Constraint Preconditioning for the coupled Stokes-Darcy System

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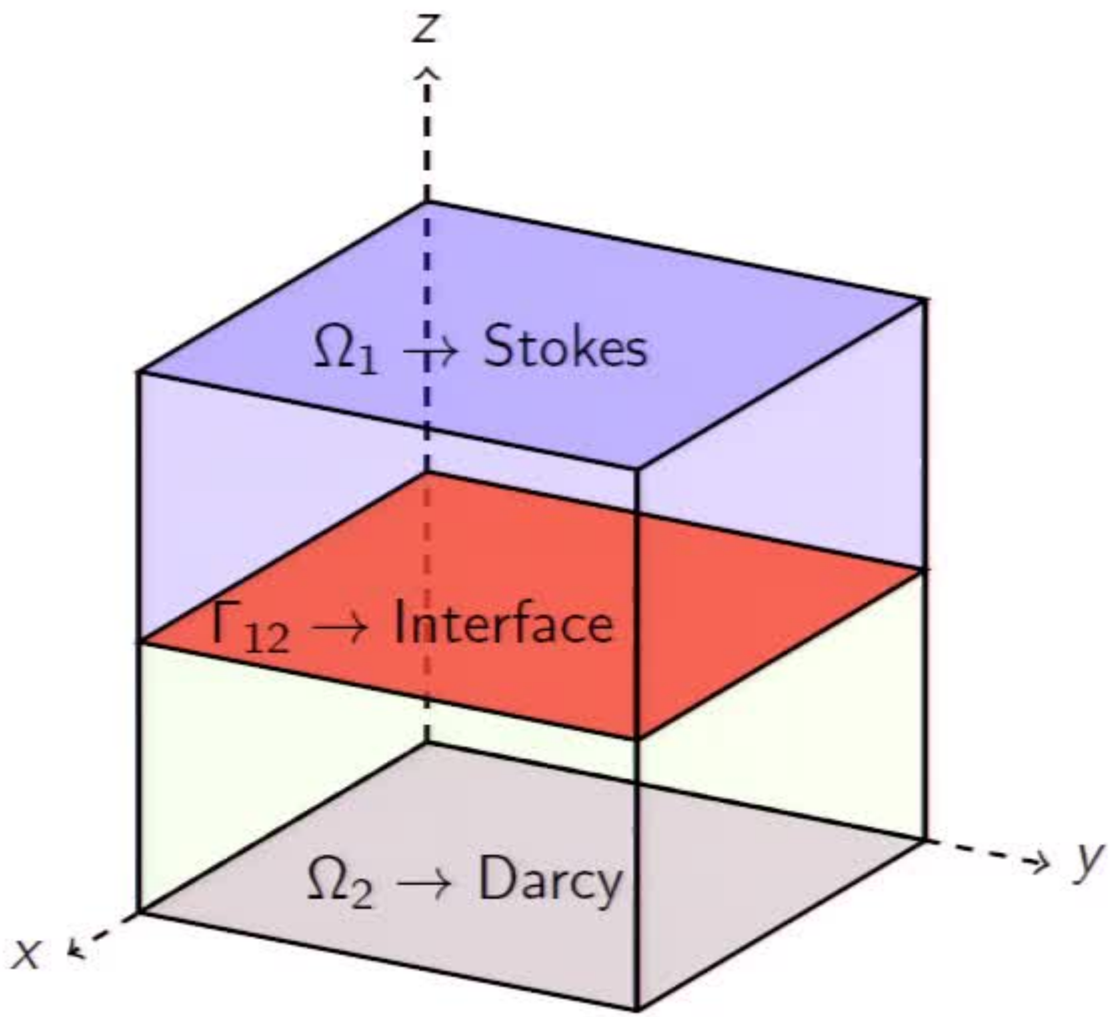
Physical setting

Describe two coupled flows

- freely flowing fluid
- porous media flow

Stokes-Darcy Flow

In this talk we consider coupled Stokes-Darcy flow in the domain $\Omega = \Omega_1 \cup \Omega_2$



Boundary
$\partial\Omega = \Gamma_1 \cup \Gamma_2$
$\Gamma_1 = \partial\Omega_1 \setminus \Gamma_{12}$
$\Gamma_2 = \Gamma_{2N} \cup \Gamma_{2D}$

The p.d.e.s are coupled together by conditions across the interface

Weak Problem Statement

Find $\mathbf{u}_1 \in \mathbf{X}_1, p_1 \in Q_1, p_2 = \varphi_2 + p_{\mathrm{D}},$ with $\varphi_2 \in Q_2$ such that

