A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem

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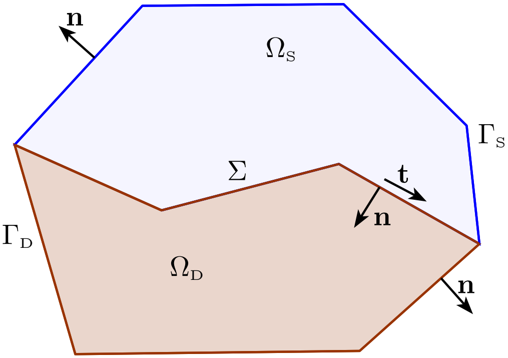
Outline

Continuous problem

Navier–Stokes/Darcy–Forchheimer coupled problem



Navier–Stokes: stress $\boldsymbol{\sigma}_{\mathrm{S}}$, velocity $\mathbf{u}_{\mathrm{S}}$, pressure p_{S}

$$\boldsymbol{\sigma}_{\mathrm{S}} = 2\mu\mathbf{e}(\mathbf{u}_{\mathrm{S}}) - p_{\mathrm{S}}\mathbb{I} \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad -\mathbf{div}\boldsymbol{\sigma}_{\mathrm{S}} + \rho(\nabla\mathbf{u}_{\mathrm{S}})\mathbf{u}_{\mathrm{S}} = \mathbf{f}_{\mathrm{S}} \quad \text{in} \quad \Omega_{\mathrm{S}},$$
$$\operatorname{div}\mathbf{u}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}}, \quad \mathbf{u}_{\mathrm{S}} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{S}}.$$

Darcy–Forchheimer: velocity $\mathbf{u}_{\mathrm{D}}$, pressure p_{D}

$$\frac{\mu}{\rho}\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}} + \nabla p_{\mathrm{D}} = \mathbf{f}_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}}, \quad \operatorname{div}\mathbf{u}_{\mathrm{D}} = g_{\mathrm{D}} \quad \text{in} \quad \Omega_{\mathrm{D}},$$
$$\mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{D}}.$$

$$\mathbf{u}_{\mathrm{S}}\cdot\mathbf{n} = \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \quad \text{on} \quad \Sigma,$$
$$\boldsymbol{\sigma}_{\mathrm{S}}\mathbf{n} + \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}(\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t})\,\mathbf{t} = -p_{\mathrm{D}}\mathbf{n} \quad \text{on} \quad \Sigma.$$

- viscosity μ, density ρ
- Forchheimer number $\mathtt{F}$
- $\mathbf{K} \in \mathbb{L}^\infty(\Omega_{\mathrm{D}}), \quad \mathbf{w}\cdot\mathbf{K}^{-1}(\mathbf{x})\mathbf{w} \geq C_{\mathbf{K}}|\mathbf{w}|^2 \quad \forall\,\mathbf{x}\in\Omega_{\mathrm{D}} \quad \forall\,\mathbf{w}\in\mathbb{R}^n$
- $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2_0(\Omega_{\mathrm{D}})$

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Variational formulation: Navier–Stokes equations

$$2\mu(\mathbf{e}(\mathbf{u}_{\rm S}),\mathbf{e}(\mathbf{v}_{\rm S}))_{\rm S} + \left\langle \frac{\alpha_d \mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\mathbf{u}_{\rm S}\cdot\mathbf{t}, \mathbf{v}_{\rm S}\cdot\mathbf{t}\right\rangle_{\Sigma} + \rho((\nabla\mathbf{u}_{\rm S})\mathbf{u}_{\rm S},\mathbf{v}_{\rm S})_{\rm S}$$

$$-(p_{\rm S}, \operatorname{div}\mathbf{v}_{\rm S})_{\rm S} + \langle \mathbf{v}_{\rm S}\cdot\mathbf{n}, \lambda\rangle_{\Sigma} = (\mathbf{f},\mathbf{v}_{\rm S})_{\rm S} \qquad \forall\, \mathbf{v}_{\rm S}\in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S})$$

$$(q_{\rm S}, \operatorname{div}\mathbf{u}_{\rm S})_{\rm S} = 0 \qquad \forall\, q_{\rm S}\in \mathrm{L}^2(\Omega_{\rm S})$$

$$\mathbf{u}_{\rm S}\in \mathbf{H}^1_{\Gamma_{\rm S}}(\Omega_{\rm S}), \quad p_{\rm S}\in \mathrm{L}^2(\Omega_{\rm S}), \quad \lambda := p_{\rm D}|_{\Sigma}$$

Variational formulation: Darcy–Forchheimer equations

$$\frac{\mu}{\rho}\big(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\big)_{\mathrm{D}} + \frac{\mathtt{F}}{\rho}\,(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - (p_{\mathrm{D}},\operatorname{div}\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} - \langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\lambda\rangle_{\Sigma} = (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}})_{\mathrm{D}} \quad \forall\,\mathbf{v}_{\mathrm{D}}\in\mathbf{H}_2$$

- $\mathbf{H}^3(\operatorname{div};\Omega_{\mathrm{D}}) := \Big\{\mathbf{v}_{\mathrm{D}}\in\mathbf{L}^3(\Omega_{\mathrm{D}}):\ \operatorname{div}\mathbf{v}_{\mathrm{D}}\in\mathrm{L}^2(\Omega_{\mathrm{D}})\Big\}$

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- $\mathbf{v}_{\mathrm{D}}\cdot\mathbf{n} : \mathbf{H}^3(\operatorname{div};\Omega_{\mathrm{D}}) \to \mathrm{W}^{-1/3,3}(\partial\Omega_{\mathrm{D}})$

$$\langle \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\partial\Omega_{\mathrm{D}}} := \int_{\Omega_{\mathrm{D}}} \mathbf{v}_{\mathrm{D}}\cdot\nabla\widetilde{\gamma}_0^{-1}(\xi) + \int_{\Omega_{\mathrm{D}}} \widetilde{\gamma}_0^{-1}(\xi)\operatorname{div}\mathbf{v}_{\mathrm{D}} \quad \forall\,\xi\in\mathrm{W}^{1/3,3/2}(\partial\Omega_{\mathrm{D}})$$

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$$\mathbf{u}_{\mathrm{D}}\in\mathbf{H}_2,\quad p_{\mathrm{D}}\in\mathrm{L}^2(\Omega_{\mathrm{D}}),\quad \lambda := p_{\mathrm{D}}|_{\Sigma}\in\mathrm{W}^{1/3,3/2}(\Sigma)$$

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$$p:=p_{\mathrm{S}}\,\chi_{\mathrm{S}}+p_{\mathrm{D}}\,\chi_{\mathrm{D}}\quad\text{with}\quad\chi_\star:=\begin{cases}1 & \text{in}\quad\Omega_\star,\\ 0 & \text{in}\quad\Omega\setminus\overline{\Omega}_\star\end{cases}$$

