A Quasi-Static Projection Method for 3-Dimensional Hypoelastoplasticity

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Harvard University, Department of Applied Mathematics.
Rycroft Group.

Bulk Metallic Glasses

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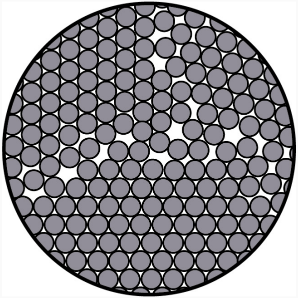

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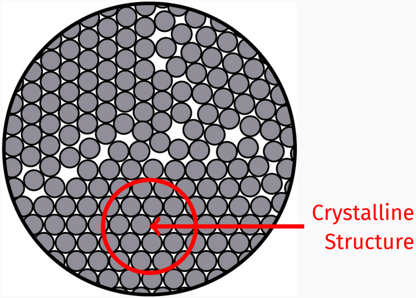

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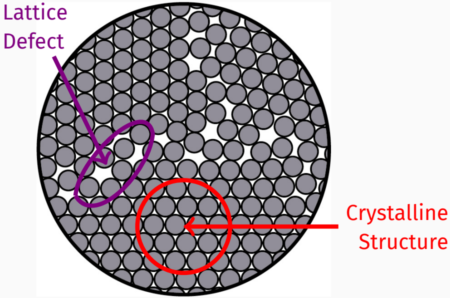

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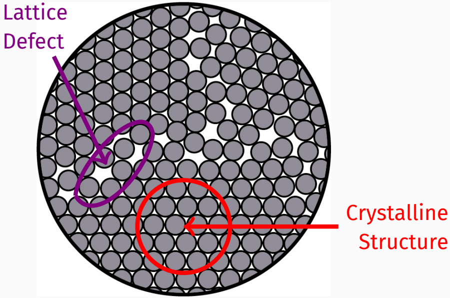

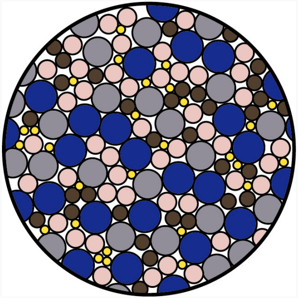

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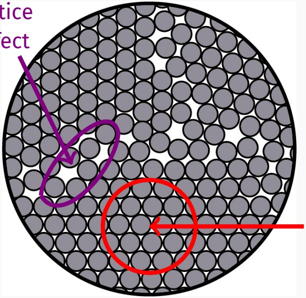

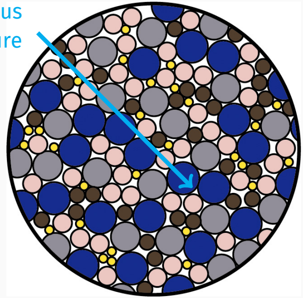

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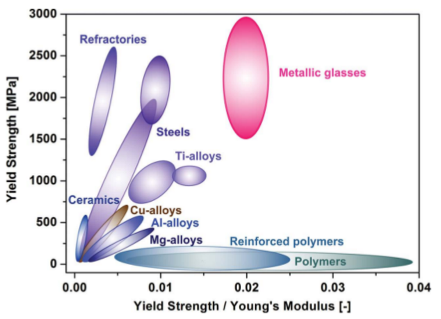

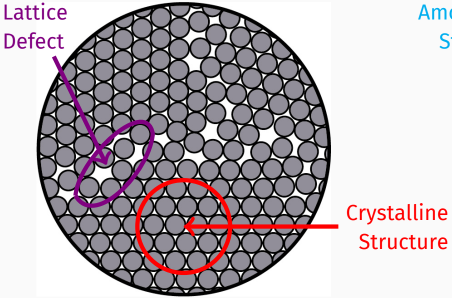

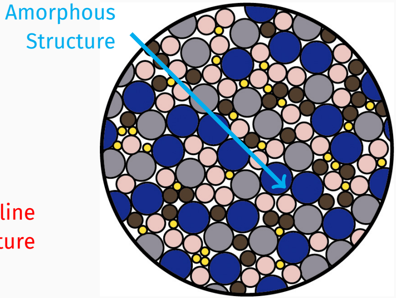

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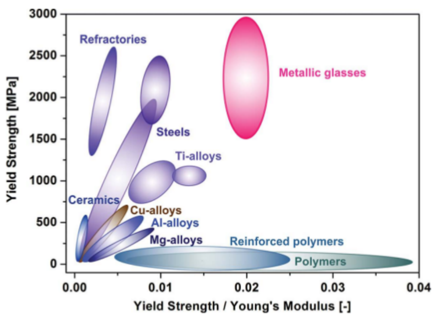
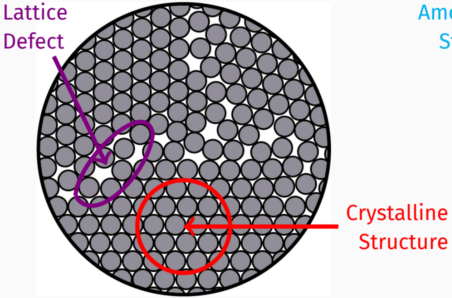
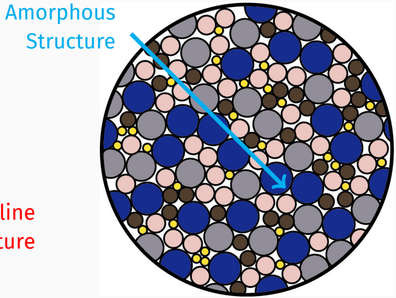
- Solid metal with atoms "frozen" into liquid-like disorder.
- Amorphous structure gives unique properties.
- Catastrophic failure: shear banding.