- $\forall \mathbf{v}_1 \in \boldsymbol{X}_1, \forall q_2 \in Q_2$

$$2\nu\big(\mathbf{D}(\mathbf{u}_1), \mathbf{D}(\mathbf{v}_1)\big)_{\Omega_1} - \big(p_1, \nabla \cdot \mathbf{v}_1\big)_{\Omega_1}$$
$$+(\varphi_2, \boldsymbol{v}_1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{G}\big(\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12}, \mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}\big)_{\Gamma_{12}} - \big(\mathbf{u}_1 \cdot \mathbf{n}_{12}, q_2\big)_{\Gamma_{12}}$$
$$+(\mathbf{K}\nabla\varphi_2, \nabla q_2)_{\Omega_2}$$
$$= \big(\mathbf{f}_1, \mathbf{v}_1\big)_{\Omega_1} + \big(f_2, q_2\big)_{\Omega_2} - \big(\mathbf{K}\nabla p_{\mathrm{D}}, \nabla q_2\big)_{\Omega_2} + \big(g_{\mathrm{N}}, q_2\big)_{\Gamma_{2\mathrm{N}}}$$

- $\forall q_1 \in Q_1 \qquad (\nabla \cdot \boldsymbol{u}_1, q_1)_{\Omega_1} = 0$

$$p_D \in H^1(\Omega_2) \text{ such that } p_D|_{\Gamma_{2D}} = g_D$$

FE Discretization

- Partition the domain Ω into finite elements (triangles,squares) with mesh width h.
- Choose finite-dimensional spaces X_1^h, Q_1^h, Q_2^h satisfying the discrete inf-sup condition
- The discrete Darcy pressure and the discrete Stokes velocity and pressure are expressed as linear combinations of the basis functions of the discrete spaces
- Assemble the system matrix and right hand side
- Solve the linear system

Linear System

Following standard finite element techniques we obtain the linear system for discretized coupled Stokes-Darcy problem

$$\mathcal{A}\mathbf{x} = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix} \begin{bmatrix} p_2 \\ \mathbf{u}_1 \\ p_1 \end{bmatrix} = \begin{bmatrix} f_{2,h} \\ \mathbf{f}_{1,h} \\ g_h \end{bmatrix} = \mathbf{b}$$

This is sparse, nonsymmetric saddle point problem and the dimension increases as $h \to 0$

$$\mathcal{A} = \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T \\ -A_{\Gamma_{12}} & A_{\Omega_1} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & B \end{bmatrix}$$

Solve iteratively with preconditioned GMRES

Preconditioning Saddle Point Problems

What preconditioners to consider?

$$\mathcal{P}_{\text{bd}} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad \mathcal{P}_{\text{lt}} = \begin{bmatrix} P_1 & 0 \\ C & P_2 \end{bmatrix}$$

Murphy, Golub, Wathen (2000)

Let $\mathcal{A} = \begin{bmatrix} A & C^T \\ C & 0 \end{bmatrix}$ be nonsingular and set $P_1 = A$, $P_2 = S = -CA^{-1}C^T$, then $\mathcal{P}_{\text{bd}}^{-1}\mathcal{A}$ and $\mathcal{P}_{\text{lt}}^{-1}\mathcal{A}$ have three and two non-zero eigenvalues, respectively.

- Krylov subspace methods converge in exact arithmetic in 3 and 2 iterations, respectively
- in practice approximations $\tilde{A}^{-1}$, $\tilde{S}^{-1}$ are used

Preconditioners

$$\mathcal{P}_{\pm} = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & 0 \\ 0 & 0 & \pm M_p \end{bmatrix} \qquad \mathcal{P}_{T_1}(\rho) = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & 0 \\ 0 & B & -\rho M_p \end{bmatrix}$$

$$\mathcal{P}_{T_2}(\rho) = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & 0 \\ 0 & B & -\rho M_p \end{bmatrix} \quad \mathcal{P}_{C}(\rho) = \begin{bmatrix} A_{\Omega_2} & A_{\Gamma_{12}}^T & 0 \\ -A_{\Gamma_{12}} & A_{\Omega_1} & 0 \\ 0 & B & -\rho M_p \end{bmatrix}$$

- M_p is the Stokes pressure mass matrix.
- M_p is spectrally equivalent to the Schur complement of the Stokes matrix matrix $S = -BA_{\Omega_1}B^T$, i.e., there exist α, β independent of h such that for all $\mathbf{x} \neq 0$, $\alpha < \frac{(\mathbf{x}, S\mathbf{x})}{(\mathbf{x}, M_p\mathbf{x})} < \beta$

[M. Cai, M. Mu, and J. Xu, 2009]

