Mathematical relations between LCS detection methods

SIAM Conference on Applications of Dynamical Systems

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Motivation: Study transport and mixing of scalar quantities

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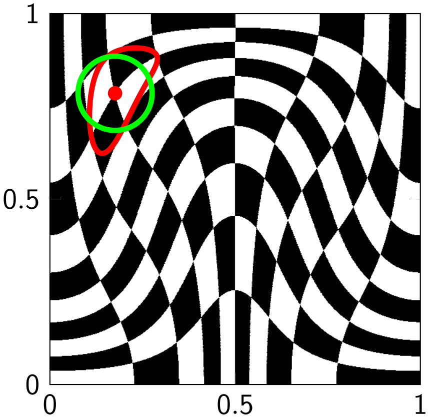
Fokker–Planck equation (FPE) (Eulerian/spatial evolution equation):

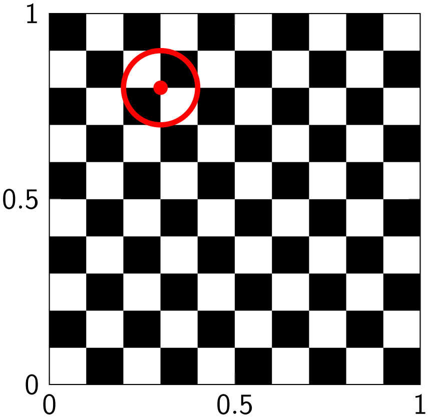
$$\boxed{\partial_t u + \operatorname{div}(u \cdot v) = \varepsilon \Delta_g u}$$

- transport/advection along velocity field v
- mixing/diffusion according to spatial geometry/distances modeled by metric g

Changing perspectives: looking from the material



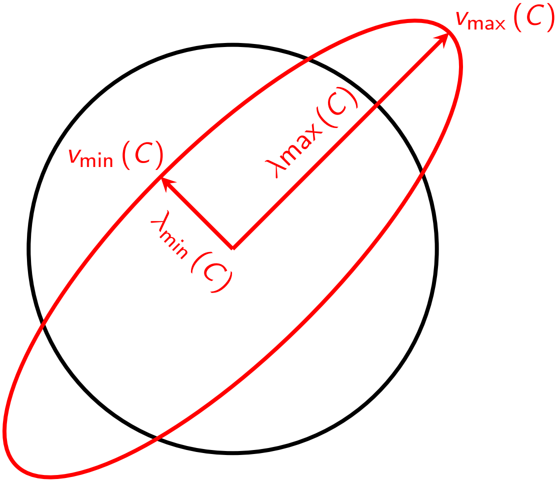
Pullback material geometry



Spatial geometry

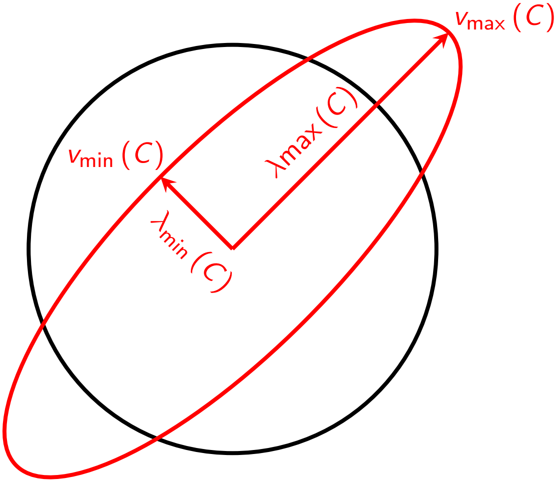
Pullback geometry: stretching and diffusivity

