Lots of recent progress: computational theorems in dynamics

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and more

Can you trust a computer?

Possible reponses:

- Can you trust a human brain?
- Quantitatively: solve $Ax = b$
  trustworthiness of x depends on condition number of A
- Need to make sure our problem is well-conditioned

prove!

quantify!

Interval arithmetic

- Computation with floating point numbers have rounding errors
- These errors can be dealt with rigorously by interval arithmetic
- Product of intervals I_1 and I_2

$$I_1 \cdot I_2 \supset \{x_1 x_2 \mid x_1 \in I_1, x_2 \in I_2\}$$

Intlab by Rump

Rigorous validation of numerical computations

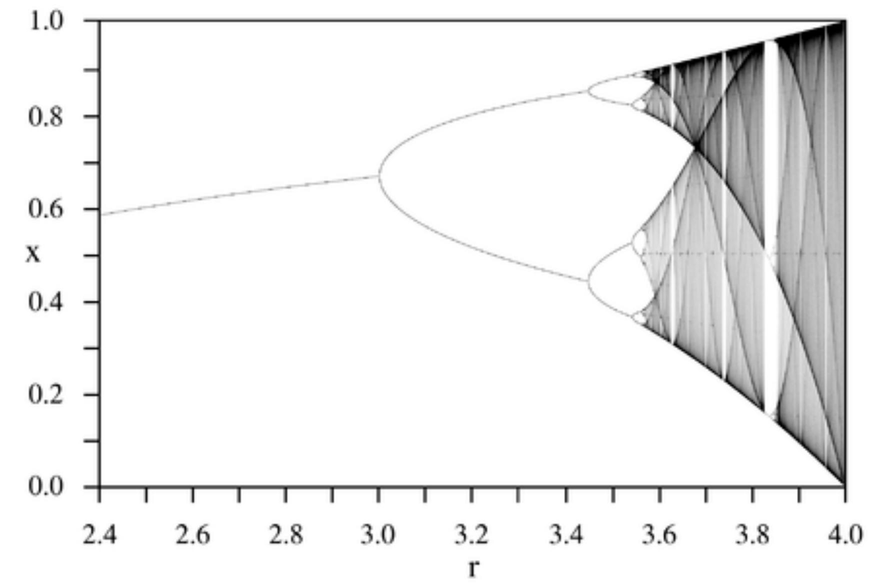
Goal: prove rigorously what we see in simulations of nonlinear dynamics

"important solutions"

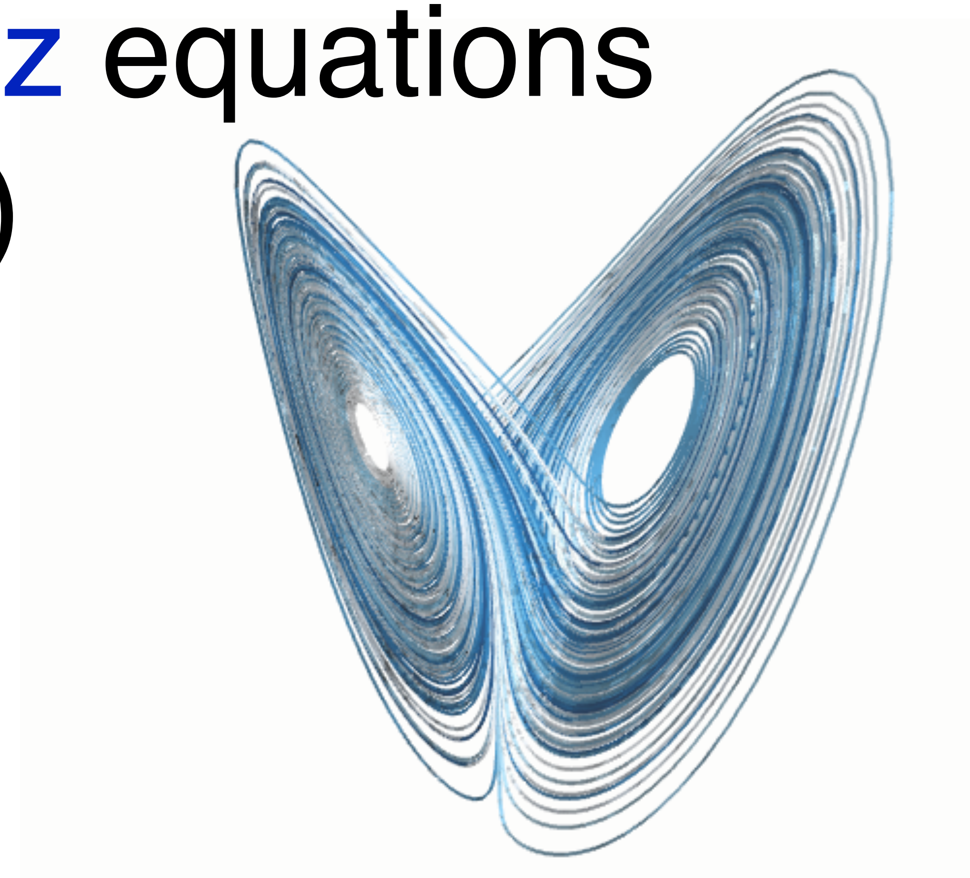
$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Computer proofs in dynamics

Feigenbaum constant is universal
(Lanford, 1982)



Chaotic attractor in Lorenz equations
(Tucker, 2002)



Some acknowledgments

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  CAPD library; rigorous integrator for ODEs
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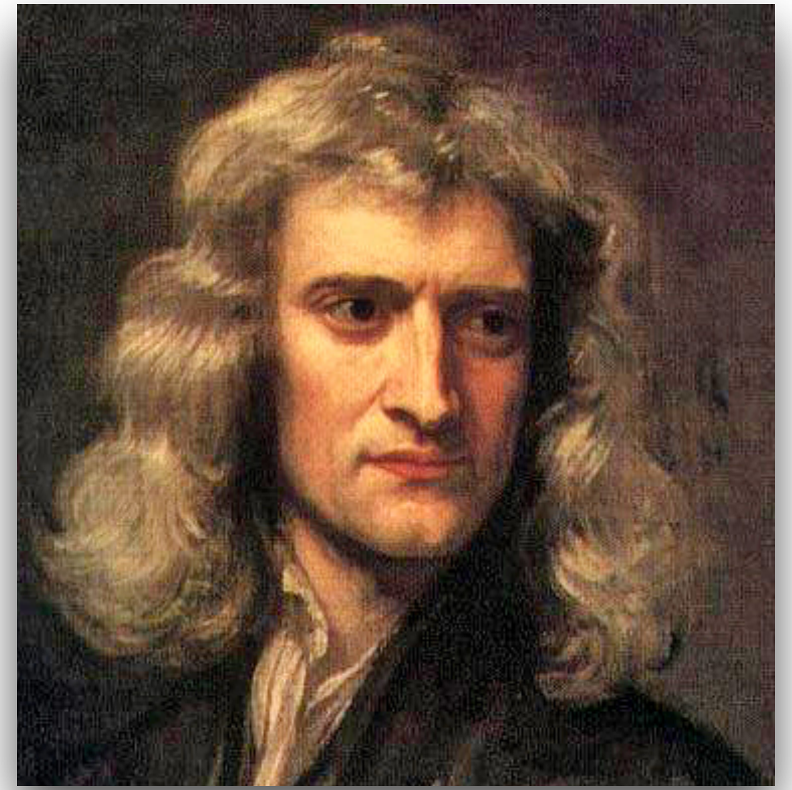
  CHOMP; Morse-Conley database
- Hans Koch
- Gianni Arioli
- Michael Plum
- Mitsuhiro Nakao
- Nobito Yamamoto
- ……

$$u' = f(u)$$

General Setup



$$F(x) = 0$$

General Setup

Functional analytic setup

Example: IVP $\begin{cases} u' = f(u) \\ u(0) = u_0 \end{cases}$

$$F(u) = \begin{bmatrix} u' - f(u) \\ u(0) - u_0 \end{bmatrix} = 0$$

$$F : C^1 \to C^0 \times \mathbb{R}$$

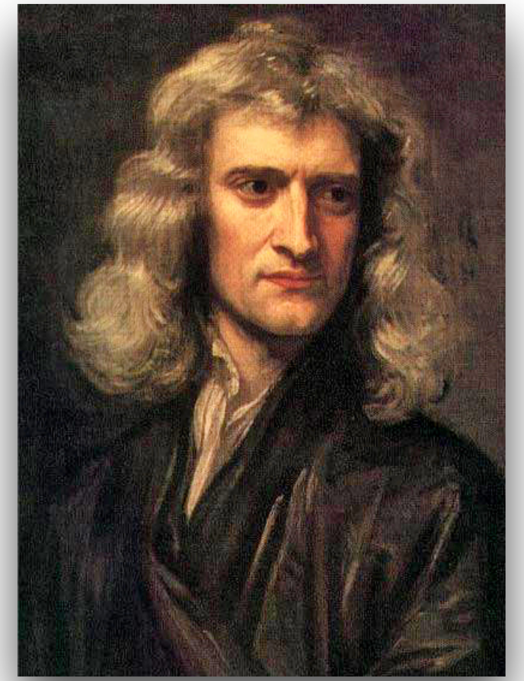
or

$$\widetilde{F}(u) = u(t) - u_0 - \int_0^t f(u(s))ds = 0$$

$$\widetilde{F} : C^0 \to C^0$$

Newton's method

finite dimensional nonlinear problem



$$F(x) = 0$$

$$x_{n+1} = \widetilde{T}(x_n)$$
$$= x_n - DF(x_n)^{-1} \cdot F(x_n)$$

$$F(\hat{x}) = 0 \iff \widetilde{T}(\hat{x}) = \hat{x}$$

$DF(\hat{x})^{-1}$ injective

$D\widetilde{T}(\hat{x}) = 0 \longrightarrow \|D\widetilde{T}(x)\|$ small near $\hat{x}$