Shear Bands

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- Localization of stress due to localization of strain.

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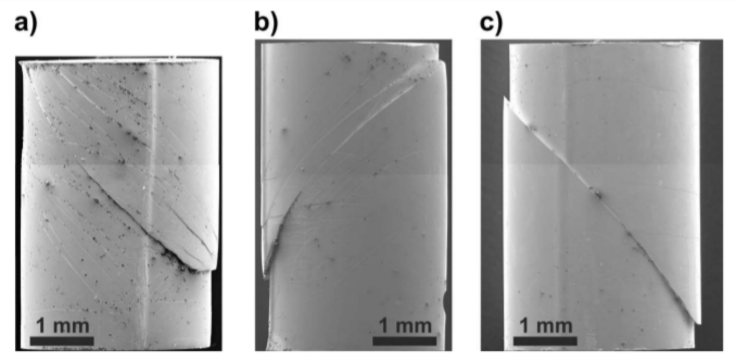

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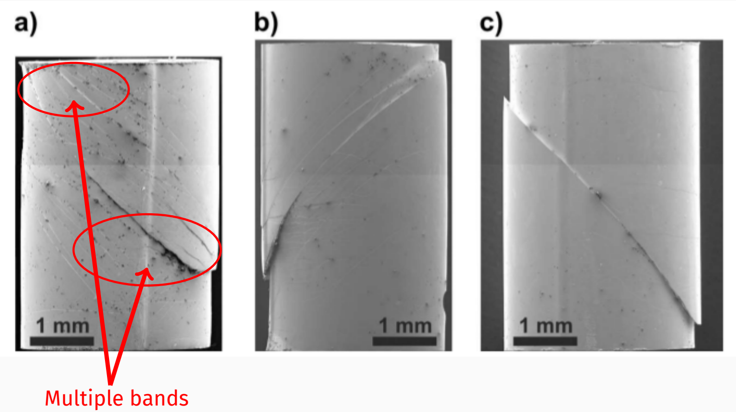

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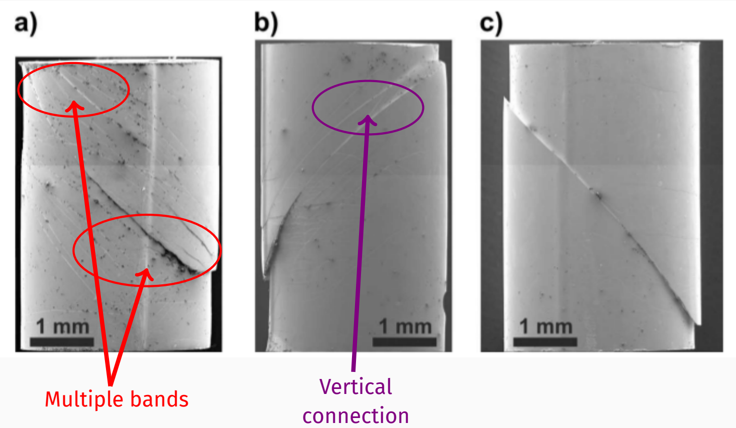

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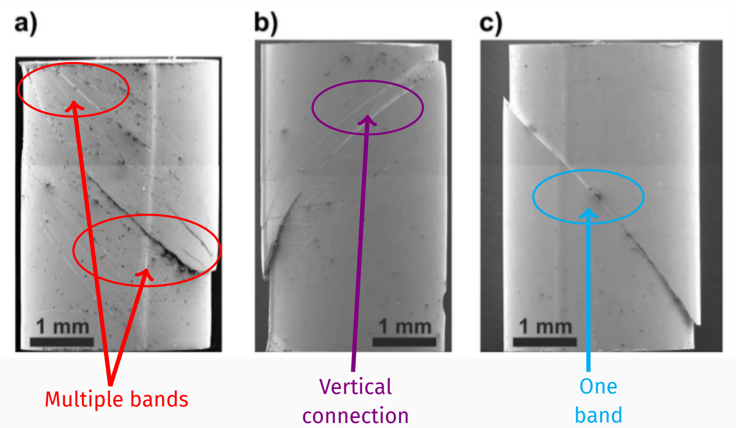
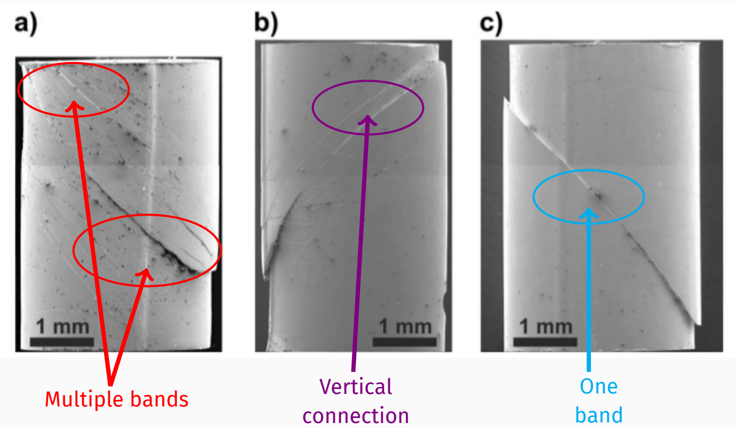


- Localization of stress due to localization of strain.
- Strain-softening instability provides positive feedback.
- Effective temperature χ quantifies localized "softness".