- $\mathbf{u}:=(\mathbf{u}_{\mathrm{S}},\mathbf{u}_{\mathrm{D}})\in\mathbf{H}:=\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})\times\mathbf{H}_2$
- $(p,\lambda)\in\mathbf{Q}:=\mathrm{L}^2(\Omega)\times\mathrm{W}^{1/3,3/2}(\Sigma)$

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$$p := p_{\mathrm{S}}\,\chi_{\mathrm{S}} + p_{\mathrm{D}}\,\chi_{\mathrm{D}} \quad \text{with} \quad \chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega}_\star \end{cases}$$

- $\mathbf{u} := (\mathbf{u}_{\mathrm{S}}, \mathbf{u}_{\mathrm{D}}) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) \times \mathbf{H}_2$
- $(p, \lambda) \in \mathbf{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{W}^{1/3,3/2}(\Sigma)$

Find $(\mathbf{u},(p,\lambda)) \in \mathbf{H}\times\mathbf{Q}$, such that

$$\begin{aligned}
[\mathbf{a}(\mathbf{u}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] + [\mathbf{b}(\mathbf{v}),(p,\lambda)] &= [\mathbf{f},\mathbf{v}] \qquad \forall\,\mathbf{v} := (\mathbf{v}_{\mathrm{S}},\mathbf{v}_{\mathrm{D}}) \in \mathbf{H},\\
[\mathbf{b}(\mathbf{u}),(q,\xi)] &= [\mathbf{g},(q,\xi)] \quad \forall\,(q,\xi) \in \mathbf{Q}.
\end{aligned}$$

$$[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}),\mathbf{v}] := [\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}]$$

$$\begin{aligned}
[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] &:= 2\mu(\mathbf{e}(\mathbf{u}_{\mathrm{S}}),\mathbf{e}(\mathbf{v}_{\mathrm{S}})) + \left\langle \frac{\alpha_d\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\mathbf{t}}}\mathbf{u}_{\mathrm{S}}\cdot\mathbf{t},\mathbf{v}_{\mathrm{S}}\cdot\mathbf{t}\right\rangle_{\Sigma},\\
[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] &:= \rho((\nabla\mathbf{u}_{\mathrm{S}})\mathbf{w}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}),\\
[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}),\mathbf{v}_{\mathrm{D}}] &:= \frac{\mu}{\rho}\left(\mathbf{K}^{-1}\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right) + \frac{\mathtt{F}}{\rho}\left(|\mathbf{u}_{\mathrm{D}}|\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}\right),
\end{aligned}$$

$$[\mathbf{b}(\mathbf{v}),(q,\xi)] := -(\operatorname{div}\mathbf{v}_{\mathrm{S}},q) - (\operatorname{div}\mathbf{v}_{\mathrm{D}},q) + \langle \mathbf{v}_{\mathrm{S}}\cdot\mathbf{n} - \mathbf{v}_{\mathrm{D}}\cdot\mathbf{n},\xi\rangle_{\Sigma}.$$

$$[\mathbf{f},\mathbf{v}] := (\mathbf{f}_{\mathrm{S}},\mathbf{v}_{\mathrm{S}}) + (\mathbf{f}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}}) \quad \text{and} \quad [\mathbf{g},(q,\xi)] := -(g_{\mathrm{D}},q).$$

Stability properties

$$\begin{aligned}
\big|[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\big| &\leq C_{\mathcal{A}_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\\
\big|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\big| &\leq \rho\, C_{\mathrm{S}}^2\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\\
\big|[\mathbf{b}(\mathbf{v}),(q,\xi)]\big| &\leq C_{\mathbf{b}}\|\mathbf{v}\|_{\mathbf{H}}\|(q,\xi)\|_{\mathbf{Q}}
\end{aligned}$$

$$\|\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}})-\mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\|_{\mathbf{H}_2'} \leq L_{\mathcal{A}_{\mathrm{D}}}\Big\{\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}+\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}\Big(\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}_2}+\|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}\Big)\Big\}$$

$$[\mathcal{A}_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] \geq 2\mu\alpha_{\mathrm{S}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}^2 \quad \forall\,\mathbf{v}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$$

$$\begin{aligned}
&[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}})-\mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}),\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}]\\
&\geq \frac{\mu}{\rho}C_{\mathbf{K}}\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}^2+\frac{\mathtt{F}}{\rho}\big(|\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}|(\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}})-|\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}|(\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}),\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\big)_{\mathrm{D}}\\
&\geq \alpha_{\mathrm{D}}\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{L}^3(\Omega_{\mathrm{D}})}^3 \quad \forall\,\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}},\mathbf{t}_{\mathrm{D}}\in\mathbf{L}^3(\Omega_{\mathrm{D}})
\end{aligned}$$

R. GLOWINSKI AND A. MARROCO, *Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires.* Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. 9 (1975), no. R-2, 41–76.

Stability properties

$$\begin{aligned} \big|[\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\big| &\le C_{\mathcal{A}_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \\ \big|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}]\big| &\le \rho\, C_{\mathrm{S}}^2\|\mathbf{w}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \\ \big|[\mathbf{b}(\mathbf{v}),(q,\xi)]\big| &\le C_{\mathbf{b}}\|\mathbf{v}\|_{\mathbf{H}}\|(q,\xi)\|_{\mathbf{Q}} \end{aligned}$$

$$\|\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}})-\mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}})\|_{\mathbf{H}_2'} \le L_{\mathcal{A}_{\mathrm{D}}}\Big\{\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}+\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}\Big(\|\mathbf{u}_{\mathrm{D}}\|_{\mathbf{H}_2}+\|\mathbf{v}_{\mathrm{D}}\|_{\mathbf{H}_2}\Big)\Big\}$$

$$[\mathcal{A}_{\mathrm{S}}(\mathbf{v}_{\mathrm{S}}),\mathbf{v}_{\mathrm{S}}] \ge 2\mu\alpha_{\mathrm{S}}\|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}}^2 \quad \forall\,\mathbf{v}_{\mathrm{S}}\in\mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$$

$$\begin{aligned} &[\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}})-\mathcal{A}_{\mathrm{D}}(\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}),\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}] \\ &\ge \frac{\mu}{\rho}C_{\mathbf{K}}\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}^2+\frac{\mathtt{F}}{\rho}\left(|\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}|(\mathbf{u}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}})-|\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}|(\mathbf{v}_{\mathrm{D}}+\mathbf{t}_{\mathrm{D}}),\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\right)_{\mathrm{D}} \\ &\ge \alpha_{\mathrm{D}}\|\mathbf{u}_{\mathrm{D}}-\mathbf{v}_{\mathrm{D}}\|_{\mathbf{L}^3(\Omega_{\mathrm{D}})}^3 \quad \forall\,\mathbf{u}_{\mathrm{D}},\mathbf{v}_{\mathrm{D}},\mathbf{t}_{\mathrm{D}}\in\mathbf{L}^3(\Omega_{\mathrm{D}}) \end{aligned}$$

R. GLOWINSKI AND A. MARROCO, *Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité, d'une classe de problèmes de Dirichlet non linéaires*. Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér. 9 (1975), no. R-2, 41–76.