$\widetilde{T}$ is a (strong) contraction

Infinite dimensions

infinite dimensional nonlinear problem

$$F(x) = 0$$

$F : X \to X'$ $\quad X, X'$ Banach spaces

Finite dimensional reduction

N-dimensional subspaces

$$X_N \subset X \qquad X'_N \subset X'$$

- truncated problem $\quad F_N : X_N \to X'_N$
- solve numerically $\quad F_N(\overline{x}_N) \approx 0$ 
- numerical "solution" $\quad \overline{x}_N = \overline{x} \in X_N \subset X$

Fixed point operator

$$F(x) = 0 \qquad F : X \to X'$$

$$T(x) = x \qquad T : X \to X$$

$$T(x) = x - DF(x)^{-1}F(x)$$

$$T(x) = x - DF(\overline{x})^{-1}F(x)$$

$$T(x) = x - AF(x)$$

$A : X' \to X$ injective

$A \approx A_N = DF_N(\overline{x}_N)^{-1}$

A "easy" (for estimates)

Contraction mapping

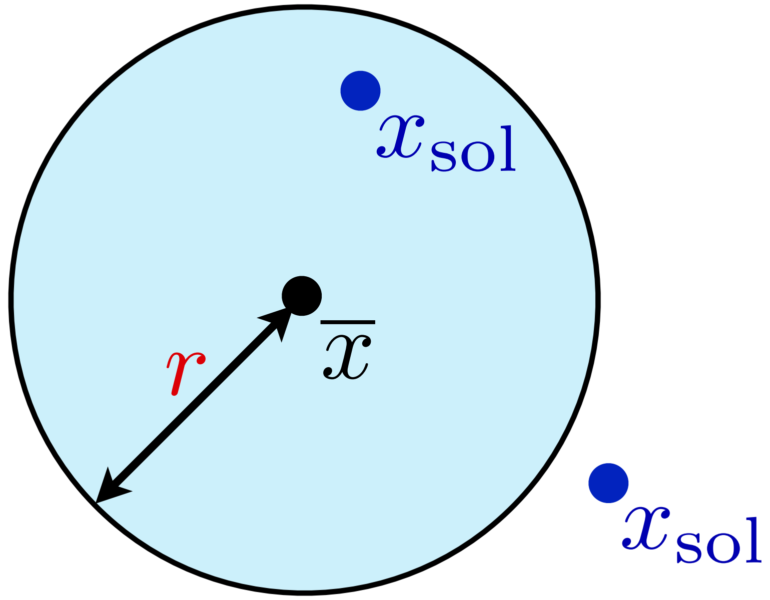
- T maps $B_r(\overline{x}) \subset X$ into itself
- $\|T(x) - T(\tilde{x})\|_X \leq \kappa \|x - \tilde{x}\|_X \qquad \kappa < 1$

Analytic estimates

$$\|T(\overline{x}) - \overline{x}\|_X \leq Y$$

$$\|DT(x)\|_{B(X)} \leq Z(r) \qquad \forall x \in B_r(\overline{x})$$

Inequality $Y + \hat{r} Z(\hat{r}) < \hat{r}$



Choices

Choices to be made:

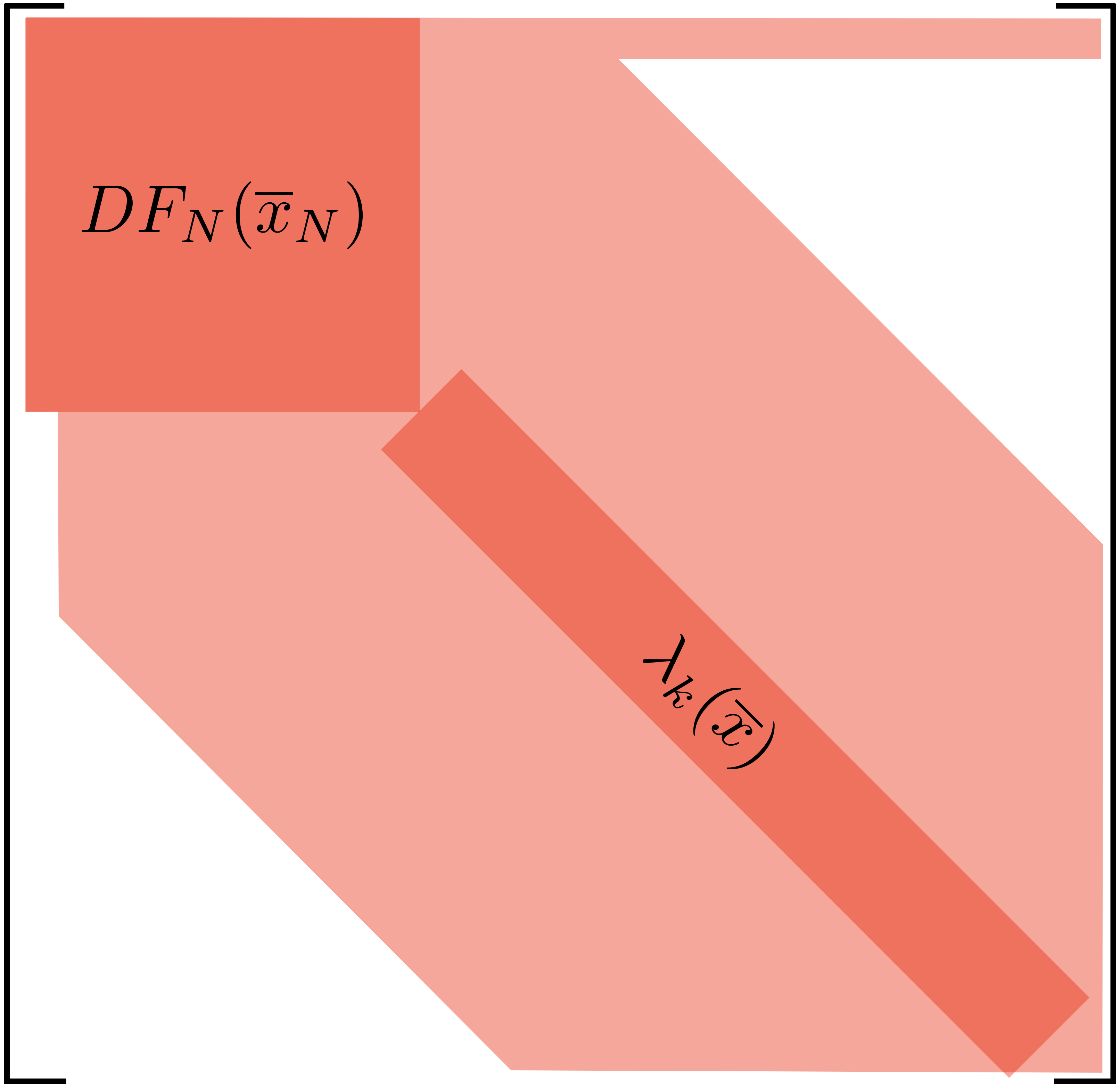
- formulation $F(x) = 0$
- X and the norm on X
- X_N and X'_N
- A: an "accurate" and "simple" approximation of $DF(\overline{x})^{-1}$

Estimates and check $Y + \hat{r}Z(\hat{r}) - \hat{r} < 0$

What is A?