Constraint Preconditioning

A constraint preconditioner has the form

$$\mathcal{P}_{con} = \begin{bmatrix} P & C^T \\ C & 0 \end{bmatrix}$$

where P is chosen to approximate the A block of the system matrix

Two Constraint Preconditioners

$$\mathcal{P}_{con_D} = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix} \qquad \mathcal{P}_{con_T} = \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ -A_{\Gamma 12} & A_{\Omega_1} & B^T \\ 0 & B & 0 \end{bmatrix}$$

e.g., [C. Keller, N. Gould, and A. Wathen, 2000]

Norm- and FOV- Equivalence

Two Useful Equivalences

1. Norm equivalence,
   $M \sim_H N \iff \exists\ \alpha, \beta$ such that $\alpha < \frac{\|M\mathbf{x}\|_H}{\|N\mathbf{x}\|_H} < \beta\ \forall \mathbf{x} \neq 0$
2. Field-of-Values (FOV) equivalence, $M \approx_H N \iff$
   $\exists\ \alpha, \beta$ such that $\alpha \leq \frac{(\mathbf{x}, MN^{-1}\mathbf{x})_H}{(\mathbf{x},\mathbf{x})_H}$ and $\|MN^{-1}\|_H \leq \beta\ \forall \mathbf{x} \neq 0$

α, β are independent of h.

Norm equivalence $\Longrightarrow \alpha < |\Lambda(MN^{-1})| < \beta$

FOV equivalence $\Longrightarrow$ H-field-of-values, $\mathcal{W}_H(MN^{-1}) \subset \mathbb{C}^+$

MINRES/GMRES Convergence

Implications of Norm/FOV-Equivalence

Norm Equivalence

- $\mathcal{A} \sim_H \mathcal{P}$
- $\mathcal{A}\mathcal{P}^{-1}$ symmetric w.r.t. $(\cdot,\cdot)_H$

The residual bound for MINRES is independent of h

[D. Loghin and A. Wathen, 2004]

FOV Equivalence

- $\mathcal{A} \approx_H \mathcal{P}$

The residual bound for GMRES is independent of h

Mesh-Independent Spectra and Field of Values

Block Diagonal/Lower Triangular

In [M. Cai, M. Mu, and J. Xu, 2009], the authors proved that for

$$\mathcal{P} \in \{\mathcal{P}_+, \mathcal{P}_-, \mathcal{P}_{T_1}, \mathcal{P}_{T_2}, \mathcal{P}_C\} \quad \text{then} \quad \mathcal{A} \sim_H \mathcal{P}$$

$\Longrightarrow$ $\Lambda(\mathcal{A}\mathcal{P}^{-1})$ bounded independent of h

Constraint

We have shown that for exact versions

$$\mathcal{P} \in \{\mathcal{P}_{con_D}, \mathcal{P}_{con_T}\} \quad \text{then} \quad \mathcal{A} \sim_H \mathcal{P} \text{ and } \mathcal{A} \approx_H \mathcal{P}$$

$\Longrightarrow$ $\Lambda(\mathcal{A}\mathcal{P}^{-1})$ bounded independent of h

$\Longrightarrow$ $\mathcal{W}_H(\mathcal{A}\mathcal{P}^{-1}) \subset \mathbb{C}^+$ bounded independent of h

2D rectangular domain

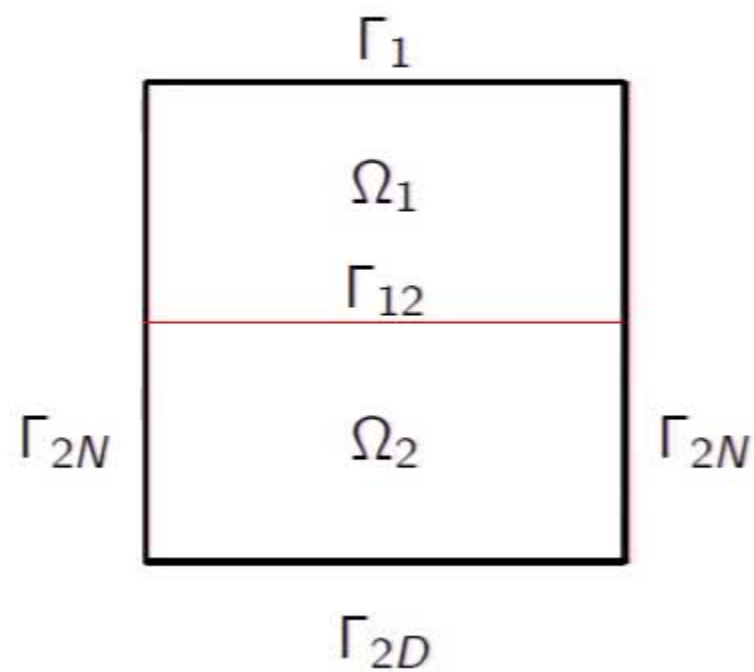
We consider a coupled Stokes-Darcy problem with $\kappa = \nu = 1$ and boundary conditions chosen to match the exact solution

$$\mathbf{u}(x,y) = \left[y^2 - 2y + 1 + \nu(2x-1),\ x^2 - x - (y-1)2\nu\right]^T$$