Analysis of the continuous problem

A fixed-point approach

M. DISCACCIATI AND R. OYARZÚA, *A conforming mixed finite element method for the Navier–Stokes/Darcy coupled problem*. Numer. Math. 135 (2017), no. 2, 571–606.

S. CAUCAO, G.N. GATICA, R. OYARZÚA, AND I. ŠEBESTOVÁ, *A fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity*. J. Numer. Math. 25 (2017), no. 2, 55–88.

B. SCHEURER, *Existence et approximation de points selles pour certains problèmes non linéaires*. RAIRO Anal. Numér. 11 (1977), no. 4, 369–400.

Define $\mathbf{T} : \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \to \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega)$,

$$\mathbf{T}(\mathbf{w}_\mathrm{S}) := \mathbf{u}_\mathrm{S} \quad \forall\, \mathbf{w}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}).$$

$$\begin{aligned}
[\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p, \lambda)] \quad &= \quad [\mathbf{f}, \mathbf{v}] && \forall\, \mathbf{v} \in \mathbf{H}, \\
[\mathbf{b}(\mathbf{u}), (q, \xi)] \quad &= \quad [\mathbf{g}, (q, \xi)] && \forall\, (q, \xi) \in \mathbf{Q}.
\end{aligned}$$

Well-definiteness of $\mathbf{T}$

Theorem

Let X_1, X_2 and Y be separable and reflexive Banach spaces, being X_1 and X_2 uniformly convex, set $X = X_1 \times X_2$. Assume that

(i) $\|a(u)-a(v)\|_{X'} \leq \gamma \sum_{j=1}^{2} \left\{ \|u_j - v_j\|_{X_j} + \|u_j - v_j\|_{X_j} \Big(\|u_j\|_{X_j} + \|v_j\|_{X_j} \Big)^{p_j - 2} \right\} \quad \forall\, u, v \in X$

(ii) $[a(u+t) - a(v+t), u - v] \geq \alpha \left\{ \|u_1 - v_1\|_{X_1}^{p_1} + \|u_2 - v_2\|_{X_2}^{p_2} \right\} \quad \forall\, u, v \in Ker(b), t \in X$

(iii) $\displaystyle\sup_{\substack{v \in X \\ v \neq 0}} \frac{[b(v), q]}{\|v\|_X} \geq \beta \, \|q\|_Y \quad \forall\, q \in Y$

Then, for each $(f, g) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} [a(u), v] + [b(v), p] &= [f, v] \quad \forall\, v \in X, \\ [b(u), q] &= [g, q] \quad \forall\, q \in Y. \end{aligned}$$

Moreover,

$$\|(u, p)\|_{X \times Y} \leq C \, \mathcal{M}(f, g)$$

$$\mathcal{M}(f, g) := \max \left\{ \mathcal{N}(f, g)^{\frac{1}{p_1 - 1}}, \mathcal{N}(f, g)^{\frac{1}{p_2 - 1}}, \mathcal{N}(f, g), \mathcal{N}(f, g)^{\frac{p_1 - 1}{p_2 - 1}}, \mathcal{N}(f, g)^{\frac{p_2 - 1}{p_1 - 1}} \right\}$$

$$\mathcal{N}(f, g) := \|f\|_{X'} + \|g\|_{Y'} + \|g\|_{Y'}^{p_1 - 1} + \|g\|_{Y'}^{p_2 - 1} + \|a(0)\|_{X'}$$

$\mathbf{v} = (\mathbf{v}_{\mathrm{S}}, \mathbf{v}_{\mathrm{D}}) \in Ker(\mathbf{b})$ implies

$$\operatorname{div} \mathbf{v}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \quad \operatorname{div} \mathbf{v}_{\mathrm{D}} = 0 \quad \text{in} \quad \Omega_{\mathrm{D}}$$

Lemma

Let $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that $\operatorname{div} \mathbf{w}_{\mathrm{S}} = 0$ in Ω_{S} and

$$\|\mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_{\mathrm{S}}}{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}.$$

Then, for each $\mathbf{t} \in \mathbf{H}$, the nonlinear operator $\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\,\cdot + \mathbf{t})$ is strictly monotone on $Ker(\mathbf{b})$.

Sketch of the proof

- $[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}+\mathbf{t}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{v}+\mathbf{t}), \mathbf{u}-\mathbf{v}] \geq 2\mu\alpha_{\mathrm{S}}\|\mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}}$
  $+\, \alpha_{\mathrm{D}}\|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}_2} + [\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}), \mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}]$
- $\big|[\mathcal{B}_{\mathrm{S}}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}), \mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}]\big| \leq \dfrac{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}{2}\|\mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}\|_{0,\Sigma}\, \|\mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}}$
- $[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}+\mathbf{t}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{v}+\mathbf{t}), \mathbf{u}-\mathbf{v}] \geq \left\{2\mu\alpha_{\mathrm{S}} - \dfrac{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}{2}\|\mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}\|_{0,\Sigma}\right\}\|\mathbf{u}_{\mathrm{S}} - \mathbf{v}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}}$
  $+\, \alpha_{\mathrm{D}}\|\mathbf{u}_{\mathrm{D}} - \mathbf{v}_{\mathrm{D}}\|^3_{\mathbf{H}_2}$

$\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D) \in Ker(\mathbf{b})$ implies

$$\operatorname{div} \mathbf{v}_S = 0 \quad \text{in} \quad \Omega_S \quad \text{and} \quad \operatorname{div} \mathbf{v}_D = 0 \quad \text{in} \quad \Omega_D$$

Lemma

Let $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that $\operatorname{div} \mathbf{w}_S = 0$ in Ω_S and

$$\|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma} \le \frac{2\mu\alpha_S}{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}.$$

Then, for each $\mathbf{t} \in \mathbf{H}$, the nonlinear operator $\mathbf{a}(\mathbf{w}_S)(\,\cdot + \mathbf{t})$ is strictly monotone on $Ker(\mathbf{b})$.