$$T(x) = x - AF(x)$$

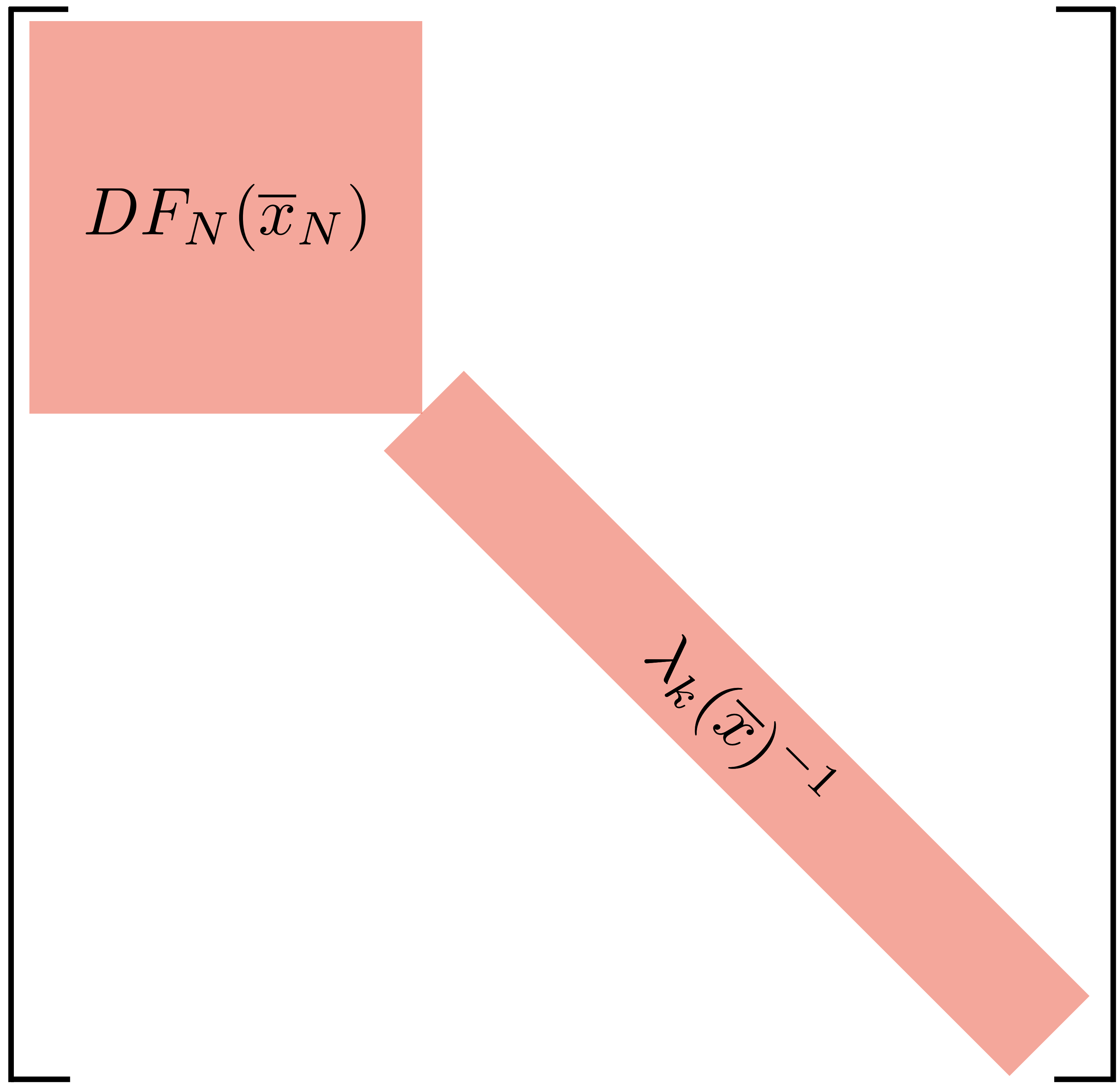
$$DF(\overline{x}) \approx$$

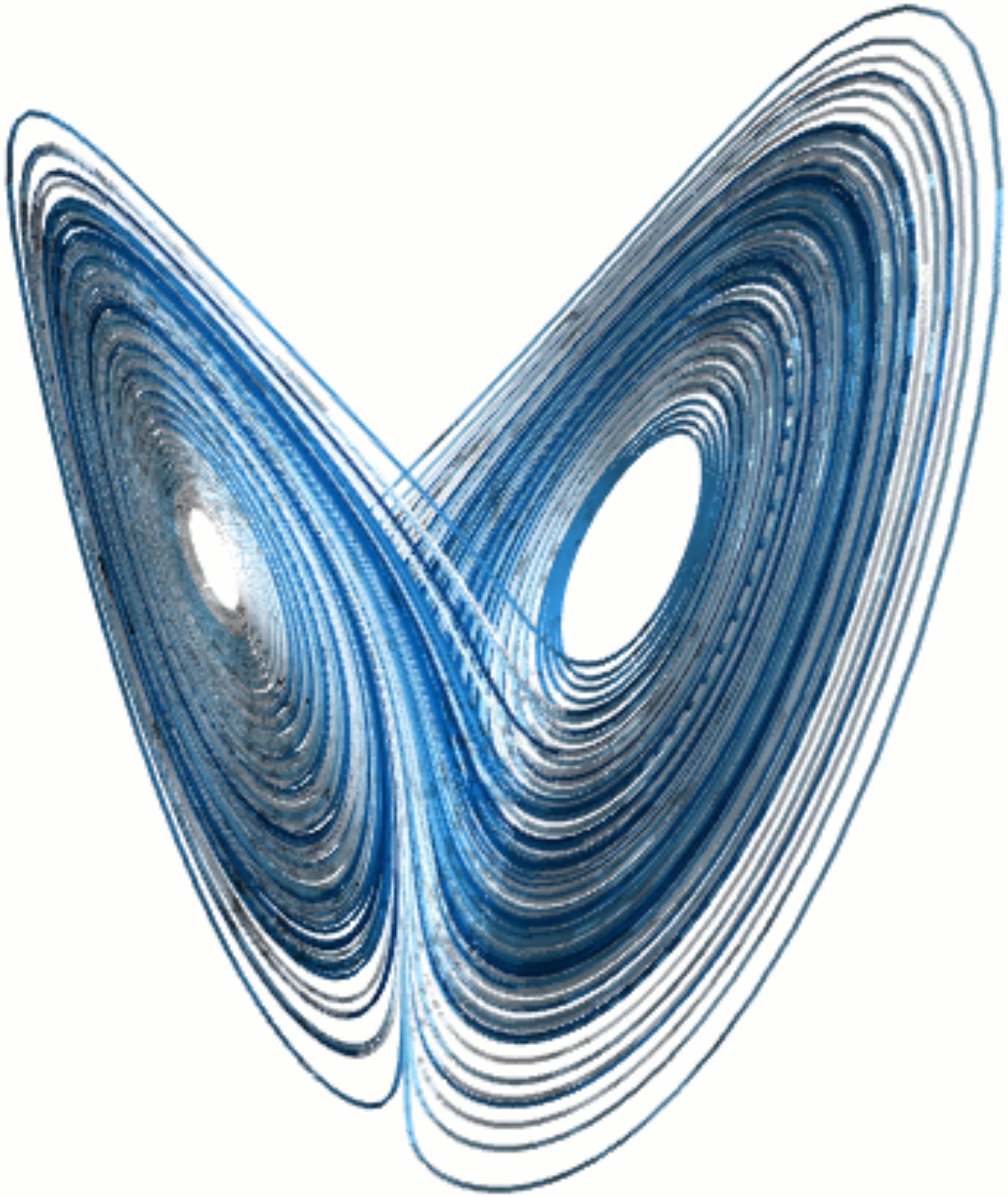


What is A?

$$T(x) = x - AF(x)$$

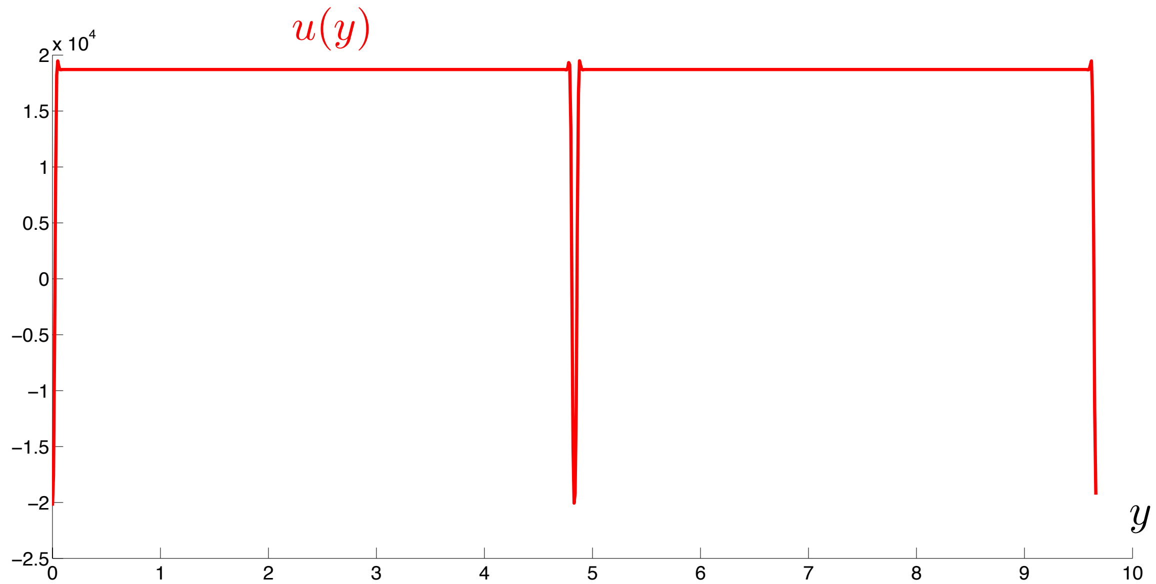
$$A =$$





Examples

Periodic solutions in ODEs



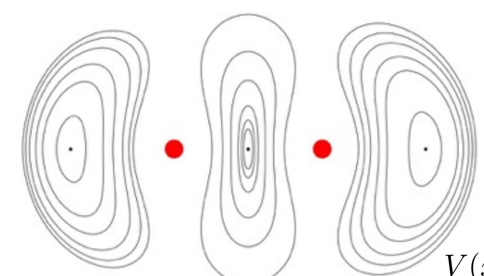
$$-u_{xxxx} - 2u_{xx} + (\lambda - 1)u - u^3 = 0$$