Continuum Theory

Linear Elasticity

Shear Transformation Zone Theory

Continuum Theory

Linear Elasticity

$$\frac{\mathcal{D}\sigma}{\mathcal{D}t} = \mathsf{C} : \mathsf{D}^{\mathrm{el}}$$

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$$\frac{d\chi}{dt} = \frac{D^{pl}\bar{s}}{s_y c_0}(\chi_\infty - \chi)$$

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A Thought-Provoking Analogy

Hypoelastoplastic Long-Time Limit

Incompressible Navier-Stokes

From Navier-Stokes to Hypoelastoplasticity

A Thought-Provoking Analogy

Hypoelastoplastic Long-Time Limit

$$\frac{\mathcal{D}\sigma}{\mathcal{D}t} = \mathsf{C} : (\mathsf{D} - \mathsf{D}^{\mathrm{pl}})$$

Incompressible Navier-Stokes

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Incompressible Navier-Stokes

$$\rho \frac{d\mathsf{u}}{dt} = -\nabla p + \nabla \cdot \mathsf{T}$$

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$$\underbrace{\frac{\mathcal{D}\sigma}{\mathcal{D}t} = \mathsf{C} : (\mathsf{D} - \mathsf{D}^{\mathsf{pl}})}_{\text{Hypoelastoplastic equation}}$$

$$\nabla \cdot \sigma \approx 0$$

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A Thought-Provoking Analogy

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$$\underbrace{\frac{\mathcal{D}\sigma}{\mathcal{D}t} = \mathsf{C} : \left(\mathsf{D} - \mathsf{D}^{\mathsf{pl}}\right)}_{\text{Hypoelastoplastic equation}}$$

$$\underbrace{\nabla \cdot \sigma}_{\text{Quasi-static constraint}} \approx 0$$

Incompressible Navier-Stokes

$$\underbrace{\rho \frac{d\mathsf{u}}{dt} = -\nabla p + \nabla \cdot \mathsf{T}}_{\text{Navier-Stokes equation}}$$

From Navier-Stokes to Hypoelastoplasticity

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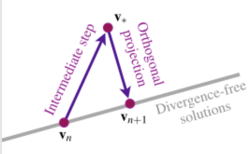

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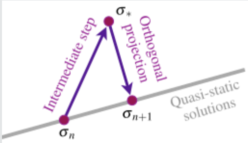

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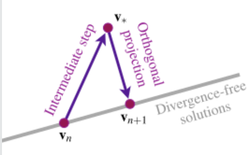

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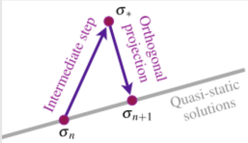

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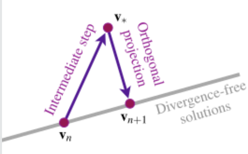

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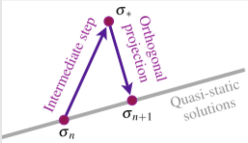

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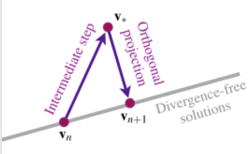

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- This analogy is independent of the plasticity model.

- Hypoelastoplastic equation:

A Quasi-Static Projection Method

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- Apply the projection step to compute σ^{n+1}.

stzpp

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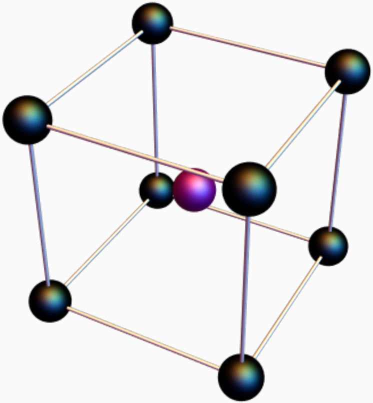

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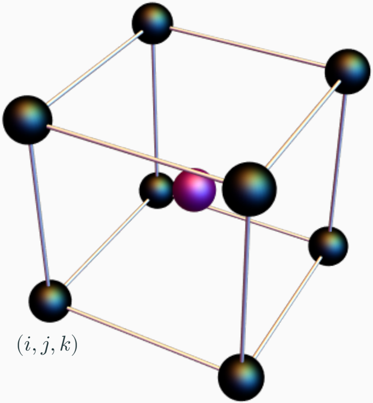

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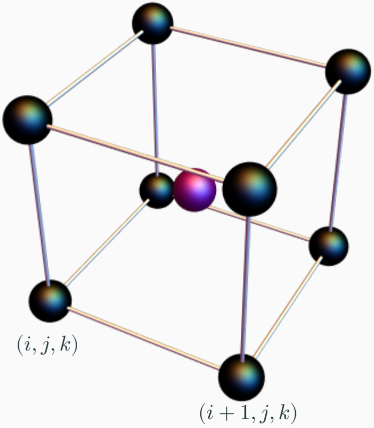

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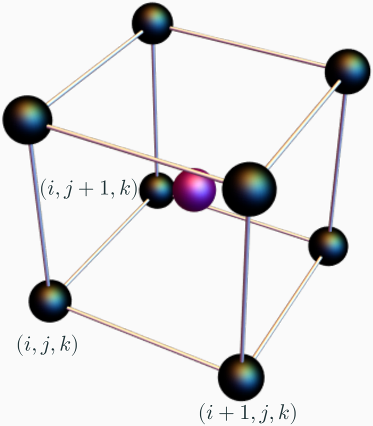

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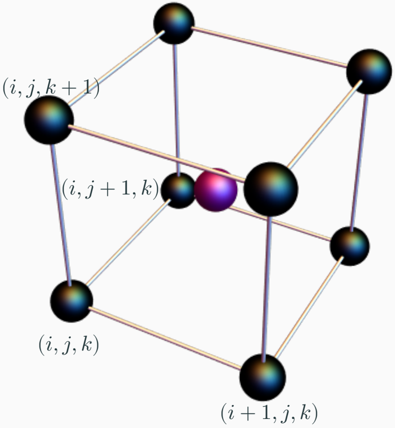

