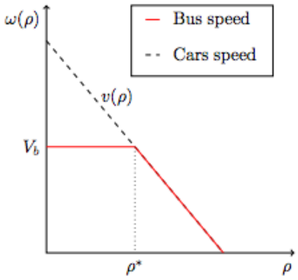
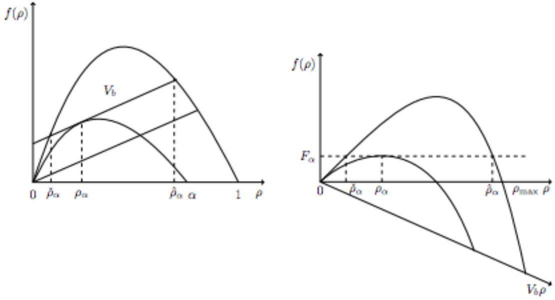


FIGURE: Bus and cars speed

A constraint on the flux f

Since $V_b < 1$, the bus can be regarded as a moving restriction of the road where the associated reduced flow f_α is defined by $f_\alpha(\rho) = \rho(1 - \frac{\rho}{\alpha})$ with $\alpha \in (0,1)$. F_α denotes the maximum value of $f_\alpha(\rho)$ with $\rho \in (0,1)$ in the bus reference frame.



(a) Fixed reference frame (b) Bus reference frame

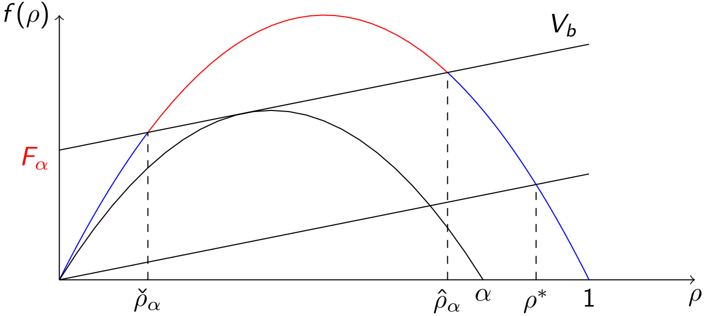
The constraint on the flux can be written as

$$f(\rho(t, y(t))) - \dot{y}(t)\rho(t, y(t)) \leq F_\alpha := \frac{\alpha}{4}(1 - \dot{y}(t))^2, \quad t \in \mathbb{R}^+ \quad \text{(Const)}$$

A strong coupled PDE-ODE system

We consider the following coupled PDE-ODE system

$$\begin{cases} \partial_t \rho + \partial_x(\rho(1-\rho)) = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \\ f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \leq F_\alpha := \frac{\alpha}{4}(1-\dot{y}(t))^2, & t \in \mathbb{R}^+, \\ \dot{y}(t) = \omega(\rho(t,y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0, & x \in \mathbb{R}. \end{cases} \quad \text{(Syst-LWR)}$$



A constraint Riemann solver $\mathcal{R}_\alpha$ for (Syst-LWR)

Let $\mathcal{R}$ the standard Riemann solver for (LWR), i.e the (right continuous) map $(t, x) \mapsto \mathcal{R}(\rho_L, \rho_R)(\frac{x}{t})$ given by the standard weak entropy solution to (LWR). The constrained Riemann solver $\mathcal{R}_\alpha$ for the coupled PDE-ODE system is defined by

- If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \geq F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$ then

$$\mathcal{R}_\alpha(\rho_L, \rho_R)(\frac{x}{t}) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho}_\alpha)(\frac{x}{t}) & \text{if } x \leq y(t) = V_b t \\ \mathcal{R}(\check{\rho}_\alpha, \rho_R)(\frac{x}{t}) & \text{if } x \leq y(t) = V_b t \end{cases}$$

- Otherwise,

$$\mathcal{R}_\alpha(\rho_L, \rho_R)(\frac{x}{t}) = \mathcal{R}(\rho_L, \rho_R)(\frac{x}{t})$$

Let $BV(\mathbb{R},[0,1])$ be the set of real-valued functions whose total variation is bounded.

Theorem (M.L Del Monache and P. Goatin, 2014)

Let $\rho_0 \in BV(\mathbb{R},[0,1])$. The Cauchy problem (Syst-LWR) *admits a solution* $(\rho, y) \in C^0(\mathbb{R}^+; L^1 \cap BV(\mathbb{R},[0,1])) \times W^{1,1}(\mathbb{R}^+,\mathbb{R})$.