$$\lambda = 3.5 \times 10^8$$

2103 Fourier modes

Hungria-Lessard-Mireles

- any polynomial vector field Queirolo MS25
- non-polynomial:
  - reformulation Lessard-Mireles-Ransford
  - interpolation estimates Kalies-Day Figueras-Haro-Luque
- less regularity Lessard-Mireles



$$\begin{cases} x'' = 2y' + \dfrac{\partial V}{\partial x} \\ y'' = -2x' + \dfrac{\partial V}{\partial y}, \end{cases}$$

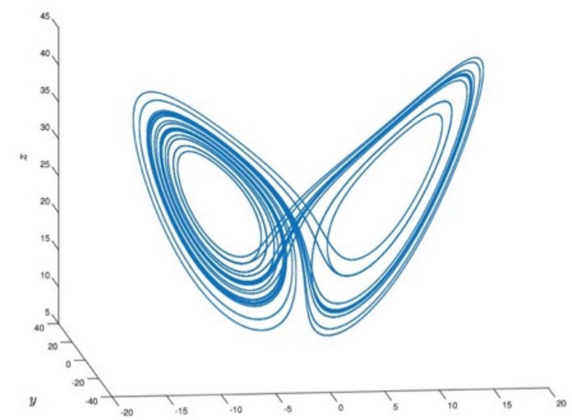
PCR3BP

$$V(x,y) \stackrel{\text{def}}{=} \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1(x,y)} + \frac{\mu}{r_2(x,y)}$$

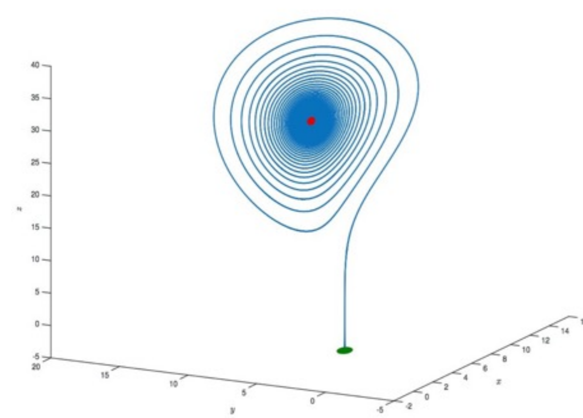
$$\begin{cases} r_1(x,y) \stackrel{\text{def}}{=} \sqrt{(x+\mu)^2+y^2} \\ r_2(x,y) \stackrel{\text{def}}{=} \sqrt{(x-1+\mu)^2+y^2} \end{cases}$$

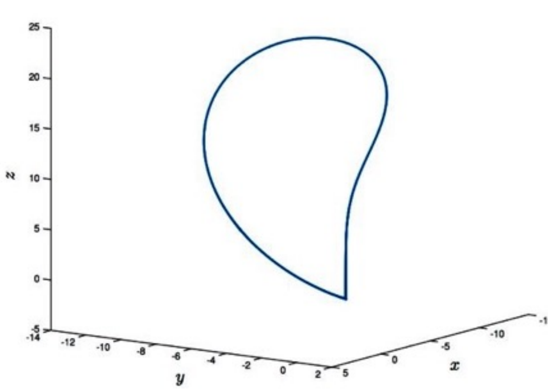
Non-periodic solutions in ODEs

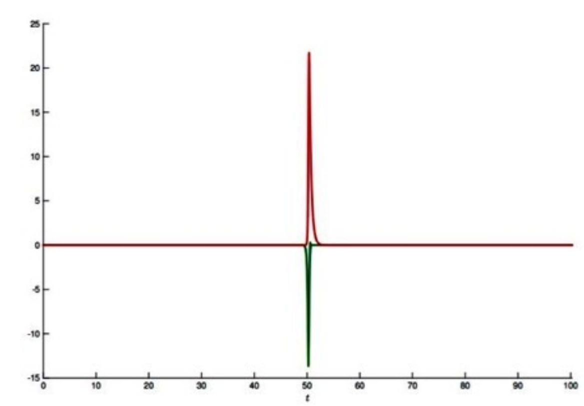
- Chebyshev series - domain decomposition
  - IVPs
  - BVPs