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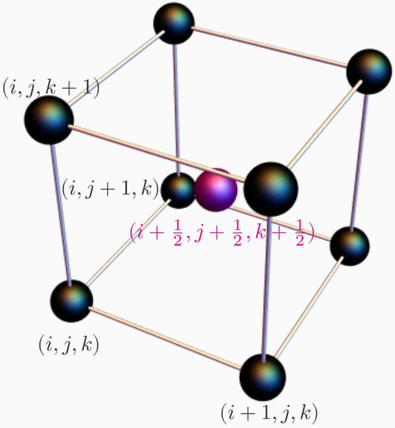

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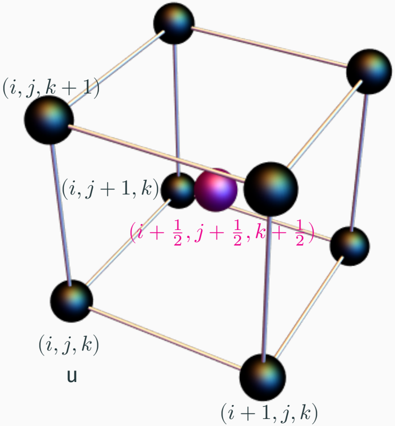

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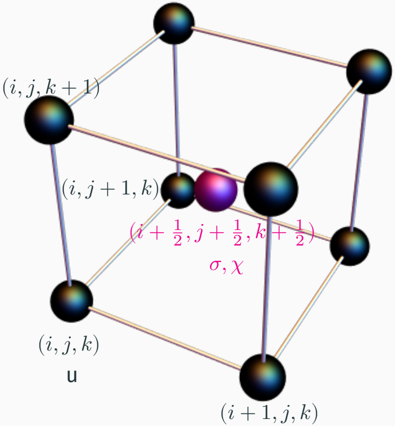

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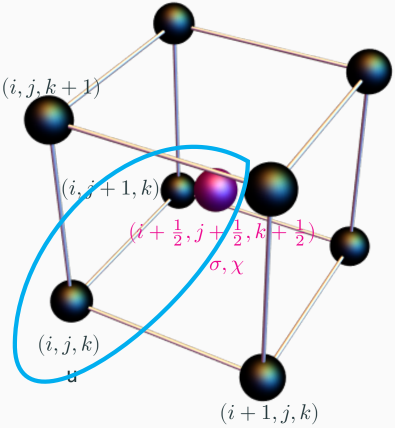

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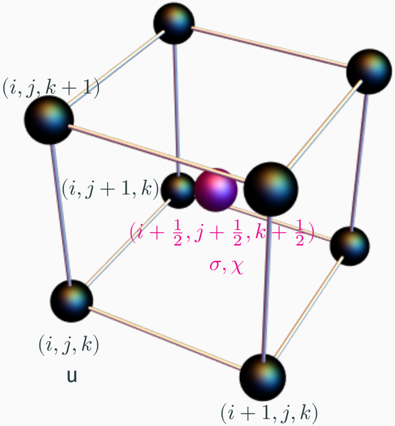

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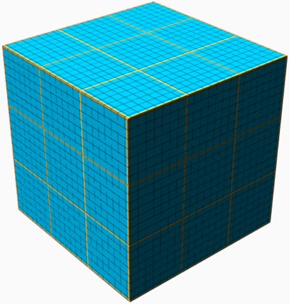
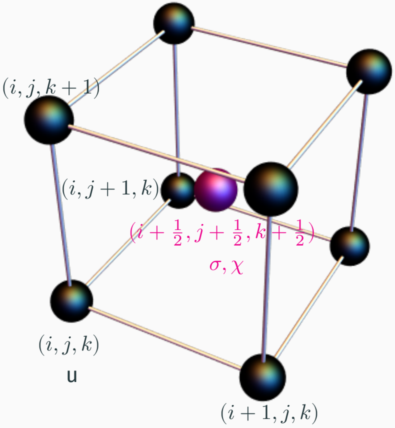
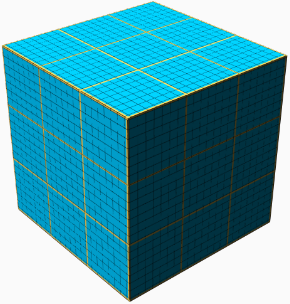
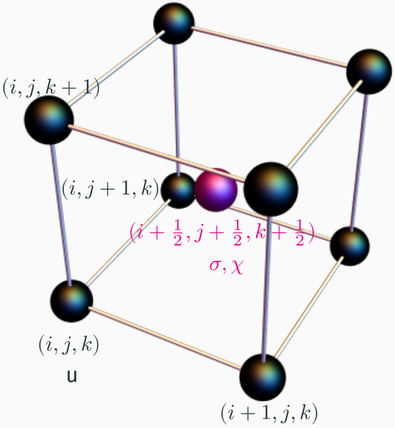




stzpp

- Same software for both quasi-static and explicit method.
- Staggered grid: σ, χ at cell centers. **u** at cell corners.
- Parallelized using domain decomposition and MPI.
- Ghost-regions pad processor subdomains with two points.