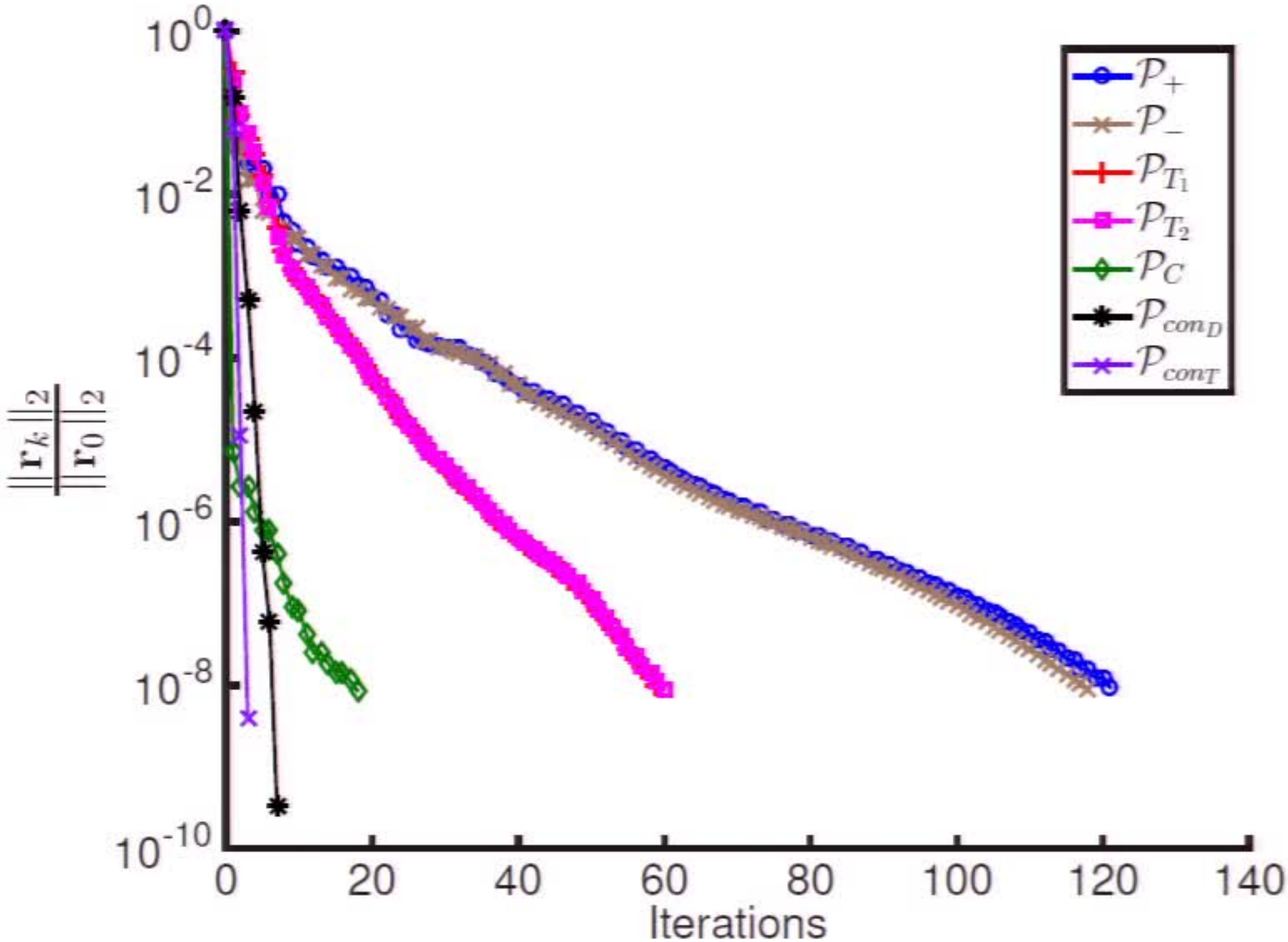
$$p_1(x,y) = 2\nu(x+y-1) + \frac{1}{3\kappa} - 4\nu^2$$

$$p_2(x,y) = \frac{1}{\kappa}\left(x(1-x)(y-1) + \frac{y^3}{3} - y^2 + y\right) + 2\nu x$$