Sheombarsing-JB

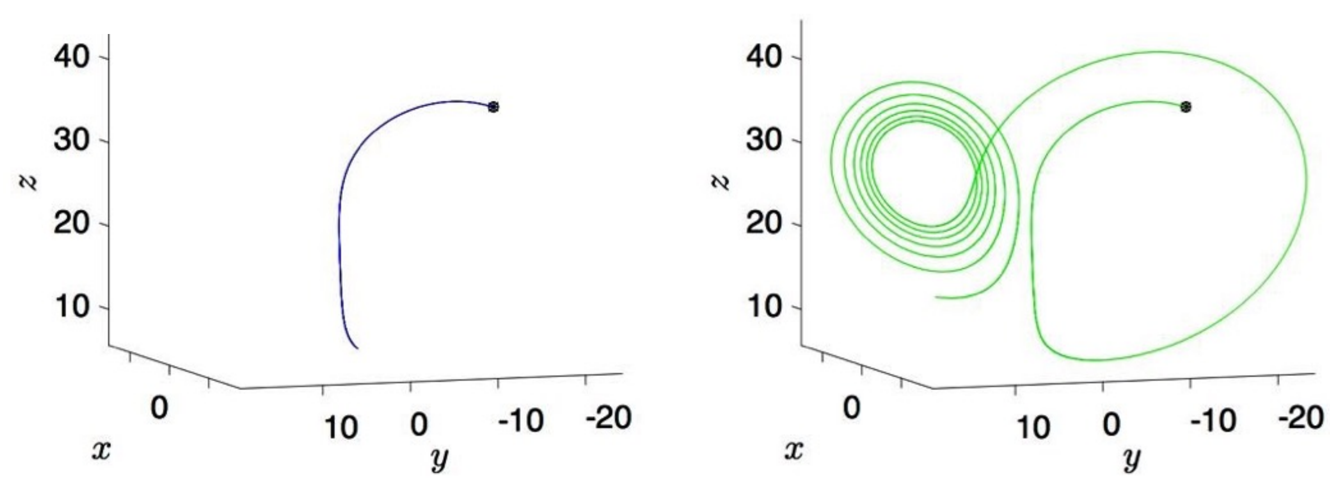






- a priori bootstrap

$$u' = f(u)$$
$$u'' = Df(u) \cdot f(u)$$



Breden-Lessard

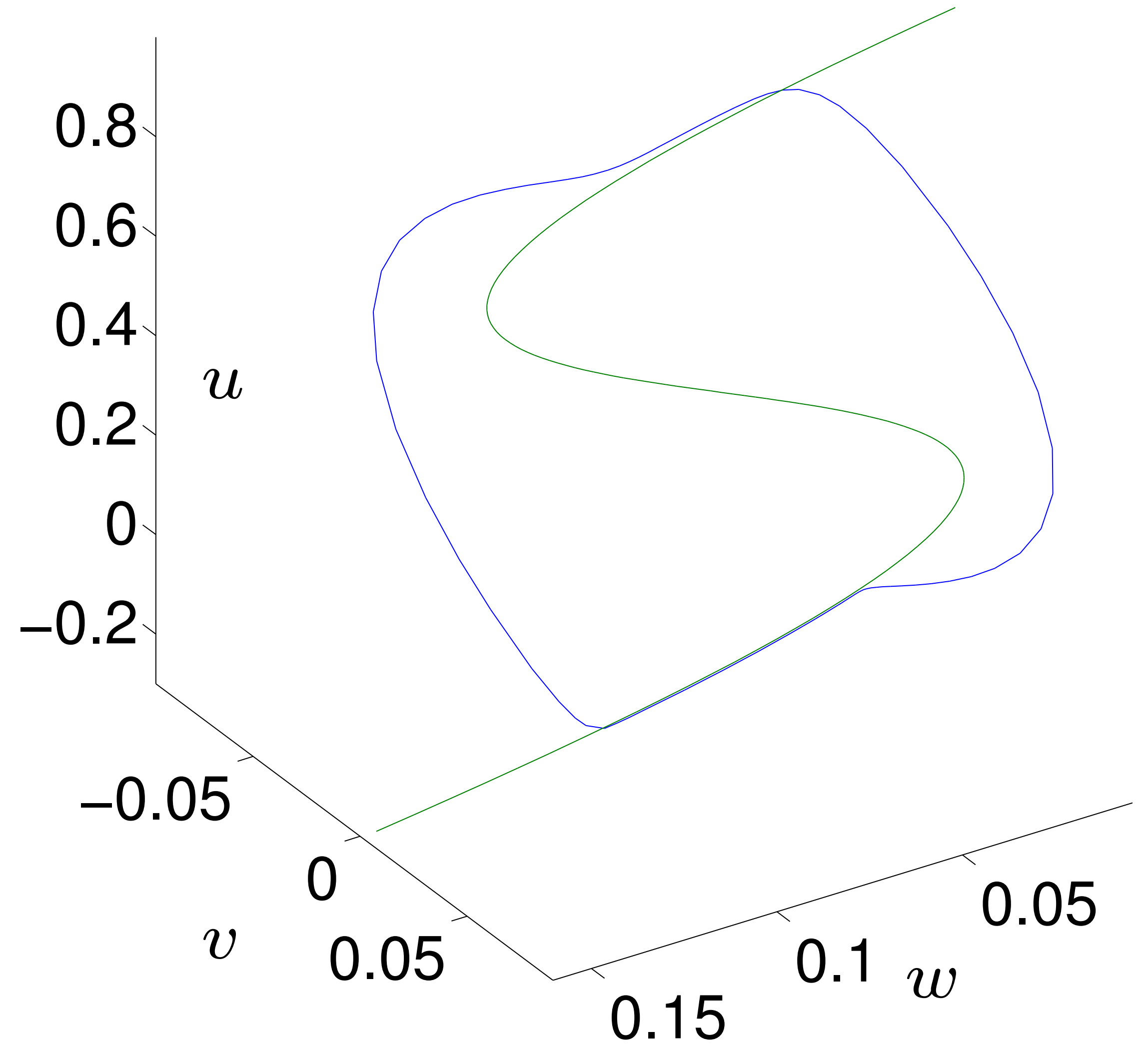
More ODEs

fast-slow systems

phase space methods

$$u' = v,$$
$$v' = \gamma(\theta v - u(u-a)(1-u) + w),$$
$$w' = \frac{\epsilon}{\theta}(u - w).$$

$$\epsilon \in (0, 1.5 \times 10^{-4}]$$



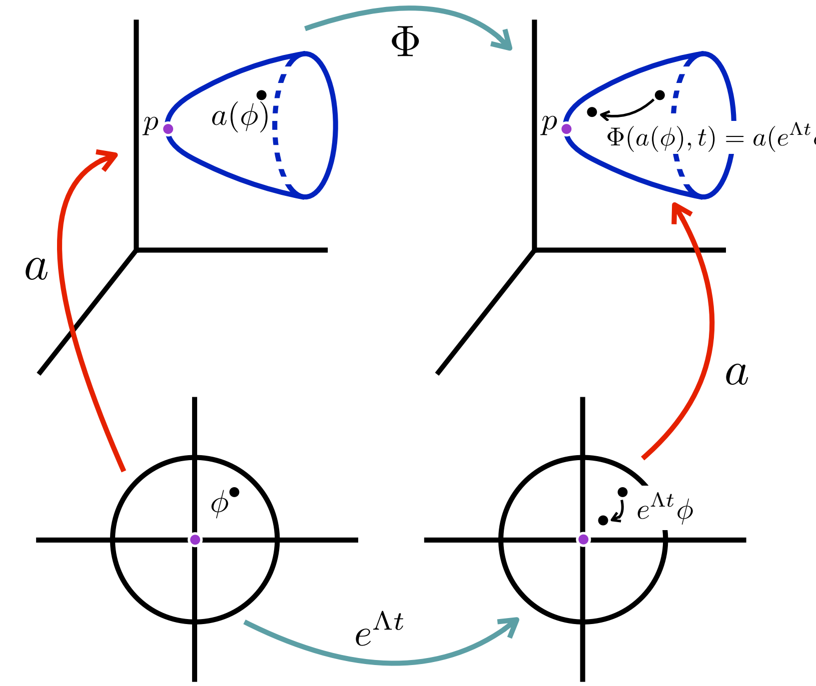
Czechowski-Zgliczynski
Matsue

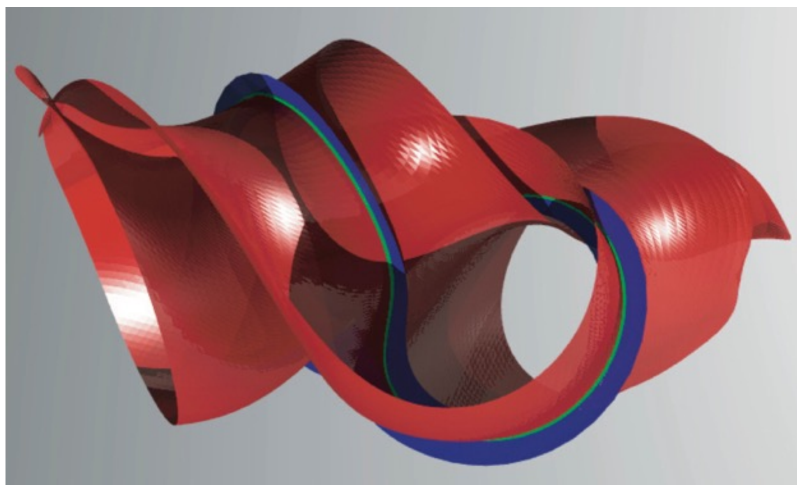
higher dimensional slow manifolds

Connecting orbits

local stable/unstable manifolds

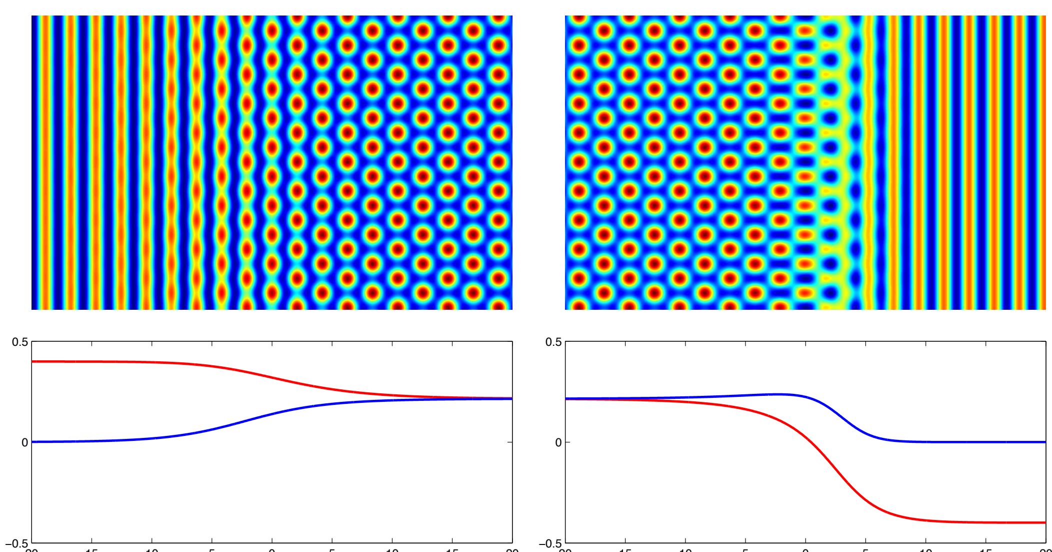
Parametrization method



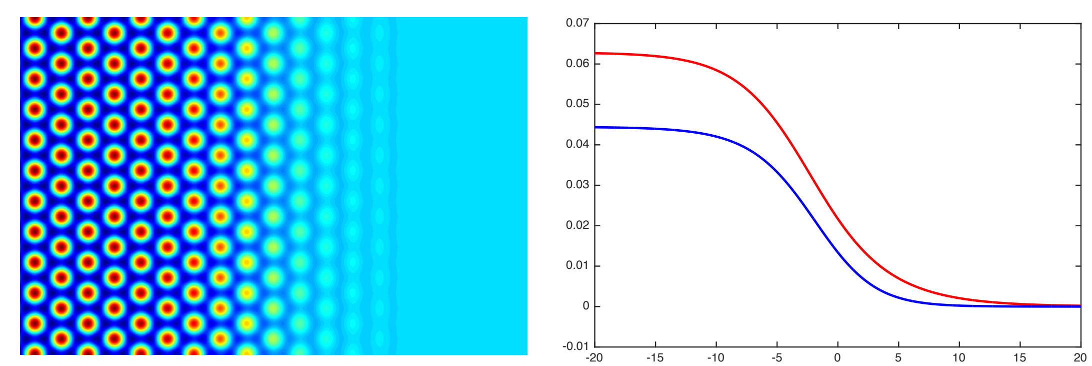


Mireles

Haro-Canadell-Figueras-Luque-Mondelo



$$\begin{cases} u'' = -\frac{1}{4}\gamma u - \frac{\sqrt{2}}{4}v^2 + \frac{3}{8}u^3 + 3uv^2, \\ v'' = -\gamma v - \frac{\sqrt{2}}{2}uv + 9v^3 + 3u^2 v, \end{cases}$$