Theorem (T.L and B. Piccoli)

The solution $(\rho, y) \in C^0(\mathbb{R}^+; L^1(\mathbb{R}) \cap BV(\mathbb{R},[0,1])) \times W^{1,1}(\mathbb{R}^+,\mathbb{R})$ *of the Cauchy problem* (Syst-LWR) *depends in a Lipschitz continuous way on the initial datum with respect to the* L^1*-topology.*

More precisely, let $T > 0$ and (ρ^0, y^0) and (ρ^1, y^1) two solutions of (Syst-LWR) with corresponding initial data (ρ_0^0, y_0^0) and (ρ_0^1, y_0^1), then there exists $C > 0$ such that

$$\|\rho^1(t) - \rho^0(t)\|_{L^1(\mathbb{R})} + |y^1(t) - y^0(t)| \leq C(\|\rho_0^1 - \rho_0^0\|_{L^1(\mathbb{R})} + |y_0^1 - y_0^0|),$$

for every $t \in [0, T]$

Wave-front tracking method

Let the mesh $\mathcal{M}_n = \{(2^{-n}\mathbb{N} \cap [0,1])\}_{i=0}^{2^n} \cup \{\rho^*, \check{\rho}_\alpha, \hat{\rho}_\alpha\}$ on $[0,1]$.

We construct a piecewise constant $(\rho^n, y_n) \in \mathcal{M}_n \times \mathbb{R}$ by the wave-front tracking method as described below

- We approximate $\rho_0 \in BV(\mathbb{R}, [0,1])$ by a piecewise constant function $\rho_0^n \in \mathcal{M}_n$
- The solution (ρ^n, y_n) solves (Syst-LWR) by means of $\mathcal{R}_\alpha$ with initial conditions (ρ_0^n, y_0) up to the first time $t_1 > 0$ where two dicontinuities collide or a discontinuity hits the bus trajectory. Each rarefaction wave is splitted into a rarefaction fan formed by rarefaction shocks that are discontinuities traveling with the Rankine-Hugoniot speed
- at $t = t_1^+$ a new Riemann problem arises and we repeat the previous strategy replacing $t = 0$ and (ρ_0^n, y_0) by $t = t_1$ and $(\rho^n(t_1, \cdot), y_0)$ respectively.

Description of all possible interactions

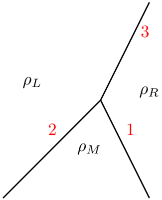


FIGURE: Two waves interact together producing a third wave

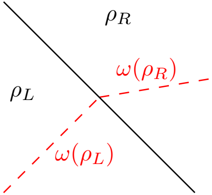


FIGURE: $\rho^* \leq \rho_R < \rho_L$ and $\rho_L - \rho_R \leq 2^{-n+1}$.

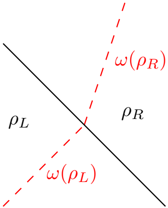


FIGURE: $\rho^* < \rho_R$ and $\rho_L \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, \rho_R]$.

Description of all possible interactions

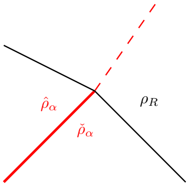


FIGURE: $\rho_L = \check\rho_\alpha$ and $\rho_R \in [\hat\rho_\alpha, 1]$

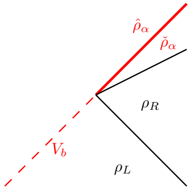


FIGURE: $\rho_L = \hat\rho_\alpha$ and $\rho_R \in [\check\rho_\alpha, \hat\rho_\alpha]$

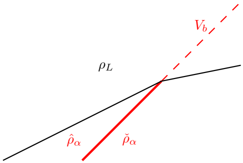


FIGURE: $\rho_R = \hat\rho_\alpha$ and $\rho_L \in [0, \check\rho_\alpha]$

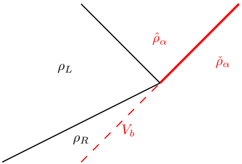


FIGURE: $\rho_L \in [\check\rho_\alpha, \hat\rho_\alpha]$ and $\rho_R = \check\rho_\alpha$

Description of all possible interactions

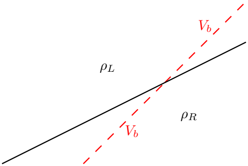


FIGURE: $\rho_L = [0, \check{\rho}_\alpha]$, $\rho_R \in [0, \check{\rho}_\alpha] \cup [\hat{\rho}_\alpha, p^*]$ and $\rho_L + \rho_R < \rho^*$

Generelized tangent vectors

The couple $(\rho^{i,n}(t,\cdot), y^{i,n}(t))$ corresponds to the wave-front tracking approximate solution of (Syst-LWR) at time t with initial data $(\rho_0^{i,n}, y_0^{i,n}) \in \mathcal{D}_C^n := \{(\rho, y) : [0,1] \times \mathbb{R} \to \mathcal{M}_n \times \mathbb{R},\ TV(\rho) \leq C)$. Let PC denotes the set of piecewise constant functions with finitely many jumps.

- We construct a particular path $\gamma_0 : [0,1] \mapsto PC$ such that $\gamma_0(0) = (\rho_0^{1,n}, y_0^{1,n})$ and $\gamma_0(1) = (\rho_0^{2,n}, y_0^{2,n})$.
- For every $\theta \in (0,1)$, $\gamma_t(\theta)$ denotes the wave-front tracking approximate solution at time t with initial data $\gamma_0(\theta)$.