- X_1^h, Q_1^h MINI-elements
- Q_2^h linear elements
- solve the linear system with GMRES
- exact preconditioners, i.e., direct solves



2D: GMRES convergence



dofs = 524545

2D: GMRES convergence (cont)

h	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_2}(0.6)$	$\mathcal{P}_C(0.6)$	$\mathcal{P}_{con_D}$	$\mathcal{P}_{con_T}$
2^{-3}	521	80 (0.12)	46 (0.04)	37 (0.03)	7 (0.01)	4 (0.01)
2^{-4}	2065	89 (0.27)	53 (0.13)	39 (0.09)	7 (0.02)	3 (0.02)
2^{-5}	8225	104 (1.26)	54 (0.64)	36 (0.42)	7 (0.16)	3 (0.12)
2^{-6}	32833	113 (7.81)	57 (3.70)	31 (2.02)	7 (0.93)	3 (0.71)
2^{-7}	131201	119 (40.3)	61 (19.9)	26 (8.70)	7 (4.63)	3 (4.04)
2^{-8}	524545	121 (288)	60 (126)	18 (36.2)	7 (28.2)	3 (26.0)

Inexact Preconditioners

For large-scale 3D problems, exact preconditioners based on factorizations are not economical

How to replace the exact block solves?

Consider

$$\mathcal{P}_+^{-1} = \begin{bmatrix} A_{\Omega_2}^{-1} & 0 & 0 \\ 0 & A_{\Omega_1}^{-1} & 0 \\ 0 & 0 & M_p^{-1} \end{bmatrix}$$

- Replace the operators $A_{\Omega_i}^{-1}$ with fast, spectrally equivalent multigrid methods (AMG).
- Replace M_p with $\text{diag}(M_p)$.

3D Coupled Flow Problems

Consider 2 coupled flow problems

- channel driven flow
- flow with an impermeable enclosure

For these problems

- use the finite element library deal.II
- X_1^h, Q_1^h discretized with Taylor-Hood elements
- Q_2^h discretized with quadratic elements
- use inexact versions of the preconditioners (calls to AMG)
- solve the linear system with GMRES

Inexact Preconditioners

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How to replace the exact block solves?

Consider

$$\mathcal{P}_+^{-1} = \begin{bmatrix} A_{\Omega_2}^{-1} & 0 & 0 \\ 0 & A_{\Omega_1}^{-1} & 0 \\ 0 & 0 & M_p^{-1} \end{bmatrix}$$

- Replace the operators $A_{\Omega_i}^{-1}$ with fast, spectrally equivalent multigrid methods (AMG).
- Replace M_p with $\mathrm{diag}(M_p)$.

Inexact Constraint Preconditioner

Block Factorized Preconditioner

$$\mathcal{P}_{con_D}^{-1} = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & BA_{\Omega_1}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{\Omega_2} & 0 & 0 \\ 0 & A_{\Omega_1} & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & A_{\Omega_1}^{-1}B^T \\ 0 & 0 & I \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} I & 0 & 0 \\ 0 & I & -A_{\Omega_1}^{-1}B^T \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{\Omega_2}^{-1} & 0 & 0 \\ 0 & A_{\Omega_1}^{-1} & 0 \\ 0 & 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -BA_{\Omega_1}^{-1} & I \end{bmatrix}$$

Inexact Inner Solves

Replace

- $A_{\Omega_2}^{-1}, A_{\Omega_1}^{-1}$ with an AMG method
- $S = -BA_{\Omega_1}B^T$ with spectrally equivalent diag$(-M_p)$

[H. Elman, D. Silvester, and A. Wathen, 2005]