Sketch of the proof

- $[\mathbf{a}(\mathbf{w}_S)(\mathbf{u}+\mathbf{t}) - \mathbf{a}(\mathbf{w}_S)(\mathbf{v}+\mathbf{t}), \mathbf{u}-\mathbf{v}] \ge 2\mu\alpha_S \|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$
  $+ \alpha_D \|\mathbf{u}_D - \mathbf{v}_D\|^3_{\mathbf{H}_2} + [\mathcal{B}_S(\mathbf{w}_S)(\mathbf{u}_S - \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S]$
- $\left|[\mathcal{B}_S(\mathbf{w}_S)(\mathbf{u}_S - \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S]\right| \le \dfrac{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}{2} \|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma} \|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$
- $[\mathbf{a}(\mathbf{w}_S)(\mathbf{u}+\mathbf{t}) - \mathbf{a}(\mathbf{w}_S)(\mathbf{v}+\mathbf{t}), \mathbf{u}-\mathbf{v}] \ge \left\{2\mu\alpha_S - \dfrac{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2}{2} \|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma}\right\} \|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$
  $+ \alpha_D \|\mathbf{u}_D - \mathbf{v}_D\|^3_{\mathbf{H}_2}$

$\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D) \in Ker(\mathbf{b})$ implies

$$\operatorname{div} \mathbf{v}_S = 0 \quad \text{in} \quad \Omega_S \quad \text{and} \quad \operatorname{div} \mathbf{v}_D = 0 \quad \text{in} \quad \Omega_D$$

Lemma

Let $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that $\operatorname{div} \mathbf{w}_S = 0$ in Ω_S and

$$\|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_S}{\rho\, C_{tr}^2 C_s^2}.$$

Then, for each $\mathbf{t} \in \mathbf{H}$, the nonlinear operator $\mathbf{a}(\mathbf{w}_S)(\cdot + \mathbf{t})$ is strictly monotone on $Ker(\mathbf{b})$.

Sketch of the proof

- $$[\mathbf{a}(\mathbf{w}_S)(\mathbf{u} + \mathbf{t}) - \mathbf{a}(\mathbf{w}_S)(\mathbf{v} + \mathbf{t}), \mathbf{u} - \mathbf{v}] \geq 2\mu\alpha_S\|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$$
  $$+ \alpha_D\|\mathbf{u}_D - \mathbf{v}_D\|^3_{\mathbf{H}_2} + [\mathcal{B}_S(\mathbf{w}_S)(\mathbf{u}_S - \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S]$$
- $$\left|[\mathcal{B}_S(\mathbf{w}_S)(\mathbf{u}_S - \mathbf{v}_S), \mathbf{u}_S - \mathbf{v}_S]\right| \leq \frac{\rho\, C_{tr}^2 C_s^2}{2}\|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma}\, \|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$$
- $$[\mathbf{a}(\mathbf{w}_S)(\mathbf{u} + \mathbf{t}) - \mathbf{a}(\mathbf{w}_S)(\mathbf{v} + \mathbf{t}), \mathbf{u} - \mathbf{v}] \geq \left\{2\mu\alpha_S - \frac{\rho\, C_{tr}^2 C_s^2}{2}\|\mathbf{w}_S \cdot \mathbf{n}\|_{0,\Sigma}\right\}\|\mathbf{u}_S - \mathbf{v}_S\|^2_{1,\Omega_S}$$
  $$+ \alpha_D\|\mathbf{u}_D - \mathbf{v}_D\|^3_{\mathbf{H}_2}$$

Lemma

$$S(q,\xi) := \sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}),(q,\xi)]}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\,\|(q,\xi)\|_{\mathbf{Q}} \quad \forall\,(q,\xi)\in\mathbf{Q}.$$

Sketch of the proof

- $q \in \mathrm{L}_0^2(\Omega), \exists\, \mathbf{z}\in\mathbf{H}_0^1(\Omega):\quad \operatorname{div}\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \leq c\|q\|_{0,\Omega}$
- $\widehat{\mathbf{v}} := (\widehat{\mathbf{v}}_{\mathrm{S}},\widehat{\mathbf{v}}_{\mathrm{D}}),\ \widehat{\mathbf{v}}_\star = \mathbf{z}|_{\Omega_\star}, \star\in\{\mathrm{S},\mathrm{D}\} \Rightarrow \widehat{\mathbf{v}}_{\mathrm{S}}\cdot\mathbf{n} = \widehat{\mathbf{v}}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ and $\|\widehat{\mathbf{v}}\|_{\mathbf{H}} \leq c\|\mathbf{z}\|_{1,\Omega}$
- $S(q,\xi) \geq \dfrac{|[\mathbf{b}(\widehat{\mathbf{v}}),(q,\xi)]|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} = \dfrac{\|q\|_{0,\Omega}^2}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\|q\|_{0,\Omega}$
- $S(q,\xi) \geq c_2\|\xi\|_{1/3,3/2;\Sigma} - c_3\|q\|_{0,\Omega}$
- $S(q,\xi) \geq \dfrac{c_1\,c_2}{c_1+c_3}\|\xi\|_{1/3,3/2;\Sigma}$

Lemma

$$S(q,\xi) := \sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}),(q,\xi)]}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\,\|(q,\xi)\|_{\mathbf{Q}} \quad \forall\,(q,\xi)\in\mathbf{Q}.$$

Sketch of the proof

- $q\in \mathrm{L}^2_0(\Omega), \exists\,\mathbf{z}\in \mathbf{H}^1_0(\Omega):\quad \operatorname{div}\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega} \leq c\|q\|_{0,\Omega}$
- $\widehat{\mathbf{v}} := (\widehat{\mathbf{v}}_{\mathrm{S}},\widehat{\mathbf{v}}_{\mathrm{D}}),\ \widehat{\mathbf{v}}_\star = \mathbf{z}|_{\Omega_\star}, \star\in\{\mathrm{S},\mathrm{D}\} \Rightarrow \widehat{\mathbf{v}}_{\mathrm{S}}\cdot\mathbf{n} = \widehat{\mathbf{v}}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ and $\|\widehat{\mathbf{v}}\|_{\mathbf{H}} \leq c\|\mathbf{z}\|_{1,\Omega}$
- $S(q,\xi) \geq \dfrac{|[\mathbf{b}(\widehat{\mathbf{v}}),(q,\xi)]|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} = \dfrac{\|q\|^2_{0,\Omega}}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\|q\|_{0,\Omega}$
- $S(q,\xi) \geq c_2\|\xi\|_{1/3,3/2;\Sigma} - c_3\|q\|_{0,\Omega}$
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Lemma

$$S(q,\xi) := \sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}),(q,\xi)]}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\,\|(q,\xi)\|_{\mathbf{Q}} \quad \forall\,(q,\xi)\in\mathbf{Q}.$$

Sketch of the proof

- $q\in \mathrm{L}^2_0(\Omega), \exists\,\mathbf{z}\in\mathbf{H}^1_0(\Omega):\quad \operatorname{div}\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega}\leq c\|q\|_{0,\Omega}$
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$$S(q,\xi) := \sup_{\substack{\mathbf{v}\in\mathbf{H}\\ \mathbf{v}\neq\mathbf{0}}} \frac{[\mathbf{b}(\mathbf{v}),(q,\xi)]}{\|\mathbf{v}\|_{\mathbf{H}}} \geq \beta\,\|(q,\xi)\|_{\mathbf{Q}} \quad \forall\,(q,\xi)\in\mathbf{Q}.$$