We obtain

$$\|\rho^{1,n}(t) - \rho^{0,n}(t)\|_{L^1(\mathbb{R})} + |y^{1,n}(t) - y^{0,n}(t)| \leq \inf_{\gamma_t} \|\gamma_t\|_{L^1(\mathbb{R})},$$

and

$$\inf_{\gamma_0} \|\gamma_0\|_{L^1(\mathbb{R})} = \|\rho_0^{1,n} - \rho_0^{0,n}\|_{L^1(\mathbb{R})} + |y_0^{1,n} - y_0^{0,n}|.$$

To prove the main theorem, it is enough to prove that

$$\|\gamma_t\|_{L^1(\mathbb{R})} \leq \|\gamma_0\|_{L^1(\mathbb{R})}$$

Generelized tangent vectors

γ_t admits shifts of waves denoted by $\xi_i(t,\theta)$ and a shift of the bus trajectory denoted by $\xi_b(t,\theta)$. Thus, for a.e $\theta \in [0,1]$ and $t \in [0,T]$,

$$\|\gamma_t\| = \int_0^1 \sum_{k\in K(n,t,\theta)} |\Delta\rho_k^n(t,\theta)\xi_k^n(t,\theta)| + |\xi_b^n(t,\theta)|\, d\theta,$$

where $\Delta\rho_k^n(t,\theta)$ are the signed strengths of the corresponding waves. To get the inequality $\|\gamma_t\|_{L^1(\mathbb{R})} \leq \|\gamma_0\|_{L^1(\mathbb{R})}$ it is enough to have

$$\sum_{k\in K(n,T)} |\Delta\rho_k^n(T)\xi_k^n(T)| + |\xi_b^n(T)| \leq C\left(\sum_{k\in K(n,0)} |\Delta\rho_k^n(0)\xi_k^n(0)| + |\xi_b^n(0)|\right),$$

A backwards in time method

We fix $T > 0$. In the sequel, $K(n,t)$ denotes the set of classical shocks at time t. We fix the wave shift $\xi_k(T)$ and the bus shift $\xi_b(T)$ with $k \in K(n,T)$.

- If no interactions occurs between $[t_1, t_2]$ then both shifts remain constant over $[t_1, t_2]$.
- For each possible interactions at time t_1, we prove that $\xi_b(t_1^+)$ (resp. $\xi_j(t_1^+)$ with $j \in K(n, t_1^+)$) can be expressed as $\xi_b(t_1^-)$ and $\xi_k(t_1^-)$ with $k \in K(n, t_1^-)$.

Thus, for every $k \in K(n,T)$ and for every $j \in K(n,0)$, there exist $W_b^1(0), W_{b,k}^2(0), W_{j,b}(0), W_{j,k}(0) \in \mathbb{R}_+^4$ such that

$$\begin{cases} \xi_b(T) = W_b^1(0)\xi_b(0) + \sum_{j\in K(n,0)} W_{j,b}(0)\Delta\rho_j(0)\xi_j(0), \\ \Delta\rho_k(T)\xi_k(T) = W_{b,k}^2(0)\xi_b(0) + \sum_{j\in K(n,0)} W_{j,k}(0)\Delta\rho_j(0)\xi_j(0) \end{cases} \quad (2)$$

A backwards in time method

From the previous equalities, we construct explicitely weight functions $(W_k^n(0))_{k\in K(n,0)}$ and $W_b^n(0)$ such that

$$\sum_{k\in K(n,T)} |\Delta\rho_k^n(T)\xi_k^n(T)|+|\xi_b^n(T)| \leq \sum_{k\in K(n,0)} |W_k^n(0)\Delta\rho_k^n(0)\xi_k^n(0)|+|W_b^n(0)\xi_b^n(0)|$$

By straighforward computations ,we have for every $k \in K(n,0)$,

$$\max(W_k^n(0), W_b^n(0)) \leq C,$$

whence the desired conclusion.

Perspectives

We can consider the generalized Aw-Rascle-Zhang (GARZ) model on each road I defined by

$$\begin{cases} \partial_t \rho(t,x) + \partial_x(\rho(t,x)v(t,x)) = 0, & (t,x) \in \mathbb{R}^+ \times I \\ \partial_t w(t,x) + v(t,x)\partial_x(w(t,x)) = 0, & (t,x) \in \mathbb{R}^+ \times I \\ v = V(\rho, w) \end{cases} \qquad \text{(GARZ)}$$

Above,

- $\rho = \rho(t,x)$ is the mean traffic density,
- $w = w(t,x)$ describes the related driver properties to the flow-density curves. For instance, w can represent the fraction of special vehicles in the total traffic stream (trucks or autonomous vehicles), the "agressivity", the "desired spacing" or "perturbation from equilibirum".
- $v = V(\rho, w)$ is the velocity. Some conditions are required on V.

Replacing (LWR) by (GARZ) in (Syst-LWR), we want to find existence and stability results for (Syst-GARZ).

Thank you for your attention