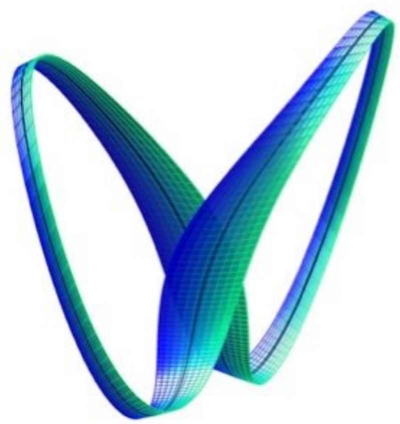
Deschênes-Lessard-Mireles-JB

Mireles-Reinhardt-JB

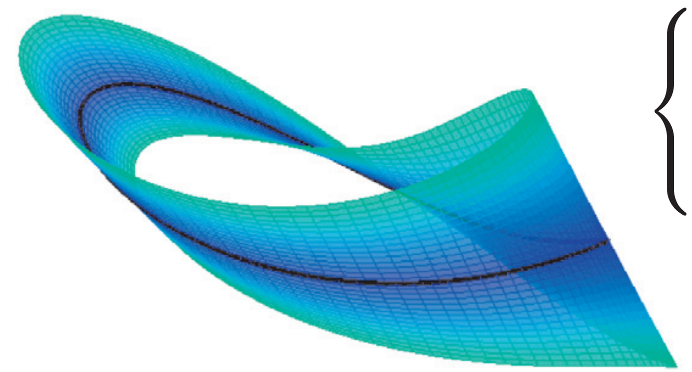
continuation through eigenvalue resonances

Local (un)stable manifolds

- of periodic orbits Castelli-Mireles-Lessard



Lorentz



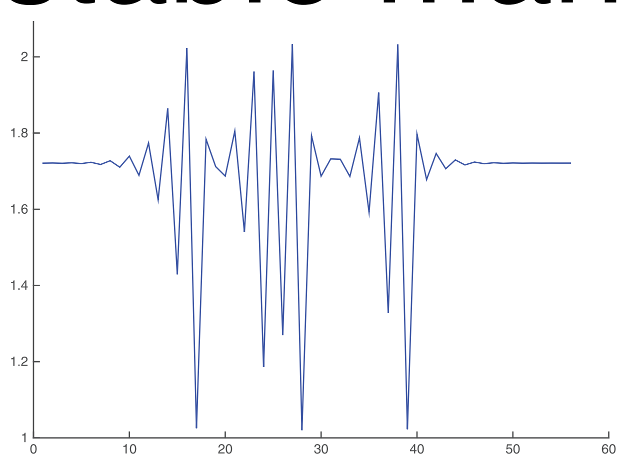
$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = \alpha x - x^2 - \beta y - z. \end{cases}$$

Arneodo

- unstable manifolds in parabolic PDEs

Mireles-Reinhardt

- stable manifolds maps in infinite dimensions



de la Llave-Mireles

Kot-Schaffer $\mathfrak{F}[u](x) := \frac{1}{\pi}\int_0^\pi K(x-y)N[u](y)\,dy$

infinite dimensional stable manifolds in PDEs

Continuation

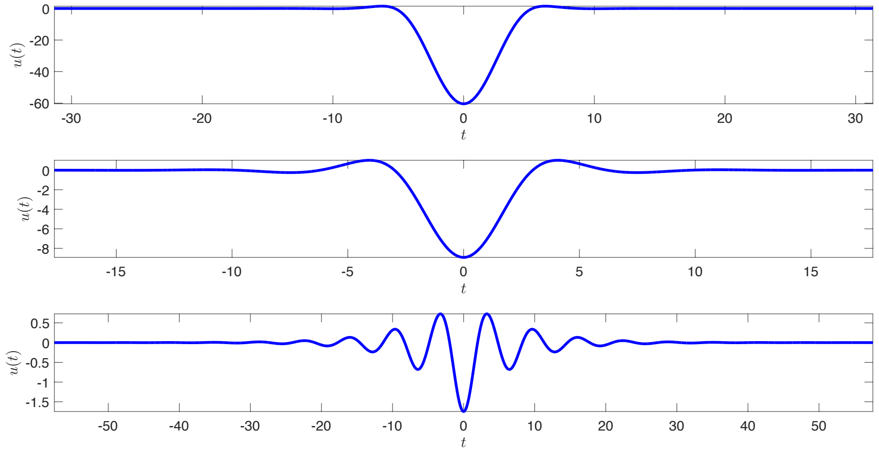
- continuation of general periodic orbits

Queirolo MS25

- continuation in 2D parameter space

Gameiro-Lessard-Pugliese

- continuation of connecting orbits



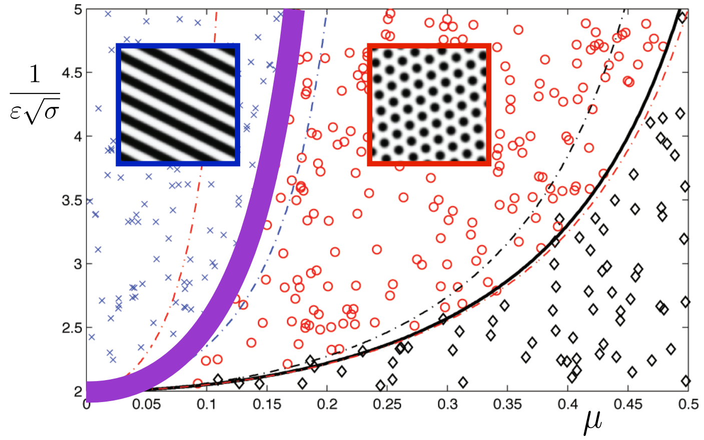
Breden-Lessard-Murray-JB

Sheombarsing-Reinhardt-JB

$$u'''' + c^2 u'' + e^u - 1 = 0$$

$$c^2 \in [0.5, 1.9]$$

- continuation in PDE



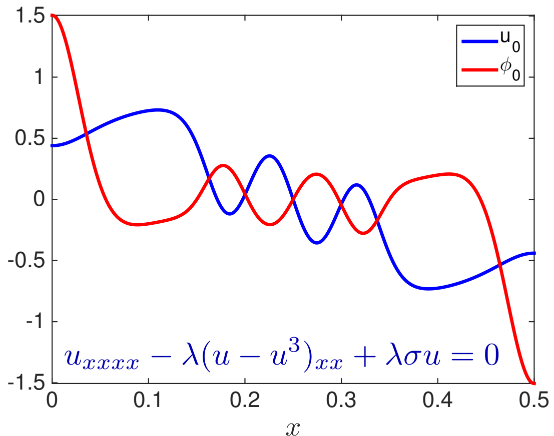
$$-\Delta\left(\varepsilon^2 \Delta u + u - u^3\right) - \sigma(u - \mu) = 0$$