Sketch of the proof

- $q\in \mathrm{L}^2_0(\Omega), \exists\,\mathbf{z}\in\mathbf{H}^1_0(\Omega):\quad \mathrm{div}\,\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega}\leq c\|q\|_{0,\Omega}$
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- $S(q,\xi) \geq \dfrac{\left|[\mathbf{b}(\widehat{\mathbf{v}}),(q,\xi)]\right|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} = \dfrac{\|q\|^2_{0,\Omega}}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\|q\|_{0,\Omega}$
- $S(q,\xi)\geq c_2\|\xi\|_{1/3,3/2;\Sigma} - c_3\|q\|_{0,\Omega}$
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Lemma

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Sketch of the proof

- $q\in \mathrm{L}^2_0(\Omega), \exists\,\mathbf{z}\in\mathbf{H}^1_0(\Omega):\quad \operatorname{div}\mathbf{z} = -q$ in Ω and $\|\mathbf{z}\|_{1,\Omega}\leq c\|q\|_{0,\Omega}$
- $\widehat{\mathbf{v}} := (\widehat{\mathbf{v}}_{\mathrm{S}},\widehat{\mathbf{v}}_{\mathrm{D}}),\ \widehat{\mathbf{v}}_\star = \mathbf{z}|_{\Omega_\star}, \star\in\{\mathrm{S},\mathrm{D}\} \Rightarrow \widehat{\mathbf{v}}_{\mathrm{S}}\cdot\mathbf{n} = \widehat{\mathbf{v}}_{\mathrm{D}}\cdot\mathbf{n}$ on Σ and $\|\widehat{\mathbf{v}}\|_{\mathbf{H}}\leq c\|\mathbf{z}\|_{1,\Omega}$
- $S(q,\xi) \geq \dfrac{\big|[\mathbf{b}(\widehat{\mathbf{v}}),(q,\xi)]\big|}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} = \dfrac{\|q\|^2_{0,\Omega}}{\|\widehat{\mathbf{v}}\|_{\mathbf{H}}} \geq c_1\|q\|_{0,\Omega}$
- $S(q,\xi)\geq c_2\|\xi\|_{1/3,3/2;\Sigma} - c_3\|q\|_{0,\Omega}$
- $S(q,\xi)\geq \dfrac{c_1\,c_2}{c_1+c_3}\|\xi\|_{1/3,3/2;\Sigma}$

Theorem

Let $\mathbf{w}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ such that $\mathrm{div}\,\mathbf{w}_{\mathrm{S}} = 0$ in Ω_{S} and

$$\|\mathbf{w}_{\mathrm{S}} \cdot \mathbf{n}\|_{0,\Sigma} \leq \frac{2\mu\alpha_{\mathrm{S}}}{\rho\, C_{\mathrm{tr}}^2 C_{\mathrm{s}}^2},$$

and let $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$. Then, $\mathbf{T}$ is well-defined. Moreover,

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} = \|\mathbf{u}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq \|(\mathbf{u},(p,\lambda))\|_{\mathbf{H}\times\mathbf{Q}} \leq c_{\mathbf{T}}\,\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})$$

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}) := \max\Big\{\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^{1/2}, \mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}), \mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}})^2\Big\}$$

$$\mathcal{N}(\mathbf{f}_{\mathrm{S}},\mathbf{f}_{\mathrm{D}},g_{\mathrm{D}}) := \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{\mathbf{L}^{3/2}(\Omega_{\mathrm{D}})} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|g_{\mathrm{D}}\|^2_{0,\Omega_{\mathrm{D}}}$$

Solvability analysis of the fixed-point equation

Lemma

Let $\mathbf{W}$ be the closed ball defined by

$$\mathbf{W} := \Big\{ \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \operatorname{div} \mathbf{v}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \quad \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \le c_{\mathbf{T}} \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \Big\}$$

and assume that the data satisfy

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \le \frac{2\mu\alpha_{\mathrm{S}}}{c_{\mathbf{T}}\, \rho\, C_{\mathrm{tr}}^3 C_{\mathrm{s}}^2}.$$

Then,

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}}) - \mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \le \frac{\rho\, C_{\mathrm{S}}}{\mu\alpha_{\mathrm{S}}} \|\mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})} \quad \forall\, \mathbf{w}_{\mathrm{S}}, \widetilde{\mathbf{w}}_{\mathrm{S}} \in \mathbf{W}.$$

Sketch of the proof

- $$\begin{aligned} [\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p - \widetilde{p}, \lambda - \widetilde{\lambda})] &= 0 \\ [\mathbf{b}(\mathbf{u} - \widetilde{\mathbf{u}}), (q, \xi)] &= 0 \end{aligned}$$
- $[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = 0$
- $$\begin{aligned} \mu\alpha_{\mathrm{S}} \|\mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}} &\le [\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = [\mathcal{B}_{\mathrm{S}}(\widetilde{\mathbf{w}}_{\mathrm{S}} - \mathbf{w}_{\mathrm{S}})(\widetilde{\mathbf{u}}_{\mathrm{S}}), \mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}] \\ &\le \rho\, C_{\mathrm{S}} \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})} \|\widetilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \end{aligned}$$

Lemma

Let $\mathbf{W}$ be the closed ball defined by

$$\mathbf{W} := \Big\{ \mathbf{v}_\mathrm{S} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \operatorname{div} \mathbf{v}_\mathrm{S} = 0 \quad \text{in} \quad \Omega_\mathrm{S} \quad \text{and} \quad \|\mathbf{v}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \leq c_\mathbf{T} \mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) \Big\}$$

and assume that the data satisfy

$$\mathcal{M}(\mathbf{f}_\mathrm{S}, \mathbf{f}_\mathrm{D}, g_\mathrm{D}) \leq \frac{2\mu\alpha_\mathrm{S}}{c_\mathbf{T}\, \rho\, C_\mathrm{tr}^3 C_\mathrm{s}^2}.$$

Then,

$$\|\mathbf{T}(\mathbf{w}_\mathrm{S}) - \mathbf{T}(\widetilde{\mathbf{w}}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} \leq \frac{\rho\, C_\mathrm{S}}{\mu\alpha_\mathrm{S}} \|\mathbf{T}(\widetilde{\mathbf{w}}_\mathrm{S})\|_{1,\Omega_\mathrm{S}} \, \|\mathbf{w}_\mathrm{S} - \widetilde{\mathbf{w}}_\mathrm{S}\|_{\mathbf{L}^4(\Omega_\mathrm{S})} \quad \forall\, \mathbf{w}_\mathrm{S}, \widetilde{\mathbf{w}}_\mathrm{S} \in \mathbf{W}.$$

Sketch of the proof

- $$\begin{aligned} [\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_\mathrm{S})(\widetilde{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p - \widetilde{p}, \lambda - \widetilde{\lambda})] &= 0 \\ [\mathbf{b}(\mathbf{u} - \widetilde{\mathbf{u}}), (q, \xi)] &= 0 \end{aligned}$$

- $[\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_\mathrm{S})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = 0$