3D: Channel driven flow

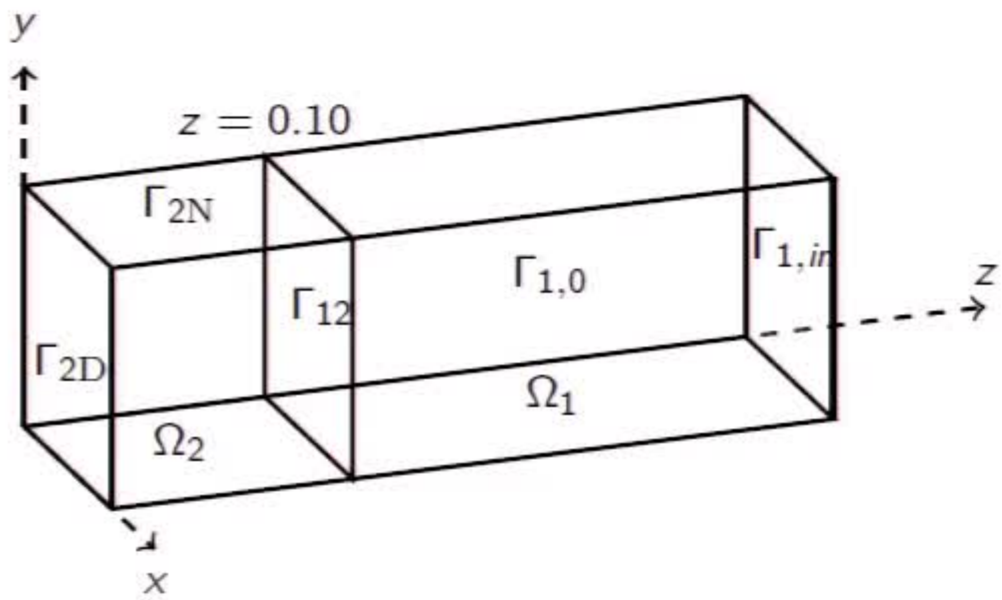
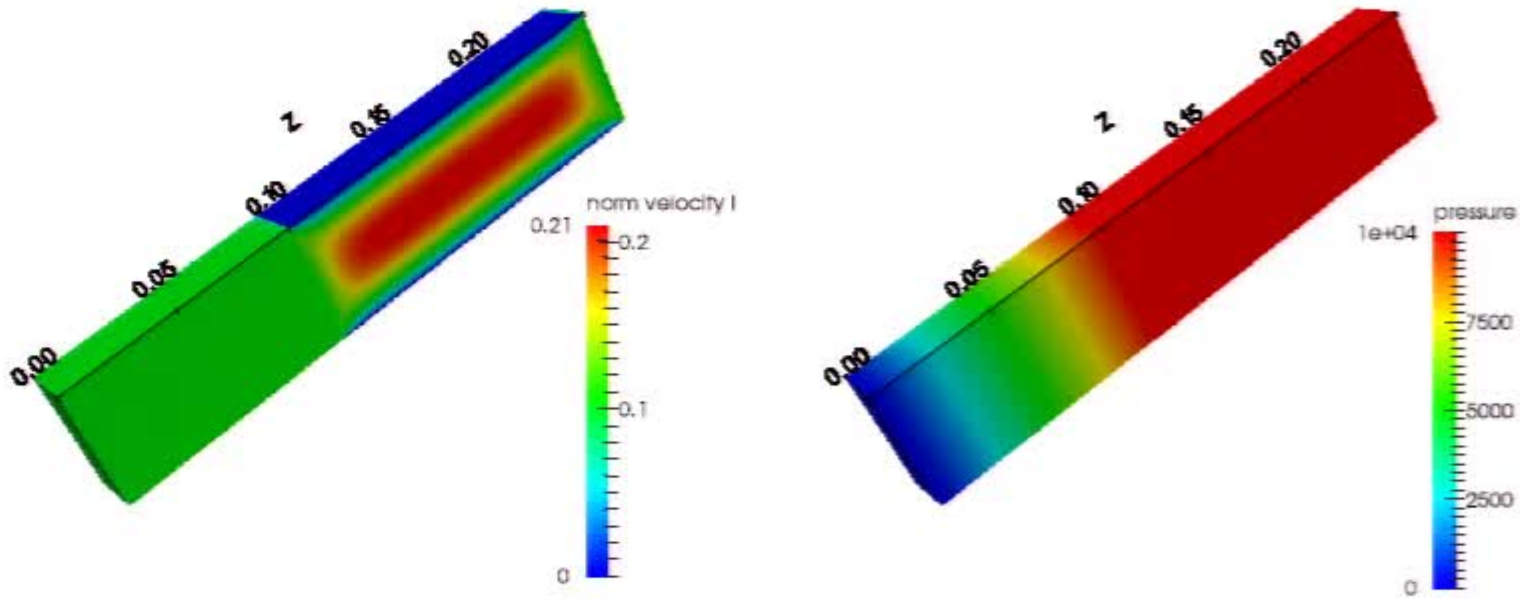


Figure: Computational domain

Computational setup

- $\Omega_1 = [0, 0.05]^2 \times [0.1, 0.25]$, $\Omega_2 = [0, 0.05]^2 \times [0, 0.1]$, and Γ_{12} is the plane $z = 0.10$
- $\mathbf{u}_1 = (0, 0, 0)^T$ on $\Gamma_{1,0}$
- $\mathbf{u}_1 = (0, 0, -0.1)^T$ on $\Gamma_{1,in}$
- $g_\mathrm{D} = g_\mathrm{N} = 0$
- $\mathbf{f}_1 = 0$, $f_2 = 0$ and $\nu = 1.0$
- κ is varied over the range $\{10^{-2}, 10^{-4}, 10^{-6}\}$