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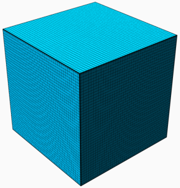

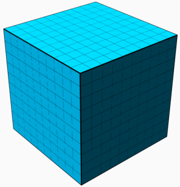

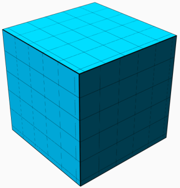

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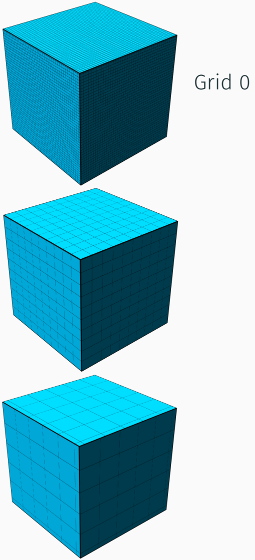

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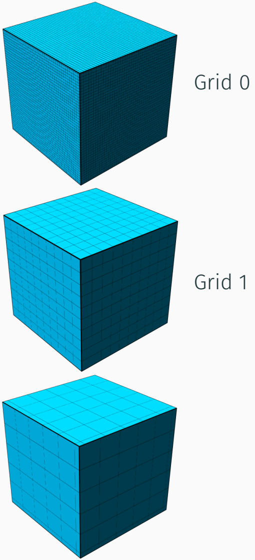

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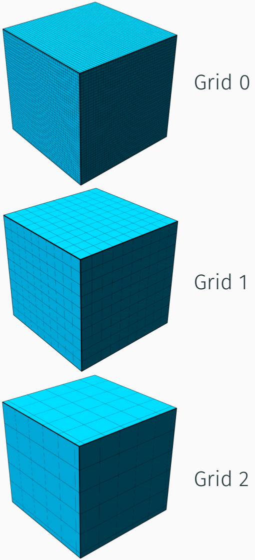

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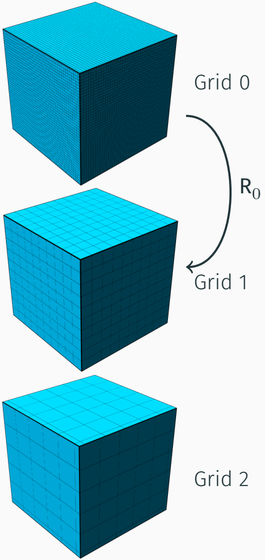

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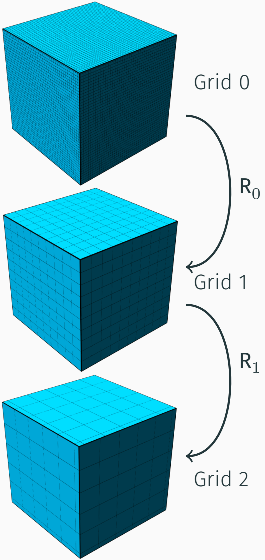

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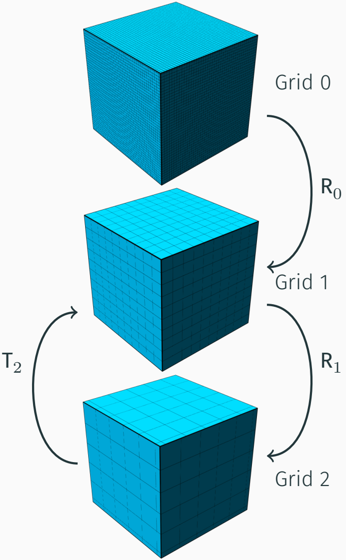

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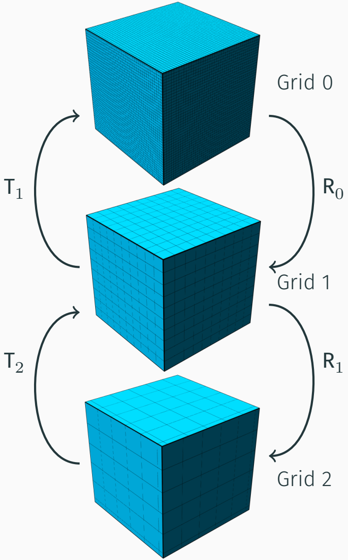

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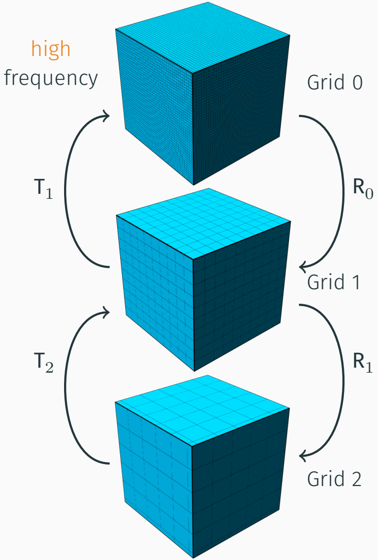

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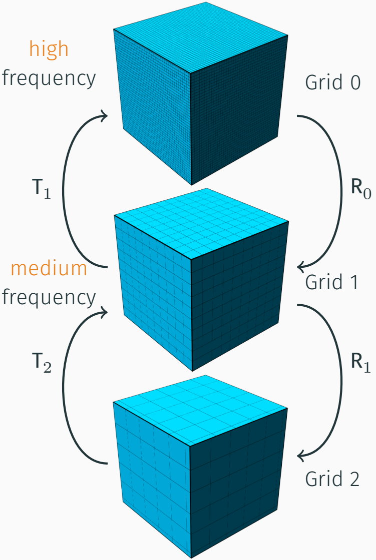

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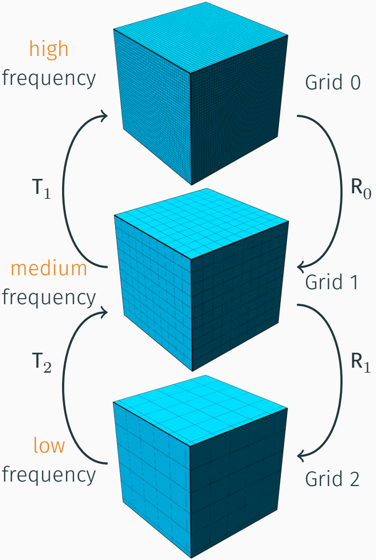