Otha-Kawasaki

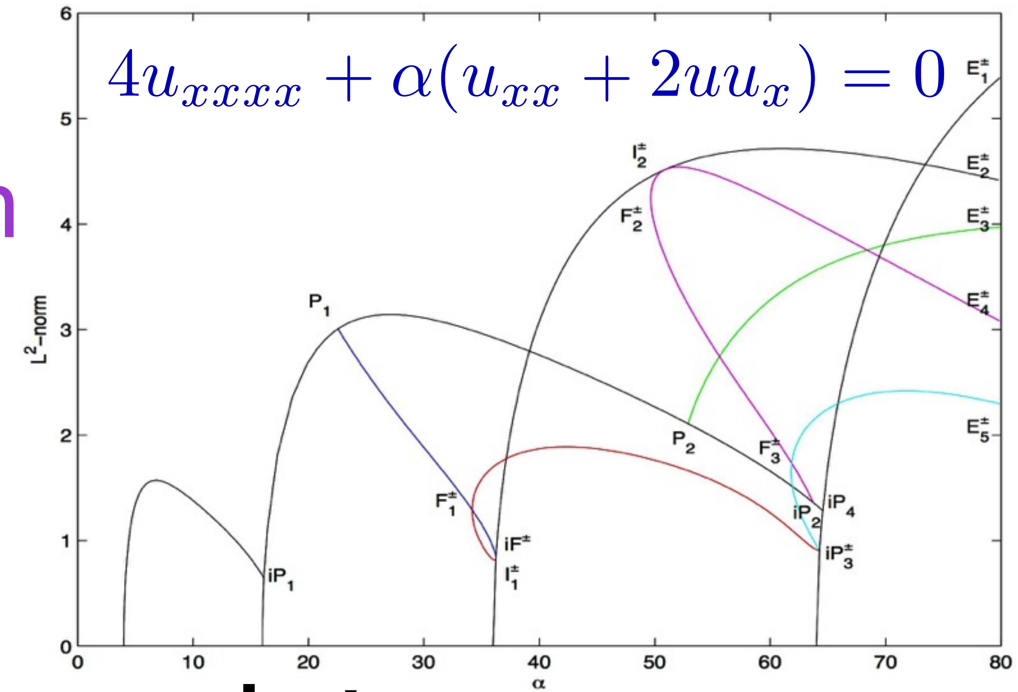
Williams-JB

Bifurcation

- saddle-node Sander-Wanner Sander MS25
- pitchfork

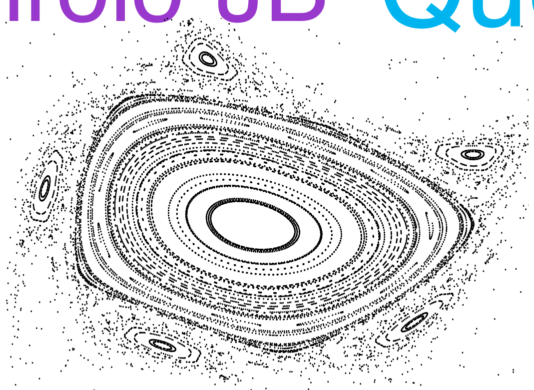


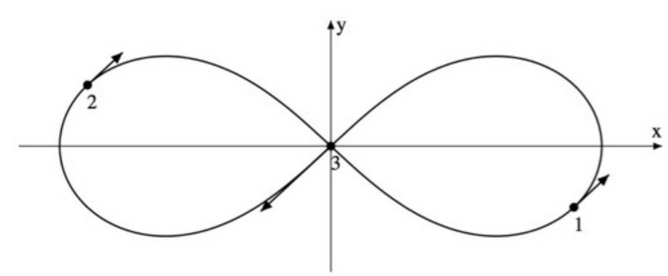
Arioli-Koch



Lessard-Sander-Wanner

- continuation of bifurcation points
- Hopf Lessard-Queirolo-JB Queirolo MS25
- KAM Kapela-Simo