Channel driven flow (cont)



(a) Norm of velocity (b) Pressure

Figure: Slice of the numerical solution along $x = 0.025$ with $\kappa = 10^{-6}$, $h = 0.00025$, dofs $= 773625$

Channel driven flow (cont)

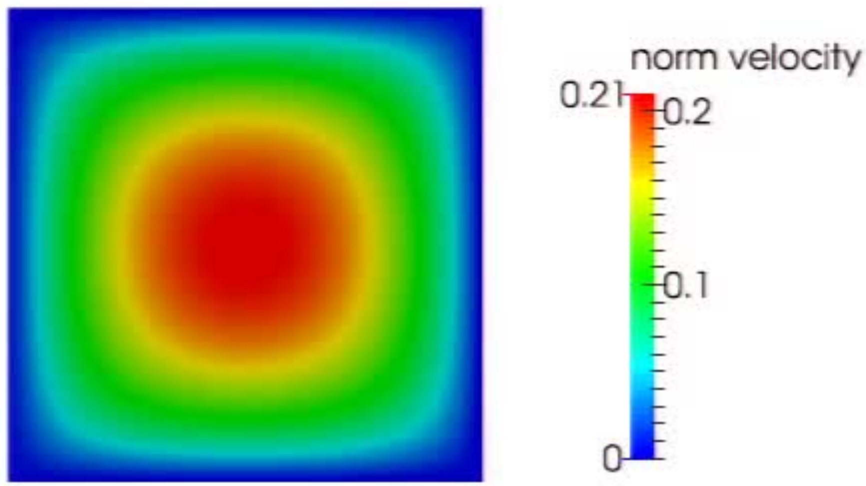


Figure: Cross section of channel flow along $z = 0.02$, $h = 0.00025$, DOFs $= 773625$

Channel driven flow (cont)

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
0.01	625	14370	154 (6.679)	58 (2.500)	47 (3.527)
0.005	5000	102535	227 (81.66)	83 (29.51)	69 (42.65)
0.00025	40000	773265	442 (1389)	139 (408.9)	117 (595.8)

(a) $\kappa = 10^{-2}$, 1 AMG cycle

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
0.01	625	14370	178 (7.756)	70 (3.013)	57 (4.260)
0.005	5000	102535	237 (85.59)	94 (33.44)	77 (47.67)
0.00025	40000	773265	443 (1394)	136 (399.7)	112 (570.3)

(b) $\kappa = 10^{-4}$, 1 AMG cycle

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
0.01	625	14370	369 (16.97)	171 (7.517)	158 (11.86)
0.005	5000	102535	558 (218.0)	229 (83.56)	189 (117.7)
0.00025	40000	773265	895 (3152)	270 (821.8)	240 (1243)

(c) $\kappa = 10^{-6}$, 1 AMG cycle

Table: GMRES iterations and CPU time (seconds)