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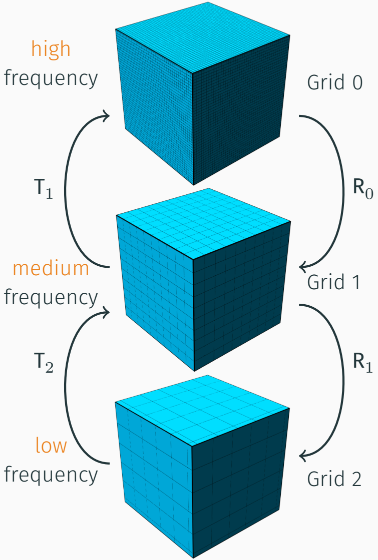

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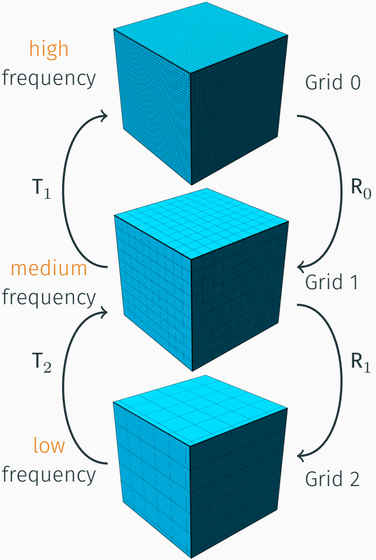

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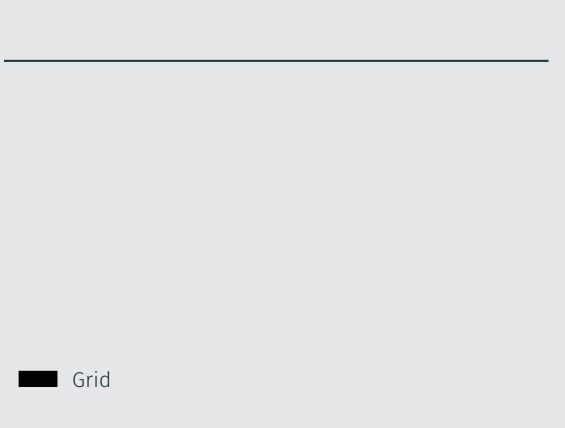

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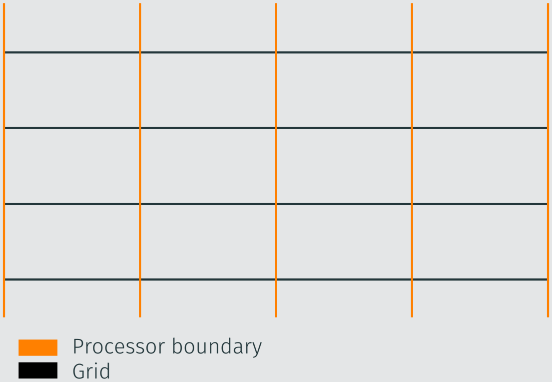

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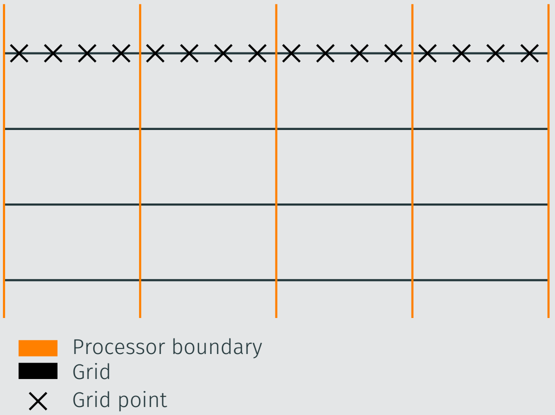

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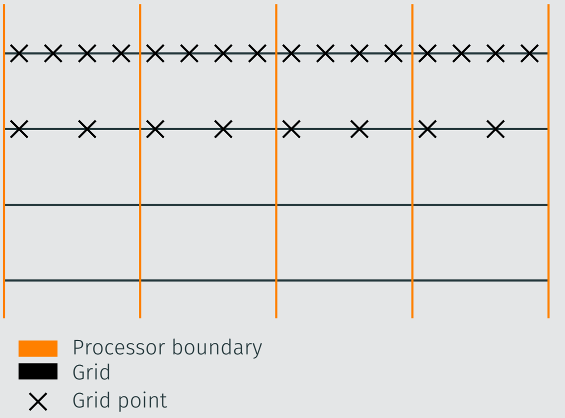

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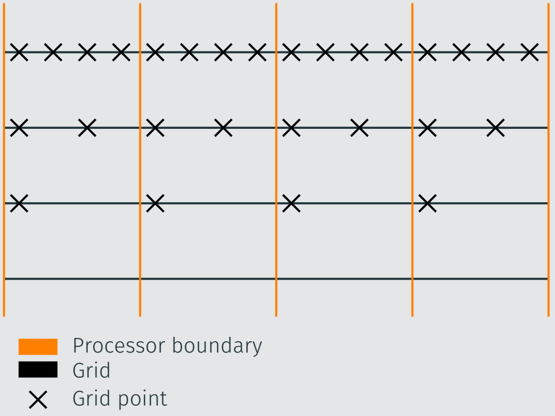

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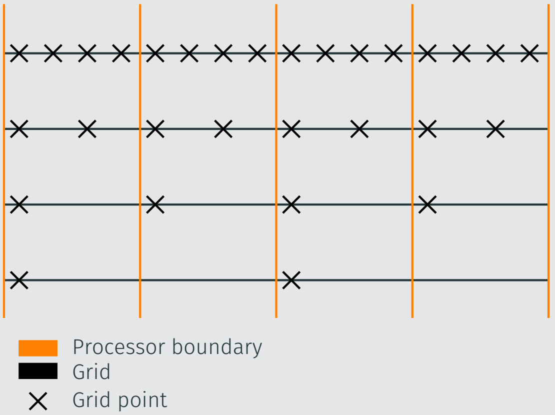