- $$\begin{aligned} \mu\alpha_\mathrm{S}\|\mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}\|^2_{1,\Omega_\mathrm{S}} &\leq [\mathbf{a}(\mathbf{w}_\mathrm{S})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_\mathrm{S})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = [\mathcal{B}_\mathrm{S}(\widetilde{\mathbf{w}}_\mathrm{S} - \mathbf{w}_\mathrm{S})(\widetilde{\mathbf{u}}_\mathrm{S}), \mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}] \\ &\leq \rho\, C_\mathrm{S} \|\mathbf{w}_\mathrm{S} - \widetilde{\mathbf{w}}_\mathrm{S}\|_{\mathbf{L}^4(\Omega_\mathrm{S})} \|\widetilde{\mathbf{u}}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \|\mathbf{u}_\mathrm{S} - \widetilde{\mathbf{u}}_\mathrm{S}\|_{1,\Omega_\mathrm{S}} \end{aligned}$$

Lemma

Let $\mathbf{W}$ be the closed ball defined by

$$\mathbf{W} := \Big\{ \mathbf{v}_{\mathrm{S}} \in \mathbf{H}^1_{\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \operatorname{div} \mathbf{v}_{\mathrm{S}} = 0 \quad \text{in} \quad \Omega_{\mathrm{S}} \quad \text{and} \quad \|\mathbf{v}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \leq c_{\mathbf{T}} \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \Big\}$$

and assume that the data satisfy

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \leq \frac{2\mu\alpha_{\mathrm{S}}}{c_{\mathbf{T}}\, \rho\, C_{\mathrm{tr}}^3 C_{\mathrm{s}}^2}.$$

Then,

$$\|\mathbf{T}(\mathbf{w}_{\mathrm{S}}) - \mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \leq \frac{\rho\, C_{\mathrm{S}}}{\mu\alpha_{\mathrm{S}}} \|\mathbf{T}(\widetilde{\mathbf{w}}_{\mathrm{S}})\|_{1,\Omega_{\mathrm{S}}} \, \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})} \quad \forall\, \mathbf{w}_{\mathrm{S}}, \widetilde{\mathbf{w}}_{\mathrm{S}} \in \mathbf{W}.$$

Sketch of the proof

- $$\begin{aligned} [\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{v}] + [\mathbf{b}(\mathbf{v}), (p - \widetilde{p}, \lambda - \widetilde{\lambda})] &= 0 \\ [\mathbf{b}(\mathbf{u} - \widetilde{\mathbf{u}}), (q, \xi)] &= 0 \end{aligned}$$
- $[\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\widetilde{\mathbf{w}}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = 0$
- $$\begin{aligned} \mu\alpha_{\mathrm{S}} \|\mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}\|^2_{1,\Omega_{\mathrm{S}}} &\leq [\mathbf{a}(\mathbf{w}_{\mathrm{S}})(\mathbf{u}) - \mathbf{a}(\mathbf{w}_{\mathrm{S}})(\widetilde{\mathbf{u}}), \mathbf{u} - \widetilde{\mathbf{u}}] = [\mathcal{B}_{\mathrm{S}}(\widetilde{\mathbf{w}}_{\mathrm{S}} - \mathbf{w}_{\mathrm{S}})(\widetilde{\mathbf{u}}_{\mathrm{S}}), \mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}] \\ &\leq \rho\, C_{\mathrm{S}} \|\mathbf{w}_{\mathrm{S}} - \widetilde{\mathbf{w}}_{\mathrm{S}}\|_{\mathbf{L}^4(\Omega_{\mathrm{S}})} \|\widetilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S}} - \widetilde{\mathbf{u}}_{\mathrm{S}}\|_{1,\Omega_{\mathrm{S}}} \end{aligned}$$

The main result

Theorem

Let $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$. Assume,

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \leq \frac{2\mu\alpha_{\mathrm{S}}}{c_{\mathbf{T}}\,\rho\, C_{\mathrm{tr}}^3 C_{\mathrm{s}}^2}$$

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) := \max\Big\{\mathcal{N}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}})^{1/2}, \mathcal{N}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}), \mathcal{N}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}})^2\Big\}$$

$$\mathcal{N}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) := \|\mathbf{f}_{\mathrm{S}}\|_{0,\Omega_{\mathrm{S}}} + \|\mathbf{f}_{\mathrm{D}}\|_{\mathbf{L}^{3/2}(\Omega_{\mathrm{D}})} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}} + \|g_{\mathrm{D}}\|_{0,\Omega_{\mathrm{D}}}^2.$$

Then the continuous problem admits a solution $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathbf{Q}$. In addition, assuming

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) < r,$$

where

$$r := \frac{\mu\alpha_{\mathrm{S}}}{c_{\mathbf{T}}\rho} \min\left\{\frac{1}{C_{\mathrm{S}}^2}, \frac{2}{C_{\mathrm{s}}^2 C_{\mathrm{tr}}^3}\right\}.$$

Then the solution is unique. In any case,

$$\|(\mathbf{u}, (p, \lambda))\|_{\mathbf{H}\times\mathbf{Q}} \leq c_{\mathbf{T}} \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}).$$

The mixed finite element scheme

Discrete spaces

$$\mathrm{BR}(T) := [\mathrm{P}_1(T)]^2 \oplus \mathrm{span}\Big\{\eta_2\eta_3\mathbf{n}_1, \eta_1\eta_3\mathbf{n}_2, \eta_1\eta_2\mathbf{n}_3\Big\}.$$

$$\mathrm{RT}_0(T) := \mathrm{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}.$$

$$\begin{aligned}
\mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) &:= \Big\{\mathbf{v} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \mathbf{v}|_T \in \mathrm{BR}(T), \quad \forall\, T \in \mathcal{T}_h^\mathrm{S}\Big\}\\
\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D}) &:= \Big\{\mathbf{v} \in \mathbf{H}_2 : \quad \mathbf{v}|_T \in \mathrm{RT}_0(T), \quad \forall\, T \in \mathcal{T}_h^\mathrm{D}\Big\}\\
\mathrm{L}_{h,0}(\Omega) &:= \Big\{q \in \mathrm{L}^2_0(\Omega) : \quad q|_T \in \mathrm{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\Big\}
\end{aligned}$$

$$\Lambda_h(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R} : \quad \xi_h|_e \in \mathrm{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h\Big\}.$$

$\prod_{e\in\Sigma_h} \mathrm{W}^{1-1/p,p}(e)$ coincides with $\mathrm{W}^{1-1/p,p}(\Sigma)$, without extra conditions when $1 < p < 2$.