3D: Impermeable enclosure

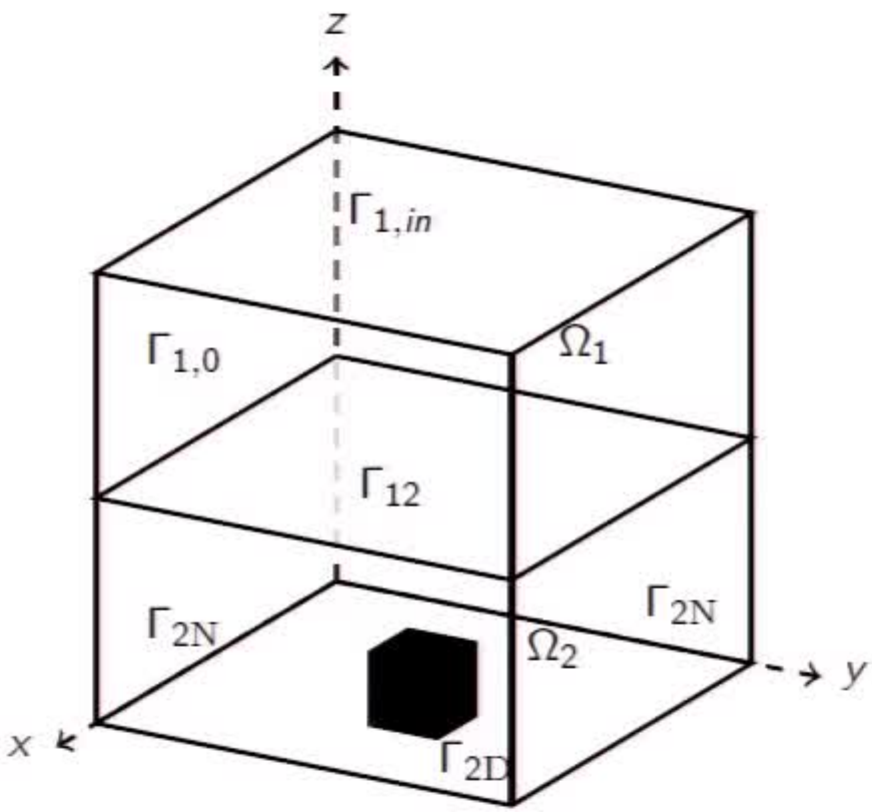


Figure: Computational domain with impermeable enclosure

Computational setup

- $\Omega_1 = [0,2]^2 \times [1,2]$, $\Omega_2 = [0,2]^2 \times [0,1]$, and Γ_{12} is the plane $z = 1$
- $\mathbf{u}_1 = (0,0,0)^T$ on $\Gamma_{1,0}$
- $\mathbf{u}_1 = (0,0,-1)^T$ on $\Gamma_{1,in}$
- $g_{\mathrm{D}} = g_{\mathrm{N}} = 0$
- $\mathbf{f}_1 = 0$, $f_2 = 0$ and $\nu = 1.0$
- $\kappa_1 = 1$ (outside black cube), $\kappa_2 = 10^{-10}$ (inside)

Impermeable enclosure (cont)

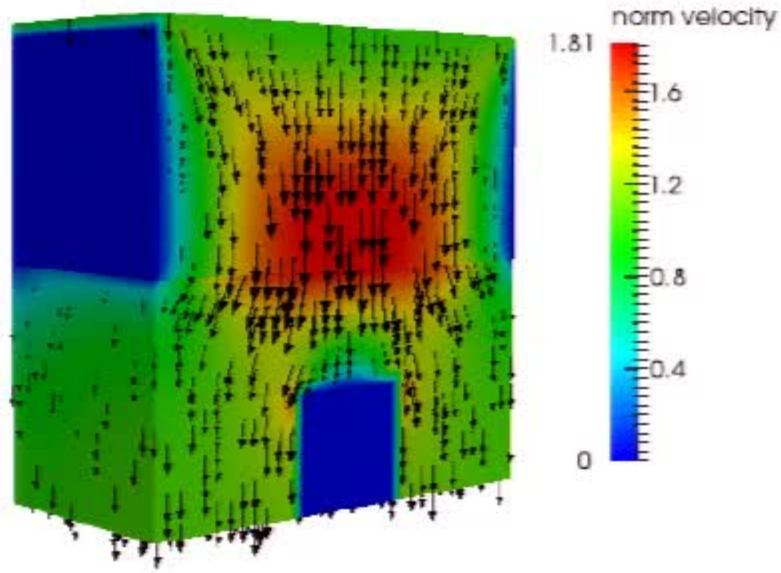


Figure: Numerical solution, $h = 2^{-4}$, DOFs $= 576213$

Impermeable enclosure (cont)

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
2^{-1}	64	1695	59 (0.312)	34 (0.188)	30 (0.276)
2^{-2}	512	10809	153 (4.638)	54 (1.637)	46 (2.386)
2^{-3}	4096	76653	250 (63.54)	64 (15.82)	56 (23.97)
2^{-4}	32768	576213	485 (1083)	116 (239.2)	99 (346.7)

(a) $\kappa_1 = 1$, $\kappa_2 = 10^{-10}$, 1 AMG cycles

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
2^{-1}	64	1695	59 (0.927)	34 (0.548)	30 (0.907)
2^{-2}	512	10809	97 (10.70)	41 (4.526)	37 (7.920)
2^{-3}	4096	76653	144 (133.3)	52 (48.48)	43 (76.80)
2^{-4}	32768	576213	267 (2027)	69 (526.2)	55 (800.7)

(b) $\kappa_1 = 1$, $\kappa_2 = 10^{-10}$, 4 AMG cycles

h	elements	DOF	$\mathcal{P}_+$	$\mathcal{P}_{T_1}$	$\mathcal{P}_{con_D}$
2^{-1}	64	1695	59 (1.763)	34 (1.044)	30 (1.767)
2^{-2}	512	10809	86 (18.82)	40 (8.894)	36 (15.81)
2^{-3}	4096	76653	116 (213.1)	49 (90.91)	38 (137.2)
2^{-4}	32768	576213	200 (3022)	61 (931.8)	44 (1307)

(c) $\kappa_1 = 1$, $\kappa_2 = 10^{-10}$, 8 AMG cycles

Table: GMRES iterations and CPU time (seconds)