- length measurement via Cauchy–Green strain tensor C



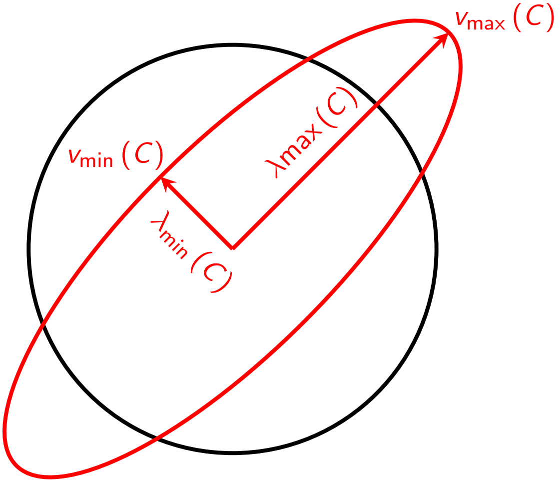
Pullback geometry: stretching and diffusivity

- length measurement via Cauchy–Green strain tensor C
- directional diffusivity via inverse CG strain tensor C^{-1}



Pullback geometry: stretching and diffusivity

- length measurement via Cauchy–Green strain tensor C
- directional diffusivity via inverse CG strain tensor C^{-1}
- strong stretching $\iff$ weak Lagrangian diffusion
- strong compression $\iff$ strong Lagrangian diffusion
- diffusion in Lagrangian coordinates is anisotropic



Black hole vortex approach Haller & Beron-Vera 2012, 2013

Seek closed curves γ, parametrized by r, that stationarize averaged tangential strain

$$\delta Q(\gamma) \stackrel{!}{=} 0, \qquad Q(\gamma) = \frac{1}{|\gamma|}\int_\gamma \sqrt{\frac{\langle r', Cr'\rangle}{\langle r', r'\rangle}} = \frac{1}{|\gamma|}\int_\gamma \frac{\|DF \cdot r'\|}{\|r'\|}$$

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If, as often observed "around" vortices, $\lambda_{\max}(C) \gg 1$, then $\lambda_{\min}(C) \ll 1$ and

$$r' \approx v_{\min}(C), \qquad \text{or equivalently} \qquad \gamma^\perp \approx v_{\max}(C)$$

$$\boxed{\text{Black hole vortices admit very small (Lagrangian) cross-diffusivity.}}$$

Lagrangian model of advection–diffusion

Eulerian:

$$\boxed{\partial_t u + \operatorname{div}(u \cdot v) = \varepsilon \Delta_g u}$$

Equivalent Lagrangian FPE (material evolution equation) [Thiffeault 2003]

$$\boxed{\partial_t w = \varepsilon \Delta_{g(t)} w}$$

$$\left.\begin{aligned} g(t) := \Phi(t)^* g = C&\text{—pullback metric, CG strain tensor,} \\ \Delta_{g(t)}&\text{—Laplace–Beltrami operator.} \end{aligned}\right\} \Rightarrow \text{evolving (material) manifold}$$

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Definition (Lagrangian Coherent Structures)

maximal material sets with minimal diffusive flux, or, metastable/almost-invariant sets under Lagrangian FPE

The dynamic Laplacian as a proxy for $\Delta_{g(t)}$

Lagrangian FPE: time-dependent & anisotropic diffusion(-only) equation

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Lemma (Froyland 2015, DK & Keller 2016, Froyland & Kwok 2016)

$\overline{\Delta}$ is an elliptic, nonpositive, selfajoint, 2nd-order differential operator.

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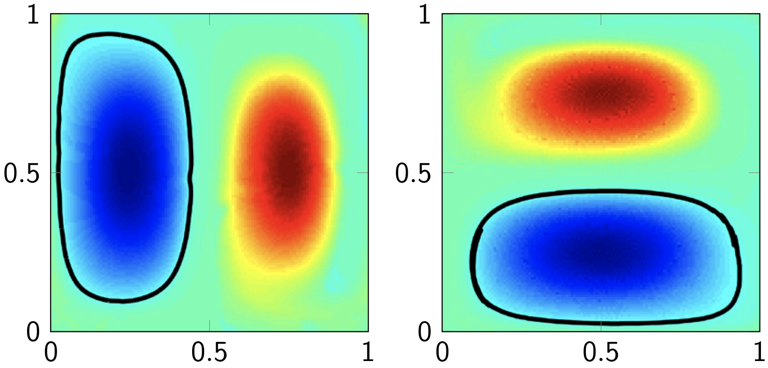
Theorem (spectral convergence to probabilistic transfer operator $\mathcal{L}_\varepsilon$, DK & Keller 2016)

$$\boxed{\mathcal{L}_\varepsilon^* \mathcal{L}_\varepsilon = I + c\varepsilon^2 \overline{\Delta} + \mathcal{O}\left(\varepsilon^4\right)}$$