center manifolds PP1: Hetebrij

bifurcations with many symmetries

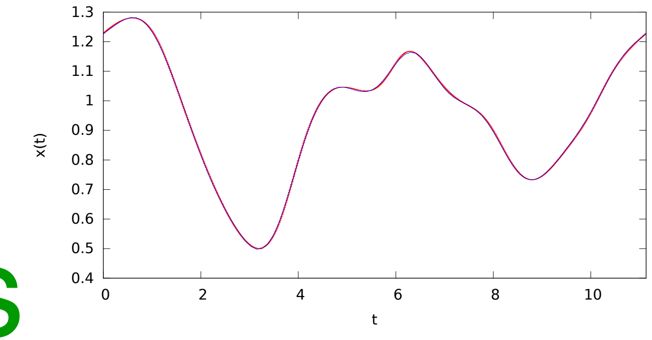
Delay equations

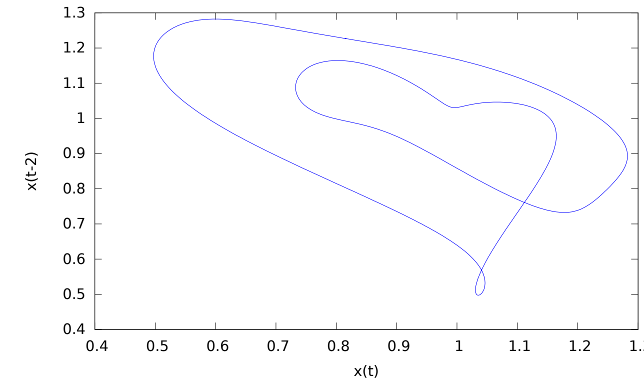
- periodic orbits

Szczelina-Zgliczynski
Groothedde-Lessard-JB

Mackey-Glass

$$\dot{x}(t) = \beta \cdot \frac{x(t-\tau)}{1+x^n(t-\tau)} - \gamma \cdot x(t)$$



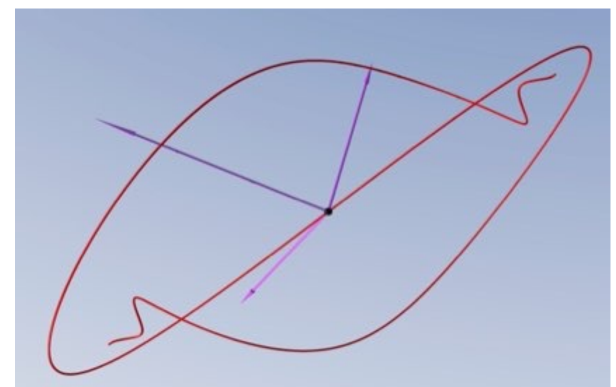


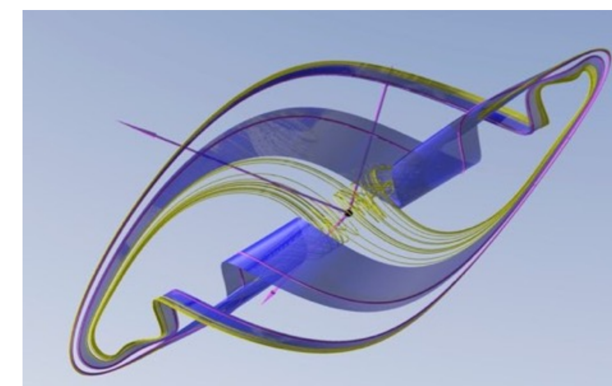
- unstable manifold

Groothedde-Mireles

Ikeda

$$u'(t) = u(t-\tau) - u(t-\tau)^3$$





Groothedde MS102

- structure of global attractor

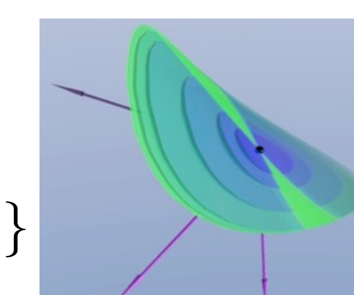
Wright

$$y'(t) = -\alpha y(t-1)\{1+y(t)\}$$

Jaquette-Lessard-Mischaikow
Jaquette-JB

Jaquette MS25

state dependent delay $y'(t) = -\alpha y(t-[1+\epsilon y(t)])\{1+y(t)\}$



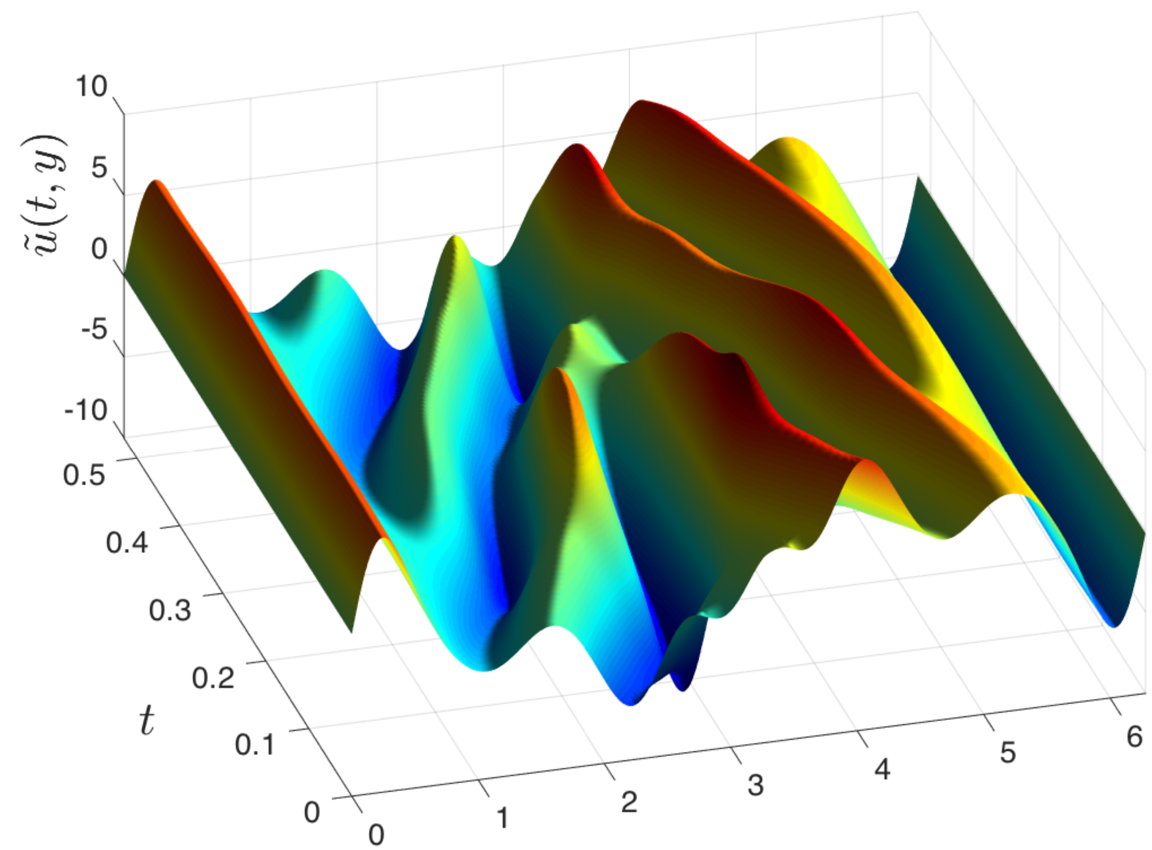
PDEs

- time-periodic solutions

Kuramoto-Sivashinsky

$$u_t = -\nu u_{yyyy} - u_{yy} + 2uu_y$$

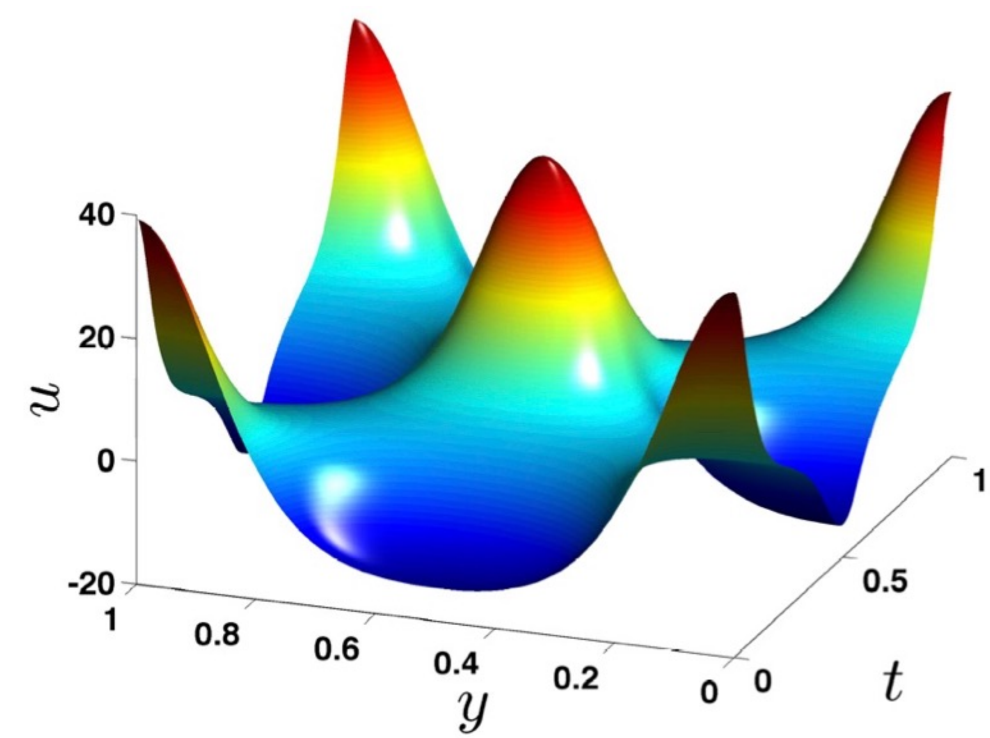
Arioli-Koch
Zgliczynski
Gameiro-Lessard
Figueras-de la Llave



Boussinesq

$$u_{tt} = \lambda u_{yyyy} + u_{yy} + (u^2)_{yy}$$

Castelli-Gameiro-Lessard



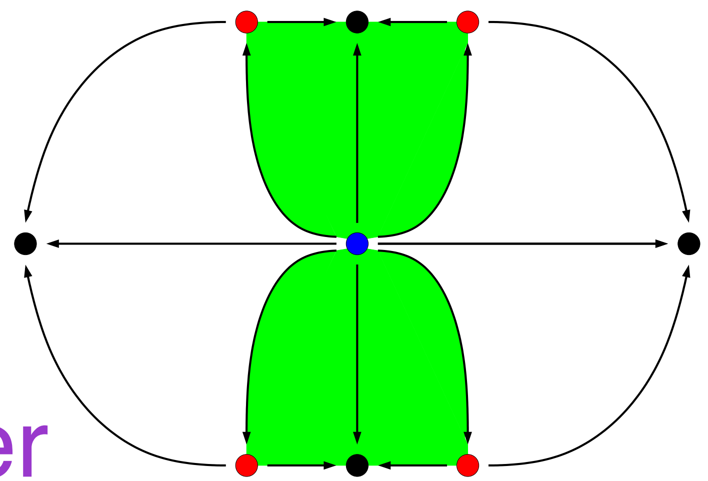
PDEs

- IVP/BVP

  Arioli-Koch
  Cyranka
  Breden-Lessard-Sheombarsing-JB

  Cyranka MS13

- connecting orbits



$$u_t = -(u_{xx} + \lambda u^3)_{xx} - \lambda\sigma(u - \mu)$$

Ohta-Kawasaki Cyranka-Wanner

- finite element methods Gonzalez PP1

general connecting orbits in PDEs

wave equations (Arioli-Koch)

Chaos

- short connections for periodic solutions
  Mireles-Murray
- hyperchaos Wilczak-Serrano-Barrio
- symbolic dynamics in PDE
  $u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x$ Wilczak-Zgliczynski
- near resonant inclination-flip bifurcation
  Fontaine-Kalies-Naudot
- general Conley index techniques
  Day MS78
- estimating topological entropy
  Frongillo MS25

Stability (PDEs)

- variational problems (gradient systems) including strongly indefinite
  Gameiro-Lessard-Vandervorst-JB
- instability of periodic orbits Gameiro-Lessard
- stability of periodic orbit Arioli-Koch
- Evans function techniques Barker MS116
  Arioli-Koch Barker-Zumbrun
- Lagrangian coherent structures & finite time Lyapunov exponents
  Kepley MS13

stability in non-variational PDEs

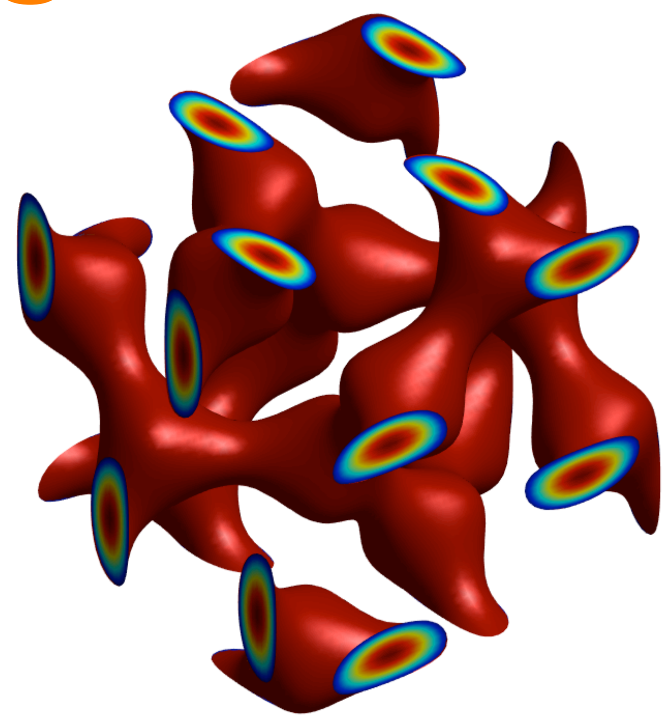
More dimensions

Ohta-Kawasaki

$$-\Delta(\epsilon^2 \Delta u + u - u^3) - \sigma(u - m) = 0$$

3D space

Williams-JB



symmetries

Navier-Stokes

$$\frac{\partial \vec{\mathbf{u}}}{\partial t} + \vec{\mathbf{u}} \cdot \nabla \vec{\mathbf{u}} - \nu \nabla^2 \vec{\mathbf{u}} + \nabla p = \vec{f}$$

$$\nabla \cdot \vec{\mathbf{u}} = 0$$

2D space x 1D time

Breden-Lessard-van Veen-JB

Lessard MS13 Breden MS60

Thanks!

Thank you

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JB van den Berg
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TODO