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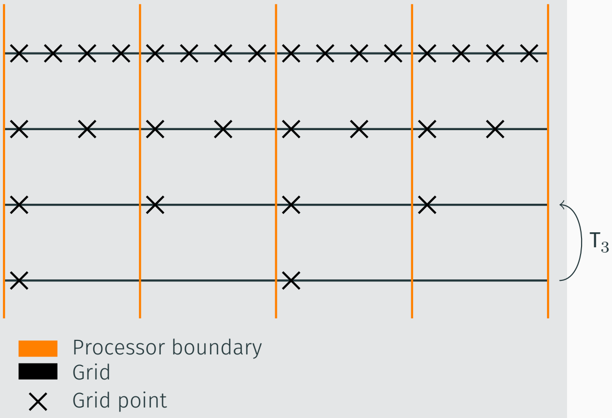

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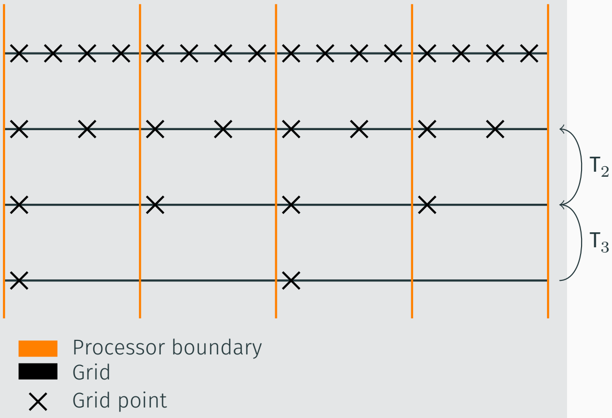

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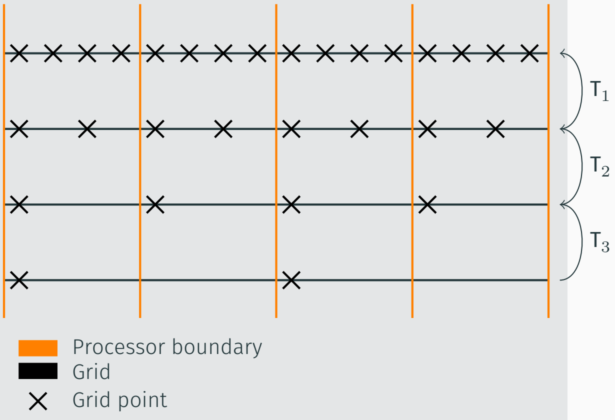

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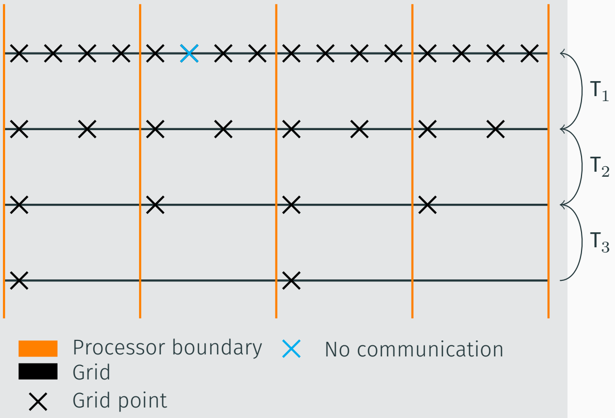

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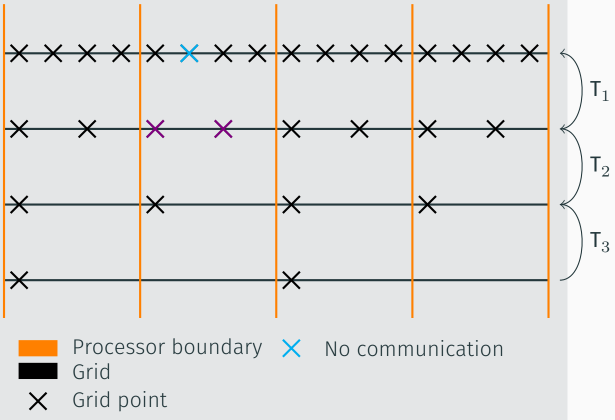

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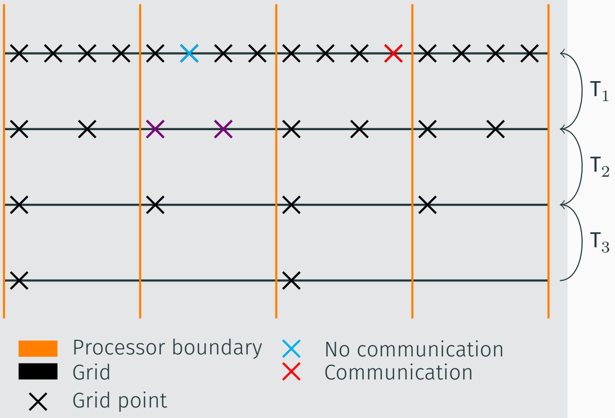