Small example: transient double gyre [Mosovsky & Meiss, 2011], figure courtesy of [Froyland, *Nonlinearity*, 2015]



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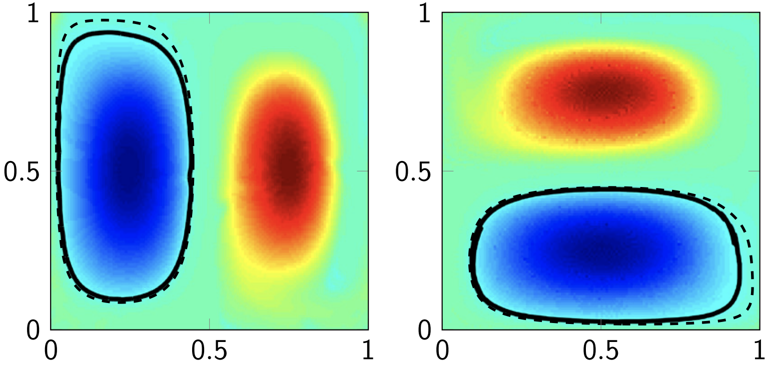

solid: level set of 2nd eigenfunction of dynamic Laplacian, minimizing $h^D(\Gamma) = 8.2$

dashed: outermost closed λ-line $h^D(\Gamma) = 6.7$

Lagrangian heat flow, initialized within coherent structure

Cylinder flow [Froyland et al., 2010]

Lagrangian heat flow, initialized within mixing region

Animation 1000 times slower than previous one!

Conclusions

- math. advection-diffusion-based relation b/w BH vortices and probabilistic TO
- within this framework strongly related objectives: minimizing Lagr. cross-diffusion
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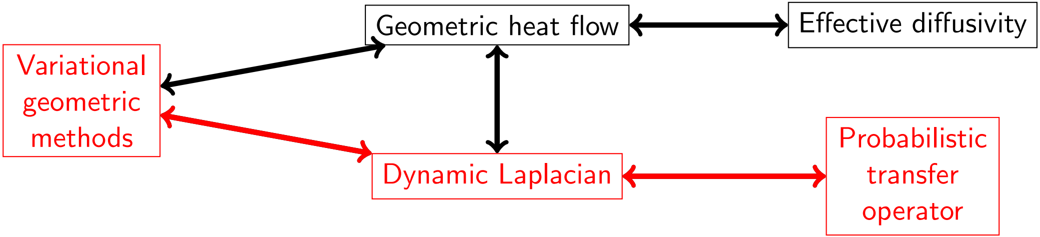

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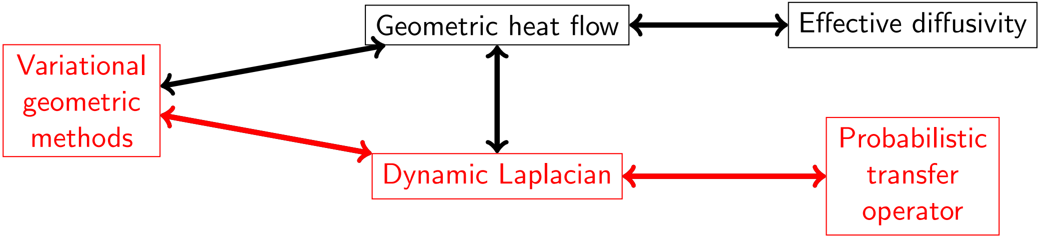

D. Karrasch and J. Keller, *preprint*, 2016, available on ResearchGate & arXiv.