P. GRISVARD, *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

Discrete spaces

$$\mathrm{BR}(T) := [\mathrm{P}_1(T)]^2 \oplus \mathrm{span}\Big\{\eta_2\eta_3\mathbf{n}_1, \eta_1\eta_3\mathbf{n}_2, \eta_1\eta_2\mathbf{n}_3\Big\}.$$

$$\mathrm{RT}_0(T) := \mathrm{span}\Big\{(1,0),(0,1),(x_1,x_2)\Big\}.$$

$$\begin{aligned}
\mathbf{H}_{h,\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) \quad &:= \Big\{\mathbf{v} \in \mathbf{H}^1_{\Gamma_\mathrm{S}}(\Omega_\mathrm{S}) : \quad \mathbf{v}|_T \in \mathrm{BR}(T), \quad \forall\, T \in \mathcal{T}_h^\mathrm{S}\Big\} \\
\mathbf{H}_{h,\Gamma_\mathrm{D}}(\Omega_\mathrm{D}) \quad &:= \Big\{\mathbf{v} \in \mathbf{H}_2 : \quad \mathbf{v}|_T \in \mathrm{RT}_0(T), \quad \forall\, T \in \mathcal{T}_h^\mathrm{D}\Big\} \\
\mathrm{L}_{h,0}(\Omega) \quad &:= \Big\{q \in \mathrm{L}^2_0(\Omega) : \quad q|_T \in \mathrm{P}_0(T), \quad \forall\, T \in \mathcal{T}_h\Big\}
\end{aligned}$$

$$\Lambda_h(\Sigma) := \Big\{\xi_h : \Sigma \to \mathbb{R} : \quad \xi_h|_e \in \mathrm{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h\Big\}.$$

$\prod_{e\in\Sigma_h} \mathrm{W}^{1-1/p,p}(e)$ coincides with $\mathrm{W}^{1-1/p,p}(\Sigma)$, without extra conditions when $1 < p < 2$.

P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.

Discrete Navier–Stokes/Darcy–Forchheimer coupled problem

Find $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, such that

$$\begin{aligned} [\mathbf{a}_h(\mathbf{u}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] + [\mathbf{b}(\mathbf{v}_h), (p_h, \lambda_h)] &= [\mathbf{f}, \mathbf{v}_h] && \forall\, \mathbf{v}_h := (\mathbf{v}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{D},h}) \in \mathbf{H}_h, \\ [\mathbf{b}(\mathbf{u}_h), (q_h, \xi_h)] &= [\mathbf{g}, (q_h, \xi_h)] && \forall\, (q_h, \xi_h) \in \mathbf{Q}_h. \end{aligned}$$

$$[\mathbf{a}_h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_h), \mathbf{v}_h] := [\mathcal{A}_{\mathrm{S}}(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{B}_{\mathrm{S}}^h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] + [\mathcal{A}_{\mathrm{D}}(\mathbf{u}_{\mathrm{D},h}), \mathbf{v}_{\mathrm{D},h}]$$

$$[\mathcal{B}_{\mathrm{S}}^h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] := \rho((\nabla \mathbf{u}_{\mathrm{S},h})\mathbf{w}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}} + \frac{\rho}{2}(\operatorname{div} \mathbf{w}_{\mathrm{S},h} \mathbf{u}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h})_{\mathrm{S}}$$

$$[\mathcal{B}_{\mathrm{S}}^h(\mathbf{w}_{\mathrm{S},h})(\mathbf{v}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}] = \frac{\rho}{2}\int_\Sigma (\mathbf{w}_{\mathrm{S},h} \cdot \mathbf{n})|\mathbf{v}_{\mathrm{S},h}|^2 \geq 0 \quad \forall\, \mathbf{w}_{\mathrm{S},h}, \mathbf{v}_{\mathrm{S},h} \in \mathbf{H}_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$$

$$\left|[\mathcal{B}_{\mathrm{S}}^h(\mathbf{w}_{\mathrm{S},h})(\mathbf{u}_{\mathrm{S},h}), \mathbf{v}_{\mathrm{S},h}]\right| \leq C_{\mathrm{sk}} \|\mathbf{w}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{u}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \|\mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}}$$

The main result

Theorem

Let $\mathbf{W}_h$ be the compact convex subset of $\mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}})$ defined by

$$\mathbf{W}_h := \Big\{ \mathbf{v}_{\mathrm{S},h} \in \mathbf{H}^1_{h,\Gamma_{\mathrm{S}}}(\Omega_{\mathrm{S}}) : \quad \|\mathbf{v}_{\mathrm{S},h}\|_{1,\Omega_{\mathrm{S}}} \leq \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \Big\}.$$

Assume that the data $\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}$, and g_{D} satisfy

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) < \widetilde{r},$$

where

$$\widetilde{r} := \frac{2\mu\alpha_{\mathrm{S}}}{\widetilde{c}_{\mathbf{T}}\rho} \min\left\{ \frac{1}{C_{\mathrm{S}}^2(2+\sqrt{2})}, \frac{1}{C_{\mathrm{s}}^2 C_{\mathrm{tr}}^3} \right\}.$$

Then, there exists a unique $(\mathbf{u}_h, (p_h, \lambda_h)) \in \mathbf{H}_h \times \mathbf{Q}_h$, which satisfies $\mathbf{u}_{\mathrm{S},h} \in \mathbf{W}_h$ and

$$\|(\mathbf{u}_h, (p_h, \lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \leq \widetilde{c}_{\mathbf{T}}\, \mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}).$$

Cea estimate

Theorem

Let $\mathbf{f}_{\mathrm{S}} \in \mathbf{L}^2(\Omega_{\mathrm{S}})$, $\mathbf{f}_{\mathrm{D}} \in \mathbf{L}^{3/2}(\Omega_{\mathrm{D}})$ and $g_{\mathrm{D}} \in \mathrm{L}^2(\Omega_{\mathrm{D}})$, such that

$$\mathcal{M}(\mathbf{f}_{\mathrm{S}}, \mathbf{f}_{\mathrm{D}}, g_{\mathrm{D}}) \leq \frac{1}{2} \min\left\{r, \widetilde{r}\right\}.$$

Then,

$$\begin{aligned}&\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\leq C \max_{i\in\{2,3\}} \left\{ \left(\inf_{\mathbf{v}_h\in\mathbf{H}_h} \left(\|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}} + \|\mathbf{u}-\mathbf{v}_h\|_{\mathbf{H}}^2 \right) + \inf_{(q_h,\xi_h)\in\mathbf{Q}_h} \|(p,\lambda)-(q_h,\xi_h)\|_{\mathbf{Q}} \right)^{\frac{1}{i-1}} \right\}.\end{aligned}$$

Rate of convergence

Theorem

Assume that:

$\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^2(\Omega_{\mathrm{S}})$, $\mathbf{u}_{\mathrm{D}} \in \mathbf{W}^{1,3}(\Omega_{\mathrm{D}})$, $\operatorname{div} \mathbf{u}_{\mathrm{D}} \in \mathrm{H}^1(\Omega_{\mathrm{D}})$, $p \in \mathrm{H}^1(\Omega)$, and $\lambda \in \mathrm{W}^{1,3/2}(\Sigma)$.