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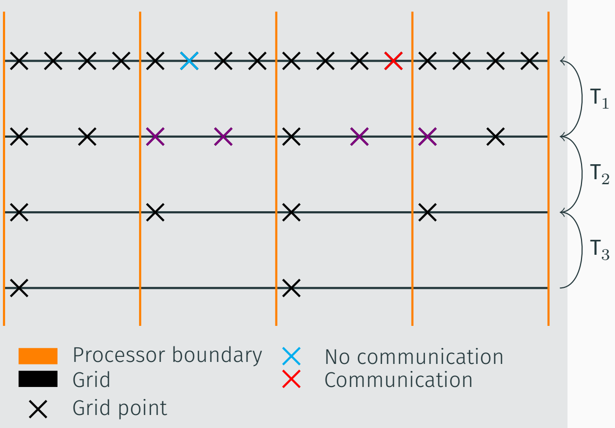

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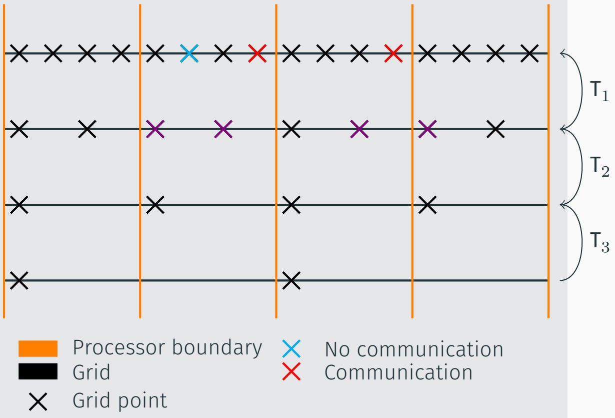

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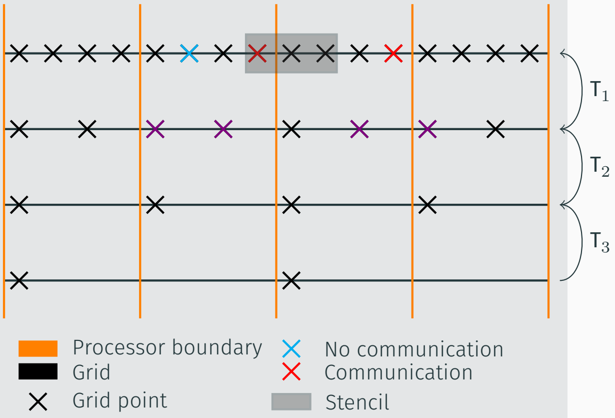

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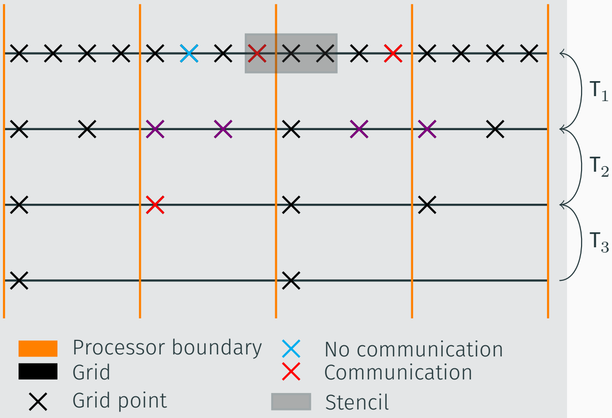

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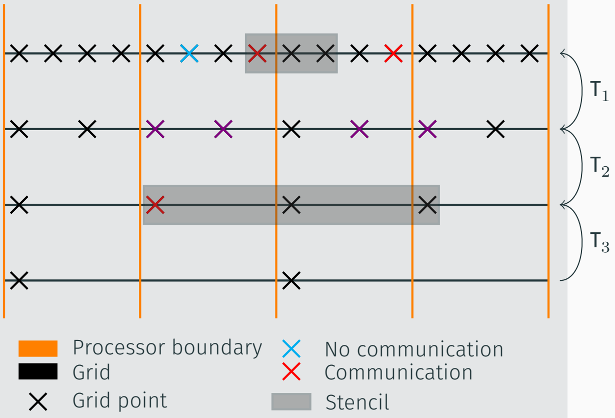

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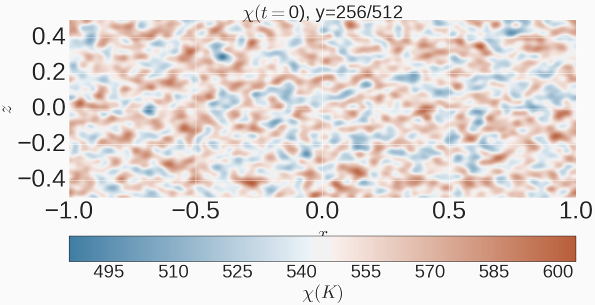

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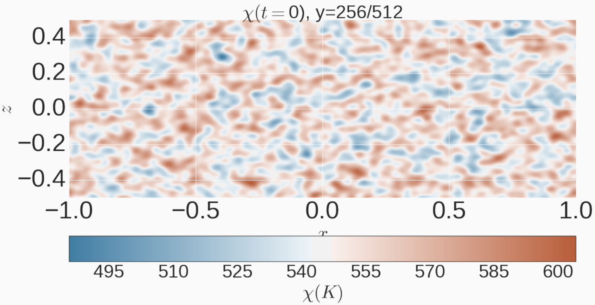

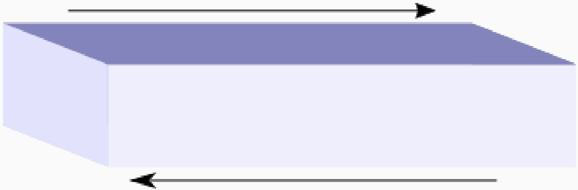

Quasi-Static Results

Conclusions, Future Directions, and Acknowledgments

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- Thanks to Chris Rycroft, collaborators Eran Bouchbinder, Michael Falk, Michael Shields, and the Department of Energy Computational Science Graduate Fellowship for funding.