Then,

$$\begin{aligned}
&\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \\
&\leq C\, h^{1/3} \max_{i\in\{2,3\}} \Bigg\{ \Big(\|\mathbf{u}_{\mathrm{S}}\|_{2,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{1,3;\Omega_{\mathrm{D}}} + \|\operatorname{div} \mathbf{u}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} \\
&+ \|\mathbf{u}_{\mathrm{S}}\|^2_{2,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|^2_{1,3;\Omega_{\mathrm{D}}} + \|\operatorname{div} \mathbf{u}_{\mathrm{D}}\|^2_{1,\Omega_{\mathrm{D}}} + \|p\|_{1,\Omega} + \|\lambda\|_{1,\frac{3}{2};\Sigma} \Big)^{\frac{1}{i-1}} \Bigg\}.
\end{aligned}$$

$$\|\xi - \xi_h\|_{1/3,3/2;\Sigma} \leq c\,\|\xi - \xi_h\|^{1-1/3}_{\mathrm{L}^{3/2}(\Sigma)} \|\xi\|^{1/3}_{1,3/2;\Sigma} \leq C\, h^{2/3}\, \|\xi\|_{1,3/2;\Sigma}$$

Rate of convergence

Theorem

Assume that:

$\mathbf{u}_{\mathrm{S}} \in \mathbf{H}^2(\Omega_{\mathrm{S}}), \quad \mathbf{u}_{\mathrm{D}} \in \mathbf{W}^{1,3}(\Omega_{\mathrm{D}}), \quad \operatorname{div} \mathbf{u}_{\mathrm{D}} \in \mathrm{H}^1(\Omega_{\mathrm{D}}), \quad p \in \mathrm{H}^1(\Omega),$ and $\lambda \in \mathrm{W}^{1,3/2}(\Sigma).$

Then,

$$\begin{aligned} &\|(\mathbf{u},(p,\lambda)) - (\mathbf{u}_h,(p_h,\lambda_h))\|_{\mathbf{H}\times\mathbf{Q}} \\ &\leq C\, h^{1/3} \max_{i\in\{2,3\}} \Bigg\{ \Big(\|\mathbf{u}_{\mathrm{S}}\|_{2,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|_{1,3;\Omega_{\mathrm{D}}} + \|\operatorname{div}\mathbf{u}_{\mathrm{D}}\|_{1,\Omega_{\mathrm{D}}} \\ &+ \|\mathbf{u}_{\mathrm{S}}\|^2_{2,\Omega_{\mathrm{S}}} + \|\mathbf{u}_{\mathrm{D}}\|^2_{1,3;\Omega_{\mathrm{D}}} + \|\operatorname{div}\mathbf{u}_{\mathrm{D}}\|^2_{1,\Omega_{\mathrm{D}}} + \|p\|_{1,\Omega} + \|\lambda\|_{1,\frac{3}{2};\Sigma} \Big)^{\frac{1}{i-1}} \Bigg\}. \end{aligned}$$

$$\|\xi - \xi_h\|_{1/3,3/2;\Sigma} \leq c\, \|\xi - \xi_h\|^{1-1/3}_{\mathrm{L}^{3/2}(\Sigma)} \|\xi\|^{1/3}_{1,3/2;\Sigma} \leq C\, h^{2/3}\, \|\xi\|_{1,3/2;\Sigma}$$

Numerical test

2D helmet-shaped domain

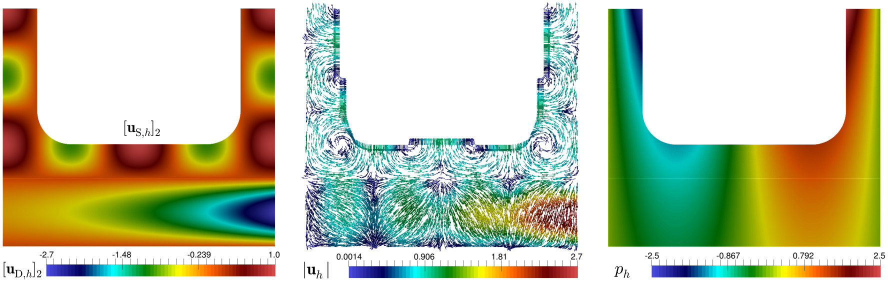
$$\Omega := \Omega_{\mathrm{S}} \cup \Sigma \cup \Omega_{\mathrm{D}}, \quad \text{where} \quad \Omega_{\mathrm{D}} := (-1,1) \times (-0.5,0), \quad \Sigma := (-1,1) \times \{0\}$$

$$\mu = 1, \quad \rho = 1, \quad \alpha = 1, \quad \mathbf{K} = \mathbb{I}, \quad \text{and} \quad \mathtt{F} = 10$$

$$\mathbf{u}_{\mathrm{S}}(x_1,x_2) = \begin{pmatrix} -\sin(2\pi x_1)\cos(2\pi x_2) \\ \cos(2\pi x_1)\sin(2\pi x_2) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{S}},$$

$$\mathbf{u}_{\mathrm{D}}(x_1,x_2) = \begin{pmatrix} \sin(2\pi x_1)\exp(x_2) \\ \sin(2\pi x_2)\exp(x_1) \end{pmatrix} \quad \text{in} \quad \Omega_{\mathrm{D}},$$

$$p_\star(x_1,x_2) = \sin(\pi x_1)\exp(x_2) + p_0 \quad \text{in} \quad \Omega_\star, \quad \text{with } \star \in \{\mathrm{S},\mathrm{D}\}.$$



N	h_{S}	$\mathrm{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathrm{e}(p_{\mathrm{S}})$	$r(p_{\mathrm{S}})$
1007	0.188	1.0274	–	0.5355	–
3790	0.109	0.5114	1.275	0.2156	1.664
14014	0.048	0.2472	0.890	0.0978	0.967
55428	0.025	0.1243	1.074	0.0483	1.103
214828	0.014	0.0620	1.129	0.0237	1.156
883963	0.008	0.0307	1.217	0.0123	1.139

N	h_{D}	$\mathrm{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathrm{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	h_{Σ}	$\mathrm{e}(\lambda)$	$r(\lambda)$	iter
1007	0.200	1.2760	–	0.1105	–	0.125	0.1930	–	7
3790	0.095	0.6135	0.984	0.0385	1.417	0.063	0.0704	1.455	8
14014	0.049	0.3115	1.037	0.0150	1.438	0.031	0.0296	1.253	9
55428	0.026	0.1566	1.081	0.0067	1.282	0.016	0.0141	1.064	9
214828	0.015	0.0784	1.184	0.0033	1.222	0.008	0.0070	1.022	9
883963	0.007	0.0393	0.982	0.0016	0.995	0.004	0.0035	1.009	9

Table: Errors and rates of convergence for the conforming $\mathrm{BR} - \mathrm{RT}_0 - \mathrm{P}_0 - \mathrm{P}_0$ scheme ($\mathtt{F} = 10$)

F	$h = 0.2001$	$h = 0.1088$	$h = 0.0494$	$h = 0.0262$	$h = 0.0146$	$h = 0.0077$
0	4	4	4	4	4	4
1	5	5	5	6	6	6
10	7	8	9	9	9	9
100	8	9	10	10	11	11

Table: Convergence behavior of our iterative method with respect to the Forchheimer number F

Some References

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Thanks for your attention!!!